Collective dynamics versus stability for self-consistent processes

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In this work we deal with two seemingly very different problems. The first of them is to analyze collective dynamics of an infinite system of chaotically wandering particles under weak local interactions. In other terms systems of this type may be represented by the so called Coupled Map Lattices with a dynamical structure of interactions. We show that depending on fine properties of the interaction one may observe both condensation/synchronization phenomenon and independent motions of individual particles in the weak interaction limit.

Another problem is to study stochastic stability of an individual system (say describing a chaotic dynamics of a single particle) with respect to perturbations depending not only on the particle's coordinates but also on the current particle's distribution. Therefore in distinction to a conventional setting when the perturbation is described by a Markov chain (and the entire perturbed system remains Markovian as well) in our case the perturbed process is no longer Markovian. The dependence of the particle's motion on the particle's distribution is emphasized in the name "self-consistent" of the processes under consideration.

Let $(X, \mathcal{B}, \rho)$ be a compact metric space and let let $\mathcal{M} = \mathcal{M}(X)$ be the set of probabilistic Borel measures on X. A (discrete time) *process* on $\mathcal{M}$ is defined by a transfer-operator $\mathcal{T}^* : \mathcal{M} \to \mathcal{M}$ satisfying the following properties:
(a) $(\mathcal{T}^*\mu)(A) \cdot \mu(A) \geq 0 \quad \forall \mu \in \mathcal{M}, A \in \mathcal{B}$ (positivity),
(b) $(\mathcal{T}^*\mu)(X) \equiv \mu(X)$ (total measure preservation).

If additionally one assumes the *linearity* of the operator $\mathcal{T}^*$ then it becomes the standard Markov operator, which describes the dynamics of measures under the action of a Markov process. Therefore the processes defined above are often called *nonlinear* Markov processes.

Denote by $\mathcal{M}^\delta \subset \mathcal{M}$ the set of δ-measures on X. It is natural to say that the process $\mathcal{T}^*$ is *deterministic* if $\mathcal{T}^* : \mathcal{M}^\delta \to \mathcal{M}^\delta$. Obviously a deterministic process $\mathcal{T}^*$ defines uniquely a deterministic map $T : X \to X$ according to the formula $Tx := y$, where $\mathcal{T}^* 1_x = 1_y$ for each $x \in X$. Here 1_x a the δ-measure supported by the point x.

We shall study a two-parameter collection of deterministic nonlinear Markov processes defined by the following nonlinear transfer-operators:

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$$\mathcal{T}^* := T^*_{\varepsilon,\gamma}\mu := Q^*_{\varepsilon,\gamma,T^*\mu}T^*\mu, \qquad Q_{\varepsilon,\gamma,\mu}x := (1-\gamma)x + \frac{\gamma}{\mu(B_\varepsilon(x))}\int_{B_\varepsilon(x)} yd\mu(y).$$

Here $0 < \varepsilon, \gamma \ll 1$ are numeric parameters, and $B_\varepsilon(x)$ is the ε-neighborhood of the point $x \in X$.

Processes defined by the transfer-operators $T^*_{\varepsilon,\gamma}$ we shall call *self-consistent*. Observe that the quantity $\frac{1}{\mu(B_\varepsilon(x))}\int_{B_\varepsilon(x)} yd\mu(y)$ is the barycenter of the distribution of mass corresponding to the restriction of the measure μ to the ε-neighborhood of the point $x \in X$. Therefore the transition from x to $Q_{\varepsilon,\gamma,\mu}x$ may be interpreted as the motion of a point-mass located at x under the attraction of the total mass in this neighborhood.

Consider now the measure μ as a distribution of a collection of particles, whose local dynamics is governed by the map T and the interaction is local in space and is defined by the map $Q_{\varepsilon,\gamma,\mu}$. A special case when the measure μ is equal to a finite sum of δ-measures explicitly corresponds to the dynamics of a finite Couple Map Lattice and was studied in very different terms and using different methods in [1, 2].

Theorem 1 *(Condensation) Let the map T satisfy Lipschitz condition with the constant $\Lambda < \infty$. Then $\exists c = c(\Lambda) > 0$ such that $\forall 0 < \varepsilon, 1-\gamma < c$ the condensation takes place: for each initial measure $\mu \in \mathcal{M}$ with the support of diameter not exceeding ε, the diameter of the support of its image $(\mathcal{T}^*)^t\mu$ vanishes with time $t \to \infty$.*

Assuming certain expanding type conditions on the map T with the constant λ and the uniqueness of the corresponding Sinai-Bowen-Ruelle (SBR) measure μ_T and that it is absolutely continuous with respect to the reference measure m (say Lebesgue measure) we may formulate the following opposite statement.

