WEAKLY SYMMETRIC HOMOGENEOUS SPACES AND SPHERICAL STEIN MANIFOLDS

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Abstract

We present several results on homogeneous spaces of reductive Lie groups and on the actions of such groups on complex manifolds. In particular, we consider weakly symmetric spaces in the sense of A.Selberg, spherical varieties, and their complex analytic counterparts. The results are motivated by geometric representation theory. The proofs are based on the properties of the Weyl involution.

1. Let G be connected Lie group, $H \subset G$ a compact subgroup, and $M = G/H$ the corresponding homogeneous space with the base point $x_0 = e \cdot H \in M$. The action $G \times M \to M$ is assumed to be effective. Thus G is identified with some group of diffeomorphisms of M. There is a G-invariant metric on M and, for any such metric, G is contained if the connected component of the isometry group of M. However, no choice of an invariant metric is required. The homogeneous space M is called weakly symmetric (with respect to G) if there exists a diffeomorphism $s : M \to M$, such that:

(1) $s\,Gs^{-1} = G$;

(2) $s^2 \in G$;

(3) $\forall\ x, y \in M \ \ \exists\ g \in G : \quad gx = sy,\ gy = sx.$

It is easy to see that if (1) is fulfilled and (3) holds for some $x, y \in M$ then (3) also holds for hx, hy with arbitrary $h \in G$. Thus, if M is symmetric and G is the connected isometry group of M, then one can take for s the geodesic symmetry at x_0. This shows that a symmetric space is weakly symmetric. The diffeomorphism s in (2) is not uniquely defined and, in all known examples, s can be chosen to be involutive.

The notion of a weakly symmetric homogeneous space was introduced by A.Selberg (1956) in connection with his celebrated trace formula. A.Selberg proved that for a weakly symmetric space M the algebra $\mathcal{D}(M)^G$ of G-invariant differential operators is commutative. This was the only property, essential for his proof of the trace formula, and so he explicitly mentioned in the inroduction to [9] that he does not know whether the assumption of weak symmetry is necessary for commutativity of $\mathcal{D}(M)^G$. We will refer to this as Selberg's question. In the same paper, A.Selberg constructed the first example of a weakly symmetric space, which is not symmetric. Namely, he showed that the unit tangent bundle over the Lobachevsky plane Λ^2 is weakly symmetric with respect to $G = \mathrm{PSL}(2, \mathbb{R}) \times \mathrm{SO}(2)$, where $\mathrm{PSL}(2, \mathbb{R})$ is the isometry group of Λ^2 and $\mathrm{SO}(2)$ acts by simultaneous rotations in all tangent planes.

2. Let $\mathbf{G}$ be a connected reductive algebraic group over $\mathbb{C}$, which will be identified with the group of its complex points. A normal algebraic $\mathbf{G}$-variety X is called spherical if a

Borel subgroup of $\mathbf{G}$ has an open orbit on X. For X affine this is the case if and only if the algebra of regular functions $\mathbb{C}[X]$ is a multiplicity free $\mathbf{G}$-module. An algebraic subgroup $\mathbf{H} \subset \mathbf{G}$ is called spherical if the homogeneous space $\mathbf{G}/\mathbf{H}$ is a spherical variety. The notion of a spherical variety has been carried over to the category of complex spaces. Namely, let G be a connected real Lie group, which is a real form of $\mathbf{G}$ in the following sense. The group G is embedded as a closed subgroup in $\mathbf{G}$ and the (real) Lie algebra $\mathfrak{g} = \mathrm{Lie}(G)$ satisfies $\mathfrak{g} \oplus i\mathfrak{g} = \mathrm{Lie}(\mathbf{G})$. Suppose that Y is a normal complex space on which G acts by holomorphic transformations. Then $\mathbf{G}$ acts on Y locally, and so $\mathrm{Lie}(\mathbf{G})$ is represented by holomorphic vector fields on Y. The complex space Y is said to be spherical with respect to G if there exists a point $y \in Y$, such that the holomorphic tangent space at y is generated by vector fields from a Borel subalgebra of $\mathrm{Lie}(\mathbf{G})$. Furthermore, if Y is contained as a G-invariant domain in a normal Stein space X acted on by $\mathbf{G}$ then X is in fact an affine spherical $\mathbf{G}$-variety; see [2]. Moreover, if G is compact and Y is itself Stein, then by a result of [3] one can always find a G-equivariant open embedding $Y \hookrightarrow X$. Here are some examples of complex spherical spaces which can serve as a motivation in the non-algebraic setting. The proofs for 2) and 3) can be found in [2].

1) Let $\mathbf{P} \subset \mathbf{G}$ be a parabolic subgroup, $X = \mathbf{G}/\mathbf{P}$ the corresponding flag variety. Any open G-orbit in X is a spherical complex manifold of G.

