IV. On the Theory of Color Mixing;

by H. Grassmann, professor in Stettin.

Translation by Dr. Jonathan Green, University of North Dakota.
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In the 87th volume of this journal, Mr. Helmholtz presents a series of partially new and ingenious observations from which he draws the conclusion that the theory of color mixing generally accepted since Newton was erroneous in its most essential points, and that there are only two prismatic colors, namely yellow and indigo, which yield white when mixed. Therefore it may not be superfluous to show how up to a certain point—namely the proposition that every color has its complementary color, which when mixed with it yields white—Newton's theory of color mixing arises from undeniable facts with mathematical evidence, so that this proposition must be regarded as one of the most well-founded in physics. I will then show how the positive observations made by Helmholtz, instead of testifying against this theory, can rather serve partly to confirm it and partly to supplement it.

Here it will be necessary to break down the color perception of which the eye is capable into its factors. First, the eye differentiates between colorless and colored light. For colorless light (white, gray), it only distinguishes greater or lesser *intensity*, and this can be specified mathematically. Likewise, for a homogeneous color, we only distinguish its greater or lesser intensity. But also for the difference of the individual homogeneous colors, we have a mathematically ascertainable measurement, which is provided to us most perfectly in the period of oscillation corresponding to each color; even popular language has designated this difference in a very fitting way with the expression *color tone* [hereafter translated as *hue*]. Thus we will be able to distinguish two things about a homogeneous color: its hue and its intensity. If you mix a homogeneous color with colorless light, the color perception is weakened by this admixture. Popular language is rich in terms that are meant to indicate this difference; the modifiers: saturated, deep, faint, pale, dull, whitish, which one adds to the color names, are meant to represent this relationship. The scientific designation that must be substituted for this popular nomenclature arises by itself from the considerations above in that every color perception of the type mentioned breaks down into three mathematically ascertainable factors: the *hue*, the *color intensity*, and the *intensity of the admixed white*. The different hues form a continuous series in such a way that if one progresses continuously beginning with one color in this series, the original color is finally repeated. In this regard, however, one circumstance should not be left unmentioned, namely the difficulty of obtaining homogeneous red light, which mediates the transition between the violet and the red of the ordinary solar spectrum, and which can only be produced through a prism under particularly favorable circumstances (around noon on sunny summer days; see Poggendorff's Annals [*Annalen der Physik*], vol. 13, p. 441). This extreme color of the spectrum, which can be understood as extreme red or as extreme violet, I will call purple. If we now finally consider an arbitrarily composed light, the eye can likewise only distinguish the three factors previously mentioned. That is, every perception of light can be imitated by mixing a homogeneous color of a certain intensity with colorless light of a certain intensity. According to this we have to distinguish three things for each perception of light: the intensity of the color, the hue, the intensity of the admixed colorless light. An apparatus could easily be constructed by means of which one would be able to specify each color according to these three factors. To give an idea of this, think of two white panels of the same type, mobile around a hinge, so that the white side of the panels is on the outside of the angle formed by the panels, and at the same time there is a divided circle to measure this angle. Now let the colored light to be tested fall on one of these panels in a plane perpendicular to the axis of rotation; white light falls on the other panel in an arbitrary direction of that plane and homogeneous light falls on it in a direction perpendicular to the same plane, and the latter is chosen so that it has the same hue as the light to be tested. By turning this latter panel around the hinge, one will be able to give any desired intensity level to the colorless and the homogeneous light, which is scattered by this panel in all directions. By then turning the first panel also, one will be able to give the light it scatters any degree of intensity that is less than the intensity of light that falls perpendicularly. In this way, if one has only made the comparison lights falling on the second panel weak enough, one will necessarily find a position of the panels such that both induce the same light perception in an eye that sees them simultaneously. Such an apparatus would therefore be sufficient to mathematically ascertain all relevant factors. Now of course, the above proposition that the eye can directly distinguish only these three factors could be called into question. And it is true that direct proof would be difficult to provide, since there is still the possibility that one eye, due to its particular organization, might be able to detect differences that another eye cannot. For our purposes, however, the fact is entirely sufficient that no observer has yet been able to indicate another factor that would determine the perception of color, and the language for describing color perceptions only knows these three factors, so that we can claim with certainty that only these three factors of color perception have been observed until now; and it is only to this assertion that we will return in the proof to be mentioned below.