Theorem 2 *(Independence) There exists an open set of absolutely continuous measures $\mathcal{M}_0 \subset \mathcal{M}$ and $c = c(\lambda)$ such that $\forall 0 < \varepsilon, \gamma < c,\ \mu \in \mathcal{M}_0$ we have*

$$(\mathcal{T}^*)^t\mu \stackrel{t\to\infty}{\longrightarrow} \mu_{\varepsilon,\gamma} \stackrel{\varepsilon,\gamma\to 0}{\longrightarrow} \mu_T.$$

Observe that this result does not imply that the measure $\mu_{\varepsilon,\gamma}$ is the SBR measure for the process $\mathcal{T}$, but only its Lyapunov stability. Namely, we are able to show that measures close enough to the measure μ_T converge weakly under dynamics to a certain measure $\mu_{\varepsilon,\gamma}$ which in turn converges to μ_T as $\varepsilon, \gamma \to 0$. To give an exact definition of the Lyapunov stability of measures one needs to specify the topology in the space of measures $\mathcal{M}$ and the type of convergence, in particular, whether one assumes the direct convergence or the convergence in Cezaro means. It turns out that different assumptions here lead to very different results. Even the existence of Lyapunov stable measures of simplest chaotic dynamical systems sensitively depends on the choice of metrics and convergence. We shall discuss these matters in detail and obtain connections between Lyapunov stable measures and SBR measures as well as ergodic properties of the corresponding dynamical systems.

Let $\rho^*(\cdot,\cdot)$ be a metric in the space $\mathcal{M}$ of probabilistic measures on X and let a map $T : X \to X$ be given. We say that a measure $\mu \in \mathcal{M}$ is *attractive* if $\exists \sigma > 0$ such that $\frac{1}{n}\sum_{k=0}^{n-1}(T^*)^k\nu \stackrel{n\to\infty}{\longrightarrow} \mu$ weakly whenever $\rho^*(\mu,\nu) < \sigma$.

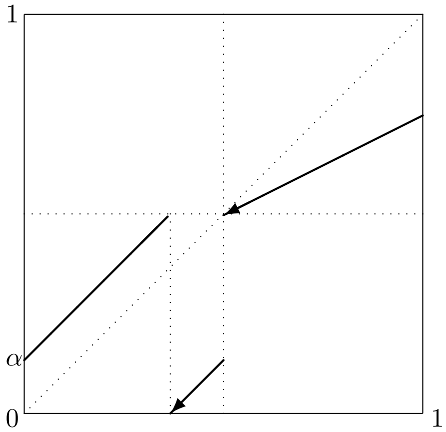

Figure 1: Example of a uniquely ergodic dynamical system having no attractive measures. $\alpha \notin \mathbf{Q}$, $\mathrm{Leb}_{|[0,1/2]}$ – SBR measure.

Theorem 3 *Unique ergodicity implies the existence of an SBR measure, but does not imply the existence of an attractive measure.*

The second result can be illustrated by a one-dimensional map

$$T_\alpha x := \begin{cases} x+\alpha & \text{if } x \in [0,1/2-\alpha] \\ x+\alpha-1/2 & \text{if } x \in (1/2-\alpha,1/2] \\ x/2+1/4 & \text{otherwise} \end{cases} .$$

shown in Fig. 1.

As a corollary we are getting a negative answer to an old question whether unique ergodicity implies the convergence of ergodic averages along all trajectories to the same limit. Indeed, in the example in Fig. 1 for $\forall x \le 1/2$ ergodic averages converge to the restriction of the Lebesgue measure to the segment $[0,1/2]$, while for $\forall x > 1/2$ they converge to the delta-measure at point 1/2 (which is not T_α-invariant).

References

[1] Blank M., *Collective phenomena in lattices of weakly interacting maps*, Doklady Akademii Nauk (Russia), 430:3(2010), 300-304.

[2] Blank M., *Self-Consistent Mappings and Systems of Interacting Particles*, Doklady Akademii Nauk (Russia), 436:3(2011), 295-298.