2) Let D be a bounded symmetric domain in $\mathbb{C}^n$ and let K be the isotropy subgroup at some point $o \in D$. Then D is a spherical complex manifold with respect to K.

3) Even if a complex manifold is acted on by a complex Lie group, there are interesting non-algebraic examples of spherical complex spaces. A compact complex homogeneous space X of a complex reductive algebraic group is spherical if and only if X is a locally trivial fiber bindle over a flag variety with a complex torus as fiber.

3. Forty years after Selberg's paper, J.Lauret constructed in [8] an example of a homogeneous space with compact isotropy subgroup, which has commutative algebra of invariant differential operators, but is not weakly symmetric. In his example, the transitive group is a semidirect product $G = K \ltimes N$, where K is compact and N is nilpotent of class 2. Thus, the answer to Selberg's question is in general negative. However, it was shown in [5] that the answer for reductive groups is positive. More precisely, we have the following two theorems.

Theorem 1. *Let $\mathbf{G}$ and $\mathbf{H} \subset \mathbf{G}$ be complex reductive algebraic groups, defined over $\mathbb{R}$. Assume that $\mathbf{G}$ is connected and $\mathbf{H}(\mathbb{R})$ is compact. Let G be the connected component of $\mathbf{G}(\mathbb{R})$ and let $H = G \cap \mathbf{H}(\mathbb{R})$. Then $M = G/H$ is a weakly symmetric homogeneous space if and only if $X = \mathbf{G}/\mathbf{H}$ is a spherical variety.*

Theorem 2. *If G and H are as in Theorem 1 then $M = G/H$ is weakly symmetric if and only if the algebra $\mathcal{D}(M)^G$ is commutative.*

Sketch of proof of Theorem 2. Consider the algebra of algebraic differential operators on $X = \mathbf{G}/\mathbf{H}$, to be denoted $\mathcal{D}_{alg}(X)$, and its subalgebra $\mathcal{D}_{alg}(X)^{\mathbf{G}}$ consisting of all $\mathbf{G}$-invariant operators. Since M is embedded into X as a totally real submanifold of maximal dimension, it is easy to show that the algebras $\mathcal{D}(M)^G$ and $\mathcal{D}_{alg}(X)^{\mathbf{G}}$ are isomorphic. It is known that $\mathbb{C}[X]$ is an irreducible $\mathcal{D}_{alg}(X)$-module. By Jackobson's Density Theorem, for any linearly independent $u_1, \ldots, u_n$ and arbitrary $v_1, \ldots, v_n$ in $\mathbb{C}[X]$

there exists $D \in \mathcal{D}_{alg}(X)$, such that $Du_i = v_i$ for all $i, \ i = 1, \dots, n$. Assume now that M is not weakly symmetric and so, by Theorem 1, X is not spherical. Let $\{p_1, \dots, p_n\}$ and $\{q_1, \dots, q_n\}$ be bases of two distinct $\mathbf{G}$-submodules in $\mathbb{C}[X]$ with the same representation matrices. Then there exists $D_1 \in \mathcal{D}_{alg}(X)$ with $D_1(p_i) = \alpha p_i, \ D_1(q_i) = \beta q_i, \ \alpha \neq \beta$. Also, there exists $D_2 \in \mathcal{D}_{alg}(X)$ with $D_2(p_i) = q_i, \ D_2(q_i) = p_i$. By unitary trick, we may assume that D_1, D_2 are $\mathbf{G}$-invariant. Obviously, D_1 and D_2 do not commute. □

3. For the future use we recall the definition of a Weyl involution. We give the definition for connected complex reductive groups, but it goes along the same lines for connected compact Lie groups. An algebraic involution $\theta : \mathbf{G} \to \mathbf{G}$ is called a Weyl involution if θ acts as inversion on a maximal torus in $\mathbf{G}$. For any $\mathbf{G}$ a Weyl involution exists and is unique up to conjugation by an inner automorphism. A Weyl involution transforms a $\mathbf{G}$-module into the dual $\mathbf{G}$-module. A Cartan involution of $\mathbf{G}$ is an antiholomorphic involution whose fixed point subgroup is a maximal compact subgroup of $\mathbf{G}$. One can always find a Cartan involution τ, commuting with the given Weyl involution. Their product $\sigma = \tau\theta = \theta\tau$ is an antiholomorphic involution defining the split real form of $\mathbf{G}$.

The main technical tool used in [5] for the proof of Theorem 1 is the construction of a Weyl involution of $\mathbf{G}$, which preserves a given reductive spherical subgroup $\mathbf{H} \subset \mathbf{G}$ and acts as a Weyl involution on the connected component $\mathbf{H}^\circ \subset \mathbf{H}$. The proof is based on the classification of reductive spherical subgroups. As a corollary, one has the following result, which is useful in the analytic setting.

