On mixture of two planar hamiltonian systems.

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1 Introduction and formulation of the main result

In several mathematical models of population dynamics evolution is described by the set of stochastically switching hamiltonian systems. See for instance [2] and references therein. Problems studied in such models usually are related to the description of asymptotical properties of trajectories of such systems. Ya.G. Sinai proposed to study similar question for deterministic switching of two planar transformations.

Setting of the problem. Let R_{1,ℓ_1} and R_{2,ℓ_2} denote two rotations around points $F_j = ((-1)^j, 0)$ for the length ℓ_j, $j = 1, 2$. Consider following transformation: $T_{\ell_1,\ell_2} : z \mapsto R_2(R_1(z))$ where $z = x + iy$. The question posed by Sinai is to describe asymptotics $|T^n_{\ell_1,\ell_2}(z)|_{\overrightarrow{n\to\infty}}$ of the trajectories of transformation T_{ℓ_1,ℓ_2}.

Rotations R_j can be considered as a section mappings of corresponding hamiltonian systems H_j with unit metric speed of rotation. In more details, let H_1 and H_2 denote two completely integrable planar Hamiltonian systems with simply connected closed invariant curves. In case of two rotations $H_j = |z - F_j|$, $j = 1, 2$. We shall say that transformation T_{ℓ_1,ℓ_2}

$$T_{\ell_1,\ell_2} : (x, y) \mapsto H_2(H_1(x, y, \ell_1), \ell_2) \tag{1}$$

is an (ℓ_1, ℓ_2)-mixture of H_1 and H_2. Here $H_j(\cdot, t)$ corresponds to the transformation along trajectories of the system H_j for the time t. In particular (ℓ_1, ℓ_2)-mixture of two rotations moves each point (x, y) by ℓ_1 units of length around first center F_1 and then moves its image by ℓ_2 units of length around second center F_2 (see Fig. 1). Negative sign of ℓ_j corresponds to the action in opposite time direction and in that particular case corresponds to the counterclockwise rotation while $\ell_j > 0$ corresponds to clockwise rotation.

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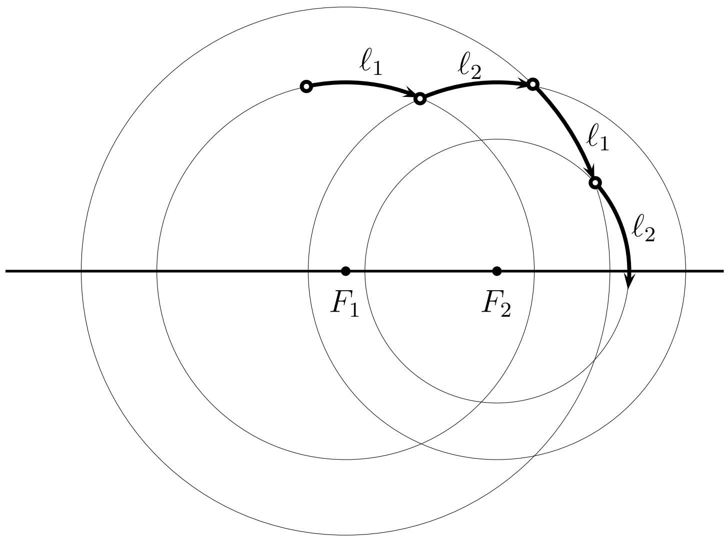

Figure 1: Mixture of two rotations with ℓ_1, $\ell_2 > 0$

Main result of this paper consists in the following proposition:

Theorem 1 *1. For $\ell_1 \neq -\ell_2$ trajectory $\{T^n_{\ell_1,\ell_2}(x,y)\}_{n=1}^{\infty}$ of any point (x,y) under the mixture* (1) *of two rotations remains bounded.*

2. If $\ell_1 = -\ell_2$ there exists trajectory going to infinity.

Proof of Theorem 1 is based of the following result from Kolmogorov-Arnold-Moser's theory:

Theorem 2 ***(see [1, §34])*** *Let T be an real-analytical (or sufficiently smooth) diffeomorphism defined in some "polar" coordinates (φ, r), $r > 0$ as*

$$\begin{cases} \varphi_1 = \varphi + \omega_0 + \omega r^{2l} + O(r^{2l+2}) \\ r_1 = r + O(r^{2l+2}) \end{cases} \tag{2}$$

for some $l \in \frac{1}{2}\mathbb{N}$. Suppose T satisfies intersection property: the image of any closed curve surrounding point $r = 0$ intersects this curve. Then T has infinitely many closed invariant curves surrounding stable point $r = 0$.

Aknowlegements Authors are deeply obliged to Ya. G. Sinai for an inspiration while posing the problem and fruitful discussions afterward. Authors appreciate very much useful remarks made by M. Balnk especially for letting us know [4] ..