The second thing we assume is: "If one continuously changes one of the two lights to be mixed (while the other remains unchanged), the perception of the mixture also changes continuously."

Specifically, we say that light perception changes continuously when the two intensities (the intensity of color and that of the admixed colorless light) change continuously, and the hue also changes continuously, provided that the intensity of the color is not zero. For if the intensity of the color is zero, the light is colorless; and therefore one hue can continuously transition into any other hue that is completely separate from it as the intensity of the color decreases steadily to zero, namely when the intensity of the other hue in turn increases steadily from zero. It hardly needs to be mentioned that the case where one or more of the factors determining perception remain the same must be included under the concept of continuity, as is customary everywhere. As far as the continuous change in hue is concerned, it is generally represented by the continuous change in the period of oscillation that determines this hue, but with the difference that the color perception of extreme violet in turn continuously connects with that of extreme red. In fact, the transition from violet through purple to red is just as continuous for the eye as between any two other colors, although observations have by no means established the limit at which the same color perception recurs with different periods of oscillation. I will designate the transition from red to orange, yellow, green, blue, violet, purple back to red as the positive transition, and the reverse as the negative. According to this, any colored light A can pass continuously into a differently colored light B in three different ways, namely either in such a way that the light's hue gradually assumes all the hues that appear on the *positive* transition from A to B, or all those lying along the *negative* transition, or finally, by the light becoming colorless once or several times during the transition. The principle of continuous transition that we have just developed must be regarded as one that is fully justified by experience, since an abrupt jump in the phenomena would have to make itself visible even to the crudest observations, and no one has observed such a jump until now.

From these assumptions, the following proposition can be derived with mathematical obviousness:

"For each color, there is another homogeneous color that, mixed with the original color, produces colorless light."

Proof. Let a be the hue of the given color. Assuming now that there is no homogeneous color that produces colorless light when mixed with the first color, then let an arbitrary homogeneous color be assumed whose hue is x and whose intensity is y. If one first lets y steadily increase from zero until the intensity of the color a disappears against it, while x remains constant, the mixture will change continuously, and since it should never give colorless light according to the assumption, its hue will also change continuously, that is, since the mixture initially has the hue a and finally has the hue x, it will continuously transition from a to x. This transition can be positive or negative. Whether the one or the other is the case will depend on the hue x. If one assumes the hue x differs infinitesimally little from a, but towards the positive transition side, that transition will also be positive. Because if it were negative, then with the increase in intensity y, all hues except those of a would have to emerge differing infinitesimally little, that is, hues that are quite different from a; let y be such an intensity at which a hue very different from a would emerge. Now it is clear that the color whose hue is a and whose intensity is y, when mixed with a, produces the hue a, while the color whose hue is x and whose intensity is y yields a completely different hue; but when the intensity y is the same, these two colors mixed with a have two infinitesimally closely bordering hues, that is, those two colors mixed with a transition continuously into each other, therefore (according to the second proposition) the mixture must also change continuously, including its hue; this was meant to be entirely different, however. Therefore the assumption that the transition from a to x should be negative leads to contradictions, that is, it is necessarily positive. For the same reason, if x, starting with a, is distant to an infinitesimally small degree on the negative side, a negative transition from a to x will occur. If one lets the hue x, beginning with a, continuously change toward the positive side so that it traverses the whole range of colors until it returns to a, then the associated transition of the mixture that is caused each time by the increase of y necessarily changes its sign somewhere, since it is initially positive and finally negative. Let a' be a hue at which this change occurs so that the transition before x reaches this hue is positive, and as soon as it has surpassed it is negative. Now if the hue x goes through this hue a' continuously, then at each value of the intensity y, the hue of the mixture must change continuously, and thus all the hues that arise by increasing the intensity y in both cases (if x lies infinitesimally close to a' to the right, or to the left) are infinitesimally close to each other. But this is impossible because the ones are on the positive transition, the others on the negative transition from a to a'. Thus the assumption that there is no homogeneous color for a that when mixed with it yields white leads to a contradiction. That means that for every color there is a homogeneous color that when mixed with it yields white. *Which was to be shown*.

I have chosen the indirect form of proof because it is the easiest way to achieve the greatest possible rigor without digressions. Incidentally, it is clear that this indirect form of proof also includes the direct assertion that the color a', where the type of transition changes, is the one that when mixed with a at some level of intensity yields colorless light.