Theorem 3. Let $\mathbf{G}$ be a connected complex reductive group and let $\mathbf{H} \subset \mathbf{G}$ be a reductive spherical subgroup. Then there exist a Weyl involution $\theta : \mathbf{G} \to \mathbf{G}$ and a Cartan involution $\tau : \mathbf{G} \to \mathbf{G}$, such that $\theta\tau = \tau\theta, \ \theta(\mathbf{H}) = \mathbf{H}$, and $\tau(\mathbf{H}) = \mathbf{H}$. Moreover, θ and τ restricted to $\mathbf{H}^\circ$ are precisely a Weyl involution and a Cartan involution of $\mathbf{H}^\circ$.

Proof. Let L be a maximal compact subgroup of $\mathbf{H}$ and let K be a maximal compact subgroup of $\mathbf{G}$ that contains L. Then K is the fixed point subgroup of some Cartan involution $\tau : \mathbf{G} \to \mathbf{G}$. By a result of [5], there is a Weyl involution $\theta : K \to K$, such that $\theta(L) = L$ and θ induces a Weyl involution of the connected component L°. For any $k \in K$ we have $\theta\tau(k) = \theta(k) = \tau\theta(k)$ by the definition of τ. Denote again by θ the unique extension to $\mathbf{G}$ of the given Weyl involution. Since $\mathbf{G}$ is connected and the relation $\theta\tau(g) = \tau\theta(g)$ holds on K, it also holds on G. Finally, L° is a maximal compact subgroup in $\mathbf{H}^\circ$, and so $\theta|_{\mathbf{H}^\circ}$ is a Weyl involution. □

4. Let Y be an irreducible reduced complex space on which a compact Lie group K acts continuously by holomorphic transformations. J.Faraut and E.G.F.Thomas gave a simple geometric condition which implies that the function algebra $\mathcal{O}(Y)$ is a multiplicity free K-module; see [6]. Namely, if there exists an antiholomorphic involution $\mu : Y \to Y$ with the property that, for every $y \in Y$, the orbit Ky is preserved by μ as a set then $\mathcal{O}(Y)$ is multiplicity free. In fact, the setting in [6] is more general. The authors consider any, not necessarily compact, group of holomorphic transformations of Y and study invariant Hilbert subspaces of $\mathcal{O}(Y)$. A simplified proof for compact groups is found in [4]. Our goal is to prove a converse theorem for a special class of complex spaces. The example of a trivial action on a compact complex manifold shows that this cannot be true in general. However, we have the following preparatory result from [4] for complex spaces with sufficiently many holomorphic functions.

Theorem 4. *Let Y be a holomorphically separable reduced complex space, K a connected compact Lie group acting on Y by holomorphic transformations, $\theta : K \to K$ a Weyl involution, and $\mu : Y \to Y$ a θ-equivariant antiholomorphic involution of Y, i.e., $\mu(ky) = \theta(k)\mu(y)$ for all $k \in K,\ y \in Y$. Then $\mathcal{O}(Y)$ is multiplicity free if and only if $\mu(y) \in Ky$ for all $y \in Y$.*

Sketch of proof. It suffices to proof the "only if" statement. Applying Stone-Weierstrass theorem to the algebra $\mathcal{A}(Y) = \mathcal{O}(Y) \cdot \overline{\mathcal{O}(Y)}$ and using the averaging process over K, one can find a special family $\{F_\lambda\} \subset \mathcal{A}(Y)^K$, which separates K-orbits. Namely, each F_λ is contained in $W_\lambda \cdot \overline{W_\lambda}$, where λ is a highest weight and W_λ is the corresponding irreducible submodule occuring in $\mathcal{O}(Y)$ (with multiplicity 1). Due to the equivariance of μ with respect to the Weyl involution, the image of W_λ under μ is precisely $\overline{W_\lambda}$. From this and from the fact that that μ composed with complex conjugation preserves a K-invariant Hermitian product on W_λ, it follows that each F_λ is μ-invariant. Therefore μ preserves each K-orbit. $\square$

5. In this section, we state and partly prove our main results. A real structure on a complex algebraic variety X is an involutive isomorphism onto the conjugate variety $\overline{X}$. Such an isomorphism is often regarded as an antiholomorphic involution $\mu : X \to X$, taking local regular functions to local conjugate regular functions.