2 Proof of main result.

Remind the notion of the composition of two hamiltonian systems with hamiltonians H_1 and H_2 as a system corresponding to vector field which is equal to linear combination of vector fields given by H_j, $j = 1, 2$. Such system also can be written in hamiltonian form with hamiltonian (see [3, Appendix 32] for details)

$$H = \ell_1 H_1 + \ell_2 H_2 \tag{3}$$

For $\ell_1 \neq -\ell_2$ invariant curves of hamiltonian H are closed and are called "ovale de Des Cartes" – algebraic curves which are defined by parametric equation

$$\ell_1|z - F_1| + \ell_2|z - F_2| = const \tag{4}$$

For particular case $\ell_1 = \ell_2$ hamiltonian H corresponds to elliptic rotation with family of confocal ellipses with foci F_1, F_2 as level-curves. If $\ell_1 = -\ell_2$ relation (4) define the family of confocal hyperbolas with the same foci.

Idea of the proof. We shall treat mixture $T_{\ell_1,\ell_2} = R_{1,\ell_1} R_{2,\ell_2}$ as a perturbation of section-mapping of composition of hamiltonian systems H_1 and H_2. For such perturbation of system (3) we apply Theorem 2.

Roughly speaking first statement of Theorem 1 can be formulated in such a way that trajectories of the transformation T_{ℓ_1,ℓ_2} cannot differ too much from the trajectories of Hamiltonian system (3). On the other hand, second statement says that even in unstable case $\ell_1 + \ell_2 = 0$ trajectories $T^n_{\ell_1,\ell_2}(z)$ follow unstable directions of the system (3).

Begining of the proof of Theorem 1. Planar topology has the following imortant property.

Lemma 1 (Topological Lemma) *Let T be $\mathbb{R}^2$ – homeomorphism and $\gamma_0 = \partial\Gamma_0$, $\gamma_1 = \partial\Gamma_1$ are its closed invariant curves such that $\gamma_0 \subset \Gamma_1$ for some open sets Γ_0, Γ_1. Then $T(\Gamma_1 \setminus \Gamma_0) \subseteq \Gamma_1 \setminus \Gamma_0$.*

Thanks to lemma 1 trajectory of any planar smooth dynamical system cannot cross invariant curve (see also [3, chapter 4, §22]). Thus for the proof of Theorem 1 it is sufficient to show that for any bounded set $\Omega \subset \mathbb{R}^2$ there exists closed invariant curve of transformation T_{ℓ_1,ℓ_2} which contain Ω in the interior. Thus first statement of Theorem 1 follows from Theorem 2 and following proposition:

Theorem 3 *Mixture* (1) *of two rotations with $\ell_1 + \ell_2 \neq 0$ for any point $z = x + iy$, $|z| > 2$ is a $O(|z|^{-3})$ perturbation of system* (3)*.*

Then if one consider a transformation T coresponding to the movement according to the system (3) by time 1, i.e $\ell_1 + \ell_2$ time of movement by the system $\frac{\ell_1}{\ell_1+\ell_2} H_1 + \frac{\ell_2}{\ell_1+\ell_2} H_2$ as a non-perturbed transformation in coordinates (r^{-1}, φ) from the formulation of Theorem 2, then by Theorem 3 T_{ℓ_1,ℓ_2} is a perturbation of T satisfying conditions of Theorem 2 and thus T_{ℓ_1,ℓ_2} have infinitely many invariant curves in any neighbourhood of the fixed point $r^{-1} = 0$ of T.

Proof of Theorem 3 goes by the next steps:

1. First consider the case $\ell_1 = \ell_2 = \ell$.

   Introduce new variables

$$\begin{cases} \xi = \dfrac{H_1 + H_2}{2} \\ \eta = \dfrac{H_2 - H_1}{2} \end{cases} \tag{5}$$

   Then

$$\begin{cases} H_1 = \xi - \eta \\ H_2 = \xi + \eta \end{cases} \tag{6}$$

   Lemma 2 *In new variables systems H_j, $j = 1, 2$ have a form*

$$H_j : \begin{cases} \dot{\xi} = (-1)^{j+1}\{\xi, \eta\} \\ \dot{\eta} = \{\xi, \eta\} \end{cases} \tag{7}$$

   while system H has the form

$$H : \begin{cases} \dot{\xi} = 0 \\ \dot{\eta} = \{\xi, \eta\} \end{cases} \tag{8}$$

   Canonical 2-form for hamiltonians H_j is $dx \wedge dy$. Corresponding Poisson bracket has the form of Jacobian:

$$\{f, g\} = \frac{\partial f}{\partial x}\frac{\partial g}{\partial y} - \frac{\partial g}{\partial x}\frac{\partial f}{\partial y} = \begin{vmatrix} \frac{\partial f}{\partial x} & \frac{\partial g}{\partial x} \\ \frac{\partial f}{\partial y} & \frac{\partial g}{\partial y} \end{vmatrix} \tag{9}$$

   In particular $\{x, y\} = 1$.