If we now examine Helmholtz's experiments, what emerges from them, at least approximately, is the color that is (along with some given color) capable of producing colorless light. For yellow, according to Helmholtz, this is indigo. This result is by no means as different from Newton's theory of color mixing as it seems at first glance. Helmholtz more precisely specified the two colors that, according to him, yield white: the yellow lies between the Fraunhofer lines D and E, and to be precise, about three times as far from E as from D. The indigo, however, lies toward G as measured from the midpoint between the lines F and G, so that any indigo that lies between the specified limits along with any yellow that is in the vicinity of the indicated point yields white. The comparison with Newton's rule of color mixing is made more difficult by the fact that the color names do not have the same meaning among different observers, as one can easily see by comparing the descriptions of the colors that are said to lie between the various Fraunhofer lines in different textbooks and treatises. Newton describes the position of the boundaries between any two of his colors exactly as they were shown in the spectrum of his prism; he also specifies the mean index of refraction and the dispersion ratio of this prism, so that all elements are present to determine the position of the Newtonian color boundaries between the Fraunhofer lines as precisely as the Newtonian specifications themselves are sufficient. According to this principle, by comparing the Fraunhofer and Newtonian measurements, and assuming that Newton's initial red and his final violet coincide with the Fraunhofer lines *B* and *H*, I found that Newton's initial orange (that is, the border between red and orange) lies between the lines *C* and *D*, distant from *C* and *D* in the ratio of 7:6; his initial yellow is located at *D* (distant from *D* in the direction of *E* by 1/11 of the interval *DE*); his initial green is located at *E* (distant from *E* by 1/11 *ED* in the direction of *D*); his initial blue is located at *F* (distant from *F* by 1/14 *FG* in the direction of *G*); his initial indigo lies between *F* and *G*, distant from *F* and *G* in the ratio of 5:3; his initial violet is in *G*.

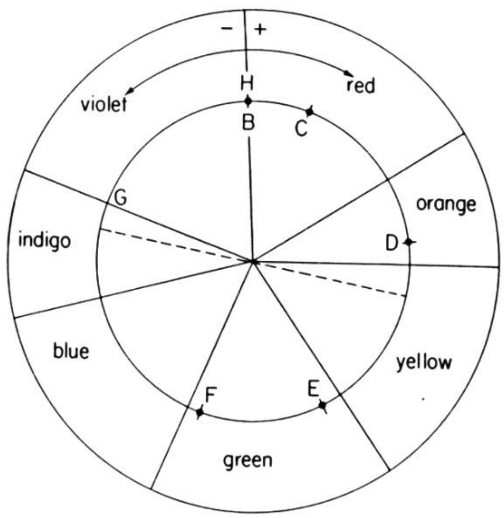

Figure 16 Plate 1.
(Figure 2 in Macadam, D. L. (1970). Sources of Color Science: Selected and Ed. by David L. Macadam. MIT Press. P. 58.)

To be sure, there is something arbitrary about the assumption that the boundaries of the Newtonian spectrum coincide with the lines *B* and *H*; but one also arrives at the same result if one assumes that the colors with the mean refractivity coincide for Fraunhofer and Newton. If one now constructs Newton's color wheel according to the rule given in his *Opticks* (*Lib. I. pars II, prop. VI*), and if one takes into it the positions of the Fraunhofer lines as they are indicated above (see Figure 16 Plate I in original source), the result is that according to the Newtonian rule, the yellow specified by

Helmholtz yields white with an indigo that lies between the Fraunhofer lines *F* and *G* and that is distant from *F* and *G* by the ratio of 15:2. In the figure, these colors are indicated by the dotted line connecting them. Thus this indigo still falls within the color boundaries between which the complementary colors of yellow lie, according to Helmholtz. One sees therefore that the above observation by Helmholtz essentially agrees with the result of Newton's experiments. For the other colors, however, Mr. Helmholtz denies the possibility of obtaining white from them through the mixture of two colors. But if we examine any of his series of experiments, for example the one about the mixing of red with the other colors, it easily yields the complementary color each time. According to him, red along with orange, yellow, and green provide the middle hues, which in this sequence, that is, beginning with red, are on the positive side according to our designation. So for example, according to him, red mixed with green produces a *pale* yellow, which transitions into red when red is predominant via orange, while when green is predominant it transitions via yellow-green into green. Likewise, red along with violet, indigo blue, and sky blue provide the intermediary hues in this series, which, beginning with red, are on the negative side according to our designation. In particular, according to him, red mixed with sky blue yields a *whitish* violet, which transitions into rose-red and carmine red when red is predominant. Therefore, according to the proposition demonstrated above, the complementary color of red must lie between green and sky blue, that is, it must be a shade of blue-green. Now it is true that Helmholtz says that when red is mixed with green-blue hues, a flesh-colored mixture is produced; but how this flesh color transitions into this when blue-green is predominant, as clearly must be the case, is not said. So a gap remains here. Moreover, flesh color is nothing but a red mixed with much white, and no other transition from this to blue-green is conceivable other than the one where the red weakens more and more until it disappears under the admixed white, and then the blue-green gradually emerges from this white (or gray); in short, the normal transition via colorless light is taking place here. The same is true of the other series of experiments. The table of complementary colors derived from them would be as follows:

Yellow,	yellow-green,	green,	green-blue,	sky blue,	indigo,
Indigo,	violet,	purple,	red,	orange,	yellow,

where the complementary colors that belong together are located below each other.

So far I have tried to make do with as few premises as possible. In order to derive the fundamental theorem of color mixing, I will now add a third premise to the previous two, namely:

"that two colors, each of which has an unvarying hue, unvarying color intensity and unvarying intensity of admixed white, also yield an unvarying color mixture, regardless of which homogeneous colors they are composed of."

This premise also seems to be sufficiently justified by the observations made so far. For the fact that the mixture of colored powders yields different results than when, instead of mixing the powders themselves, one mixes the light emanating from them, cannot provide any grounds for objection, especially since Helmholtz discovered the reason for this deviation.

Now let *a* be a homogeneous color, and let *a*' be the homogeneous color that yields white when mixed with *a*. For the sake of clarity, one may imagine *a* and *a*' represented by two equally long but oppositely directed line segments (Figure 17, Plate I) that originate from one point. Furthermore, let *b* be a color that, when mixed with *a*, yields as much white as when mixed with *a*'; and in order to express this equal relationship of *b* to *a* and to *a',* let *b* be represented by a line segment perpendicular to *a* and *a*'. Furthermore, let the intensity of the color *b* be chosen so that if *b*' is the color that yields white along with *b*, the intensity of the light produced by this mixture is equal to the intensity of the light produced by the mixture of *a* and *a*'. Let this be represented graphically by making the line segment that expresses the color *b* the same length as *a* and *a*', while the complementary color of *b* is represented by the line segment *b*' with the same length as *b* but of opposite direction. Let us assume that of the two colors *b* and *b*', the color *b* is the one that, beginning from *a*, lies on the positive transition side. It is evident that if the color *a* is given, then *a*', *b*, *b*' can be found by observation. If, for example, *a* is yellow, then *a*' is indigo; the various shades of green and blue lie on the positive transition from *a* to *a'*; mixed with yellow (*a*), green-yellow yields a very small admixture of white, while mixed with indigo (*a*'), it yields a very significant admixture of white. Beginning from green-yellow, if one proceeds on the positive side, the admixture of white will gradually increase when mixed with yellow, and decrease when mixed with indigo. So a hue will lie along the transition that, when mixed with the yellow, yields as much white as when mixed with indigo. If for example this is green, then *b* will be green and *b*' will be purple. It is now evident that by mixing two of these four colors at a time, one must obtain all the hues. For all intensity levels of the homogeneous colors to be mixed *a* and *b*, *b* and *a*', *a*' and *b*', *b*' and *a*, these hues are to be found by observation. We assume that the intensities of the two colors to be mixed are represented by the lengths of the associated line segments so that if one color has the hue *a*, for example, and its intensity has the same ratio to that of *a* as the ratio of *m* to 1, then let that color be represented by a line segment that has the same direction as *a* but has *m* times the length. After the two colors to be mixed have been geometrically represented in this way, let *the geometric sum* be constructed from these line segments – that is, the diagonal of the parallelogram that has these two line segments as sides[1] – and let it be stipulated that this sum or diagonal is meant to represent the color of the mixture,

[1] The concept of this geometric sum was first developed by me in my *Ausdehnungslehre* [extension theory] (Leipzig 1844) and by Möbius in his *Mechanik des Himmels* [mechanics of the heavens] (Leipzig 1843).

namely its direction represents the hue and its length represents the intensity of the color.

