ISOMONODROMIC CONFLUENCE OF IRREGULAR SINGULARITIES

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Consider a family of systems of p ordinary differential equations on the Riemann sphere

$$\frac{dy}{dz} = A(z,t)y, \quad A(z,t) = \sum_{j=1}^{n} \sum_{k=1}^{r_j+1} \frac{A^j_{-k}(t)}{(z-a_j(t))^k}, \quad \sum_{j=1}^{n} A^j_{-1}(t) = 0, \tag{1}$$

where $A(z,t)$ depends holomorphically on $t \in D(t^0)$, $D(t^0)$ is a small neighbourhood of t^0 in a parameter space.

The system (1) is allowed to have Fuchsian singularities and irregular singularities, whose Poincare rank is minimal. The leading terms $A^i_{-r_i-1}$ are allowed to have equal eigenvalues, i.e. resonant irregular singularities are allowed.

Definition 1. *Family* (1) *is called an admissible deformation of the system* $\frac{dy}{dz} = A(z,t^0)y$, *if the following conditions hold.*

1. $a_1(t),...,a_n(t)$ *are holomorphic and* $a_i(t) \neq a_j(t), \forall t \in D(t^0)$.

2. All Poincare ranks of (1) *are minimal and don't depend on* t.

3. Stokes sectors are deformed by a parallel translation.

Definition 2. *Admissible deformation* (1) *of the system* $\frac{dy}{dz} = A(z,t^0)y$ *is called isomonodromic, if for any* $t \in D(t^0)$ *there is a fundamental solution* $Y(z,t)$ *of* (1) *such, that the following conditions are true.*

1. A monodromy representation, defined by $Y(z,t)$, *is equal to the monodromy representation of the system* $\frac{dy}{dz} = A(z,t^0)y$.

2. There is a set of fundamental solutions $Y^i_1(z,t) = Y(z,t),...,Y^i_{N_i}(z,t)$ *for any irregular singularity* $a_i(t)$ *such that according set of the Stokes matrices dose not depend on* t.

The following theorem is well known for deformations with nonresonant irregular points and it can be generalized for any admissible deformations.

Theorem 1. *The dmissible deformation* (1), $t = (a_1,...,a_n)$, *is the isomonodromic one if and only if, there is a meromorphic differential* 1*-form* ω *on* $\mathbb{CP}^1 \times D(t^0)$ *with singularities along* $\{z - a_i = 0\}$ *such that*

1. $\omega = A(z,t)dz$ *for any fixed value* $t \in D(t^0)$;

2. $d\omega = \omega \wedge \omega$.

Theorem 2. *If* $t = (a_1,...,a_n)$ *and a differential* 1*-form* ω *on* $\mathbb{CP}^1 \times D(t^0)$ *determines the isomonodromic deformation* (1), *then the general view of* ω *is*

$$\begin{aligned} \omega = &\sum_{i=1}^{n} \sum_{k=1}^{r_i+1} \frac{A^i_{-k}(t)}{(z-a_i)^k} d(z-a_i) + \\ &\sum_{i=1}^{n} \sum_{j=1}^{n} \left(\frac{\Phi^i_{M_i,j}(t)}{(z-a_i)^{M_i}} + ... + \frac{\Phi^i_{1,j}(t)}{z-a_i} \right) da_j + \sum_{j=1}^{n} \Psi_j(t) da_j, \end{aligned} \tag{2}$$

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where matrices $\{\Phi^i_{k,j}(t), \Psi_j(t)\}$ *are holomorphic on* $D(t^0)$.

This theorem can be extended to the more general set of parameters t.

Theorem 2 helps to proof a statement about confluences of irregular singularities. Recall that a question about confluence of regular singularities first was proposed by V. Arnold in [2]. And it was answered completely by A. Bolibruch [3, 4].

Definition 3. *Let* (1) *be an isomonodromic deformation. If the limit of the coefficient matrix of* (1) *exists*

$$B(z) = \lim_{a_1,\dots,a_m \to 0} A(z,t), \tag{3}$$

then the system

$$\frac{dy}{dz} = B(z)y \tag{4}$$

is called a resultant system of the isomonodromic confluence of the singularities $a_1, ..., a_m$.

Definition 4. *Isomonodromic confluence is called normalized if the differential* 1*-form* ω *determined by* (1) *satisfies the identity*

$$\omega(z,t)|_{z=\infty} \equiv 0. \tag{5}$$

Theorem 3. *Let* $p = 2$. *The resultant system of the normalized isomonodromic confluence of fuchsian singularities and irregular non-resonant singularities can't have irregular ramified singular point.*

An irregular singular point is a ramified one if the fundamental solution of the (1) has a ramified function in exponent. A system with ramified irregular singular points is more complicated for examination. And to build an isomonodromic deformation of any system with ramified points is undecided issue yet.

References

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