   The only difficulty in the proof of lemma 2 is to check, that expressions (5) indeed correspond to some variables. In the case of two rotations it is obviously so since variables (ξ, η) are in fact elliptic and hyperbolic coordinates of the point $z \in \mathbb{C}_+$ and thus $\mathbb{C}$ can be considered as 2–folded covering of (ξ, η)–plane. More general case will be discussed in section **??**. Once this fact is checked proof of the lemma follows from the next computation:

$$\{H_1, \xi\} = \frac{1}{2}\{H_1, H_2\}$$

$$\{H_2, \eta\} = -\frac{1}{2}\{H_2, H_1\}$$

$$\{H, \eta\} = \{\xi, \eta\} = \frac{2}{4}\{H_2, H_1\}$$

■

2. Variables $x = \mathrm{Re}(z)$ and $y = \mathrm{Im}(z)$ has the following expressions in variables (ξ, η)

$$\begin{cases} x = \xi\eta \\ y = \pm\sqrt{(\xi^2-1)(1-\eta^2)} \end{cases} \tag{10}$$

The latter expression is well-defined since for $|z| > 2$ it follows $|\xi| > 1$ and $|\eta| < 1$ since $F_j = (-1)^j + 0i$ and thus $|F_1 - F_2| = 2$.

Since $H_j = \sqrt{(x+(-1)^j)^2 + y^2}$ all terms in Jacobians (9) can be easily computed

$$\begin{aligned} \frac{\partial H_j}{\partial x} &= \frac{1}{2H_j}(2(x+(-1)^j)) = \frac{x+(-1)^j}{H_j} \\ \frac{\partial H_j}{\partial y} &= \frac{y}{H_j} \end{aligned} \qquad (j = 1, 2)$$

Finally from (10) it follows

$$\{\xi, \eta\} = \{H_1, H_2\} = -\frac{2y}{H_1 H_2} = -\frac{2\sqrt{(\xi^2-1)(1-\eta^2)}}{\xi^2-\eta^2} \tag{11}$$

3. On the level-curve $\xi = const$ variable η is a bounded function having same number of zeros for any level-curve. Thus η can be interpreted as a cosine of some function of arc-length parameter ψ corresponding to the level-curve and so pair (ξ, ψ) has a form of polar coordinates. Fortunately, in particular case of two rotations $\cos\psi = \eta$ and thus

$$\dot{\psi} = -\frac{1}{\sin\psi}\dot{\eta} \tag{12}$$

Since $\sin\psi = \dfrac{y}{\sqrt{\xi^2-1}}$ in variables (ξ, ψ) systems (7), (8) are non-degenerate.

4. Introduce new variable $\rho = \dfrac{1}{\xi}$. Then

$$\dot{\rho} = -\rho^2\dot{\xi} \tag{13}$$

and proof of Theorem 3 follows from the lemma

Lemma 3 *In variables (ρ, ψ) Poisson bracket $\{H_1, H_2\}$ for $\rho < \frac{1}{2}$ (i.e. $\xi > 2$) is $O(\rho)$.*

Proof of Lemma 3 consists in the computation $-\dfrac{2\sqrt{(\xi^2-1)(1-\eta^2)}}{\xi^2-\eta^2} = \rho\dfrac{\sqrt{1-\rho^2}}{1-\rho^2\cos^2\psi}$

■

From Lemma 3 and relation (13) it follows that in (ρ, ψ)– variables systems H_1 and H_2 correspond to $O(\rho^3)$ movement in ρ. Equations (7)–(8) get a form

$$
\begin{aligned}
&H_j : \begin{cases} \dot{\rho} = \rho^3 \cdot \frac{(-1)^{j+1}\sqrt{1-\rho^2}\sin\psi}{1-\rho^2\cos^2\psi} \\ \dot{\psi} = \rho \cdot \frac{\sqrt{1-\rho^2}}{1-\rho^2\cos^2\psi} \end{cases} \qquad j = 1, 2 \\
&H \ : \begin{cases} \dot{\rho} = 0 \\ \dot{\psi} = \rho \cdot \frac{\sqrt{1-\rho^2}}{1-\rho^2\cos^2\psi} \end{cases}
\end{aligned} \tag{14}
$$

Notice that equations for $\dot{\psi}$ coincide for three Hamiltonians. Thus movement for time ℓ under the action H_1 and then for time ℓ under the action of H_2 deviate from the movement for time 2ℓ under the action H by the quantity of order $O(\rho^3)$ in variable ρ. Since rhs of equations (14) for $\dot{\psi}$ are of order $O(\rho)$ then angular deviation of the mixture (1) from composition (3) is also of order $O(\rho^3)$. For $\ell_1 = \ell_2$ Theorem 3 is proven.