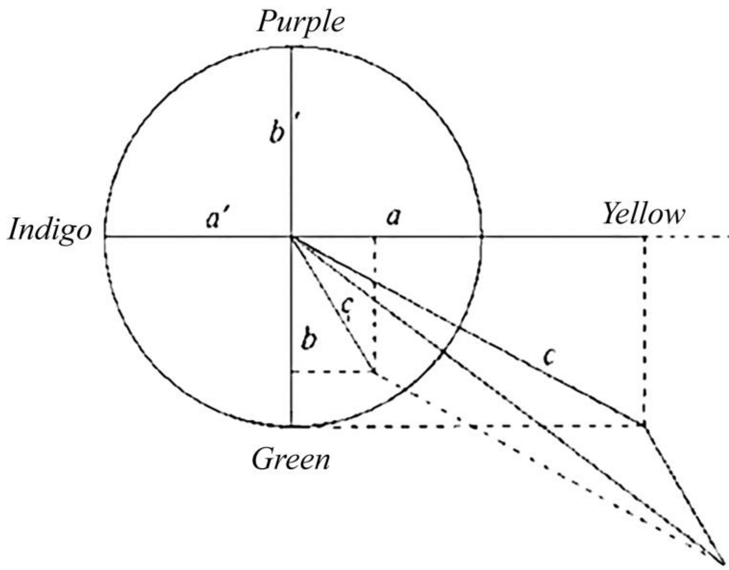

Figure 17 Plate 1.

Once this has been done, the hue and intensity of any mixture of colors can be found from now on through simple construction. To be precise, one only needs to specify the line segments that represent the hue and the color intensity of the colors to be mixed, and then add them geometrically, that is, combine them like forces, and then the geometric sum (the resultant of those forces) represents the hue and the color intensity of the mixture. It follows directly from this that the order in which one adds geometrically (combines the forces) is unimportant to the result. In fact, let the colors represented by the line segments a, b, a', b' be taken as a basis according to the above specification, and let αa be understood (when α is positive) as a color that has the hue a and whose color intensity has the ratio to that of a like α has to 1, or (when α is negative) let αa be understood as a color that has the hue of the complementary color a' and whose color intensity in turn has the ratio to that of a' like α has to 1. Let the same be true of the second color taken as a basis b and its complementary color b'. Of the two colors e and e_1 whose color when mixed one is seeking, if the one can be represented by the mixture of colors αa and βb and the other by the mixture of the colors $\alpha_1 a$ and $ßb_1$, then (always irrespective of the admixed white) the mixture of c and c_1 can be represented as the mixture of the four colors αa, βb, $\alpha_1 a$, $\beta_1 b$. But αa mixed with $\alpha_1 a$ yields $(\alpha + \alpha_1)a$ and βb mixed with $\beta_1 b$ yields $(\beta + \beta_1)b$. Thus the mixture of c and c_1 can also be represented by the mixture of the two colors $(\alpha + \alpha_1)a$ and $(\beta + \beta_1)b$. Since these latter ones, however, have the hues taken as a basis a, b or a', b', their mixture is represented by the geometric sum of the line segments, that is, by the line segment $(\alpha + \alpha_1)a + (\beta + \beta_1)b$, that is, by $(\alpha a + \beta b) + (\alpha_1 a + \beta_1 b)$, that is, by the geometric sum of two line segments that, taken individually, represent the colors to be mixed.

We can express this law, which necessarily follows from the three basis assumptions, and which only requires a simple but complete series of observations to determine the color series, in another way. Specifically, if one draws a circle around the origin point of the line segments with radius *a*, and instead of each line segment draws the point at which it meets the circumference, provided with a weight that is proportional to the length of the line segment, then given two colors, their color mixture can be found in the following way: One represents each of the colors to be mixed by this type of weighted point on the circumference, specifically so that the associated radius indicates the hue, and the associated weight expresses the color intensity, and then one determines the center of gravity. Then the line segment drawn from the midpoint to this center of gravity indicates the hue and, after it is multiplied by the sum of the weights, the color intensity as well. The identity of this determination with the earlier one can easily be seen from the following construction of the center of gravity, which has been proven in my extension theory: One finds the center of gravity of the points A, B, C..., which are respectively provided with the weights α, β, γ..., by drawing from an arbitrary point O the line segments OA, OB, OC..., multiplying them by α, β, γ,... (that is, changing their length in the ratio of 1: α, 1: β, 1 : γ... without changing their direction) and forming the geometric sum from the line segments thus obtained, and then dividing this by $\alpha + \beta + \gamma +$..., and then the end point of the line segment thus obtained is the center of gravity that is sought.