Theorem 5. *Let $\mathbf{G}$ be a connected complex reductive group, X a non-singular affine spherical $\mathbf{G}$-variety, and $\sigma : \mathbf{G} \to \mathbf{G}$ an antiholomorphic involution defining the split real form. Then X has a σ-equivariant real structure.*

Theorem 6. *Let K be a connected compact Lie group, $\theta : K \to K$ a Weyl involution, and Y a Stein manifold on which K acts by holomorphic transformations. Assume that Y is spherical with respect to K. Then there exists a θ-equivariant antiholomorphic involution $\mu : Y \to Y$. Any such involution μ preserves each K-orbit on Y.*

Theorem 7. *Let Y be a Stein manifold on which a connected compact Lie group K acts by holomorphic transformations. Then Y is spherical if and only if there exists an antiholomorphic involution $\mu : Y \to Y$, preserving each K-orbit.*

Proof of Theorem 7. If there exists μ with the above properties then, according to the result of J.Faraut and E.G.F.Thomas, $\mathcal{O}(Y)$ is a multiplicity free K-module. Since Y is a Stein manifold, it follows from [3] that Y is spherical. The converse statement results from Theorem 6. $\square$

Proof of Theorem 6. There is a K-equivariant open embedding $\iota : Y \hookrightarrow X$, where X is an affine spherical variety of the complexified group $\mathbf{G} = K^{\mathbb{C}}$. Moreover, according to [3], the embedding $\iota : Y \hookrightarrow X$ can be chosen in such a way that $\mathbf{G} \cdot \iota(Y) = X$. Therefore X is non-singular. By Theorem 6 we have an antiholomorphic involution $\mu : X \to X$, which is σ-equivariant with respect to $\mathbf{G}$. Since σ and θ agree on K, μ is θ-equivariant with respect to K. By Theorem 4, μ preserves each K-orbit. Therefore, $\iota(Y)$ is μ-invariant, and we get $\mu : Y \to Y$ by restriction. $\square$

Proof of Theorem 5. A well-known application of Luna's Slice Theorem displays X as a vector bundle. Namely, it is shown in [7] that

$$X = \mathbf{G} \times_{\mathbf{H}} V,$$

where $\mathbf{H}$ is a spherical reductive subgroup of $\mathbf{G}$ and V is a spherical $\mathbf{H}$-module. We sketch the proof for $\mathbf{H}$ connected and refer to [1] for the general case. We may assume that $L \subset \mathbf{H}$ and $K \subset \mathbf{G}$ are chosen and that θ, τ and σ are constructed as in the proof of Theorem 3. Assume we have an antilinear involution $\nu : V \to V$, satisfying

$$\nu(hv) = \sigma(h)\nu(v) \tag{$*$}$$

for all $h \in H,\ v \in V$. Consider the antiholomorphic involution $\tilde{\mu} : G \times V \to G \times V$, defined by

$$\tilde{\mu}(g, v) = (\sigma(g), \nu(v)).$$

For $h \in H$ let $t_h(g, v) = (gh^{-1}, hv)$. Then $t_h, h \in H$ define an action of H on $G \times V$, and X is the geometric quotient with respect to this action. It follows from $(*)$ that $\tilde{\mu} \cdot t_h = t_{\sigma(h)} \cdot \tilde{\mu}$. Thus $\tilde{\mu}$ gives rise to a self-map $\mu : X \to X$. Clearly, μ is an antiholomorphic involution with the required property.

In order to find an antilinear involution $\nu : V \to V$ satisfying $(*)$, recall that H and L are connected. Since θ defines a Weyl involution of L, it follows that the L-modules $\overline{V}$ (the complex conjugate) and V^θ (the θ-twisted module) are isomorphic. In other words, there is an antilinear isomorphism $\nu : V \to V$, such that

$$\nu(lv) = \theta(l)\nu(v) \tag{$**$}$$

for all $l \in L, v \in V$. It remains to find an involution with the same properties. This is reduced to an irreducible representation. Take a maximal (compact) torus $T \subset L$, on which θ acts as inversion. There is a weight vector $v \in V$, such that the corresponding weight $\lambda : T \to \mathbb{C}^*$ has multiplicity one. For the map ν constructed above and for $t \in T$ we have

$$t^{-1}\nu(v) = \theta(t)\nu(v) = \nu(tv) = \nu(\lambda(t)v) = \overline{\lambda(t)}\nu(v),$$

or, equivalently,

$$t\nu(v) = \lambda(t)\nu(v).$$

Therefore $\nu(v) = av$, where $a \in \mathbb{C}^*$. Applying ν again, we get $\nu^2(v) = |a|^2 v$. Since V is irreducible and ν^2 is an automorphism of V commuting with the group action, we have $\nu^2 = |a|^2 \cdot \mathrm{Id}$ by Schur's Lemma. Replacing ν by $|a|^{-1}\nu$, we obtain an antilinear involution with the property $(**)$ which is equivalent to $(*)$. $\square$

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