5. Certainly composition of H_1-action for the time ℓ and H_2-action for the time $(-\ell)$ (i.e. movement for time ℓ with Hamiltonian $(-H_2)$) deviate from initial point (ρ, ψ) of order $O(\rho^3)$. For variable ρ proof is the same as in step 4, and for variable ψ one should notice that for H_1 action ψ grows with the law H_1 for time ℓ and then decreases for the same time ℓ with the law which differs from $-H_1$ by the value of order $O(\rho^3)$. So mixture (1) in that particular case is a perturbation of identical transformation. Thus no KAM-theory results can be applied for this case.

6. Case $|\ell_1| \neq |\ell_2|$ can be deduced from the case $\ell_1 = \ell_2$ by lemma

Lemma 4 *Mixture* (1) *of two rotations with* $|\ell_1| \neq |\ell_2|$ *is also a* $O(\rho^{-3})$ *perturbation of the system* (3).

Proof. Decompose

$$
\begin{aligned}
\ell_1 &= \frac{\ell_1 + \ell_2}{2} + \frac{\ell_1 - \ell_2}{2} \\
\ell_2 &= \frac{\ell_2 - \ell_1}{2} + \frac{\ell_1 + \ell_2}{2}
\end{aligned}
$$

Then composition of H_1-action for the time ℓ_1 and H_2-action for the time ℓ_2 equals composition of H_1-action for the time $\frac{\ell_1+\ell_2}{2}$, H_1-action for the time $\frac{\ell_1-\ell_2}{2}$, H_2-action for the time $-\frac{\ell_1-\ell_2}{2}$ and H_2-action for the time $\frac{\ell_1+\ell_2}{2}$. Composition of second and third actions thanks to step 5 is $Id + O(\rho^3)$. Identical transformation commute with H_2 and thus up to order $O(\rho^3)$ one gets composition of actions by H_1 and H_2 for the same time $\frac{\ell_1+\ell_2}{2}$. Then from step 4 it follows that trajectories of this composition differ from the trajectories of H-action for the time $(\ell_1 + \ell_2)$ by $O(\rho^3)$ for both coordinates ρ and ψ.

■

Since $\rho = \xi^{-1}$ and $\xi = O(|z|)$ for $|z| > 1$ Theorem 3 is proven.

From Theorems 3 and 2 it follows that the mixture (1) has infinitely many nested invariant curves expanding to infinity. Thanks to Lemma 1 trajectory of any point cannot cross an invariant curve and thus remains bounded for all time. This ends the proof of the first statement of Theorem 1

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Proof of second statement of Theorem 1. Let us remind that for the case $\ell_1 = -\ell_2$ mixture (1) is an $O(|z|^{-3})$ perturbation of identical transformation and thus KAM – theory methods are not applicable to that case.

Consider an arc $\widehat{z_0, z_1}$ of length $|\ell_1|$ with center at $F_1 = -1 + 0i$ from some point $z_0 = x_0 + iy_0$, $x_0 > 0$ which intesects straight line $x = 0$ in its middle point. Distance from z_1 to the line $x = 0$ equals $-\mathrm{Re}(z_1) = -x_1 = x_0$. After rotation H_2 distance from $z_2 = H_2(z_1)$ to the axis wil be greater then x_1 since $|z_1 - F_2| > |z_1 - F_2|$. Thus $\mathrm{Re}(z_2) > \mathrm{Re}(z_0)$. Similarly, for $z_3 = H_1(z_2)$ we get $\mathrm{Re}(z_3) < \mathrm{Re}(z_1)$. At each step distance from the image of the point z_0 to the axis will be greater then those on previous step. Incidentally, arc obtained on each iteration should intersect an axis since $|x_j| < \ell$. Thus because of convexity of the circle, sequence $\{y_j\}$ — coordinates of the intersection of the axis and arc $\widehat{z_j, z_{j+1}}$ is increasing. Suppose, that this sequence has some finite limitting point y_∞. Then for y_∞ one gets $T_{\ell_1,\ell_2}(0 + iy_\infty) = 0 + iy_\infty$ which is definitely impossible.

Similar considerations can be done for any point $z \in \mathbb{C}$. Role of the axis $x = 0$ in this case will play some hyperbola with foci F_j. Statement 2 of Theorem 1 is proven.

■

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