Finally, as far as the admixture of colorless light is concerned, an additional premise is necessary. It is simplest to assume:

“that the total light intensity of the mixture is the sum of the intensities of the mixed lights.”

In this regard, by the total light intensity, I mean the sum of the intensity of the color (as I have specified it above) and the intensity of the admixed white. In doing so, I do not set the intensity of the white or of each individual color proportional to the square of the oscillation intensity, but to the oscillation intensity itself, so that when two white lights or lights of the same color are mixed, the intensity of the mixture is the sum of the intensities of the mixed lights. This fourth premise is not to be regarded as so well-founded as the earlier ones, although it certainly emerges as the most probable due to theoretical considerations. To draw inferences from this hypothesis, we will set the intensity of the color represented by the line segment *a* equal to 1 and assume that the various homogeneous colors whose intensity is 1 are represented by points on the circumference so that the weight of these points, in accordance with the above, must also be set equal to 1. Now let (Figure 18 Plate 1) *A* and *B* be two points on the circumference that thus represent homogeneous colors of

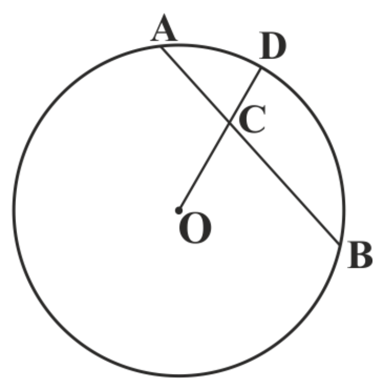

Figure 18 Plate 1

intensity 1. Now if the colors αA and βB are mixed, that is, two homogeneous colors whose intensities are α and β and whose hues are A and B, then the sum of the intensities is $\alpha + \beta$. In order to determine the color of the mixture, in accordance with the above, we must seek the center of gravity of the points A and B provided with the weights α and β. If the center of gravity is C and the center of the circle is O, then if the radius of the circle is set to 1, in accordance with the above, the color intensity is equal to $(\alpha + \beta)OC$. Let the point where OC extended meets the circumference be D, and then the total intensity is $\alpha + \beta$, or,
since the radius is set to 1, $(\alpha + \beta)$OD. According to the stipulated premise, this total intensity should be equal to the intensity of the color plus the intensity of the admixed white, thus the latter is equal to $(\alpha + \beta)$OD - $(\alpha + \beta)OC$, that is, = $(\alpha + \beta)CD$. Thus the intensity of the admixed white is equal to the distance of the center of gravity from the circumference multiplied by the sum of the weights. From this it then further follows that if one constantly thinks of the entire mass as united in the center of gravity (in which case one calls the center of gravity thus provided with such a weight the *geometric sum* of the individual points with their weights[2]), then every perception of light is precisely represented according to its three factors by a point with a certain weight. The direction in which this point lies from the center, or also the point at which this direction meets the circumference, represents the hue, while the weight of the point represents the total light intensity; the distance from the center multiplied by this weight represents the intensity of the color, and the distance from the circumference multiplied by the weight represents the intensity of the admixed white. If by the color saturation of a light we understand the intensity of its color divided by the total light intensity, then the color
saturation is represented by the simple distance of the point from the center. If one has represented two or more colors to be mixed in this way, then the mixture is completely represented by the geometric sum of the weighted points representing the individual colors. It can be seen that this law, derived here in a purely mathematical way from four sufficiently well-founded premises, agrees in its essential features with Newton's empirical rule as he presents it at the place indicated. But the way in which Newton distributes the homogeneous colors around the circumference of his circle requires a thorough revision, for which the experiments of Mr. Helmholtz have made only the first beginning. Only when there is sufficient light on this can one venture to answer the interesting question concerning the law by which the vibrations of the ether belonging to the various colors combine in the nerves or in the sensory apparatus to form simple color perceptions, a question on whose answer the idea of different colors and colorless light substantially depends.

Stettin, 19 February 1853.

Translation by Dr. Jonathan Green, University of North Dakota.

[2] See my *Ausdehnungslehre* [extension theory] and Möbius' *Barycentric Calculus*.