ORNSTEIN-ZERNIKE ASYMPTOTICS AND THE LOCAL LIMIT THEOREM

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1 Ornstein-Zernike Asymptotics

In statistical physics and probability theory we often find problems of the following type: we have an operator $\mathcal{T}$ acting on the Hilbert space $\ell_2(\mathbb{Z}^d)$, $d = 1, 2, \ldots$, as

$$(\mathcal{T}f)(x) = \sum_{y \in \mathbb{Z}^d} (a(y - x) + c(x; y)))\, f(y), \quad f \in \ell_2(\mathbb{Z}^d) \tag{1.1}$$

and we are interested in the correlations induced by $\mathcal{T}$, i.e, of the scalar products

$$\left(\mathcal{T}^t f^{(1)}, f^{(2)}\right)_{\ell_2(\mathbb{Z}^d)}, \qquad t = 1, 2, \ldots \tag{1.2}$$

for $f^{(1)}, f^{(2)}$ in some suitable class of functions, when t is large.

A general example is that of a stationary Markov chain $\{\xi_t : t \in \mathbb{Z}\}$, describing a random field on $\mathbb{Z}^d$, with state space Ω, and invariant measure ν. Quantities of the type

$$\langle \Phi(\xi_0), \Phi(\xi_t)\rangle := \langle \Phi(\xi_0)\Phi(\xi_t)\rangle - \langle \Phi(\xi_0)\rangle^2, \tag{1.3}$$

where $\Phi \in L_2(\Omega, \nu)$ and $\langle \cdot \rangle$ denotes averaging with respect to the probability distribution of the chain, may represent the time correlation for a random walk in a Markov environment, the space correlation for a Gibbs state, or other quantities of physical interest.

If S is the stochastic operator of the Markov chain and $(\cdot, \cdot)$ is the scalar product in $L_2(\Omega, \nu)$, the correlation (1.3) can be written as

$$\left((S^t(\Phi - \langle \Phi \rangle)), \Phi - \langle \Phi \rangle\right). \tag{1.4}$$

In many models (see, e.g. [1]) we can represent $L_2(\Omega, \nu)$ as a direct (in general non-orthogonal) sum of invariant (under S) subspaces

$$L_2(\Omega, \nu) = \mathcal{H}_0 + \mathcal{H}_1 + \mathcal{H}_2 + \mathcal{H}_3, \tag{1.5}$$

where $\mathcal{H}_0$ is the subspace of the constants, and $\mathcal{H}_1, \mathcal{H}_2$ are the so-called "one-particle" and "two-particle" subspaces. Moreover the restrictions $S_i := S/\mathcal{H}_i$, $i = 0, \ldots, 3$, are such that the maximal absolute values of the spectra $\kappa_i := \max_{\lambda \in \sigma(S_i)} |\lambda|$ are decreasing

$$1 = \kappa_0 > \kappa_1 > \kappa_2 > \kappa_3. \tag{1.6}$$

*Based on joint work with R. A. Minlos and A. Pellegrinotti

In such cases if we expand element Φ according to (1.5)

$$\Phi = \langle \Phi \rangle + \Phi_1 + \Phi_2 + \Phi_3, \qquad \Phi_i \in \mathcal{H}_i, \quad i = 1, 2, 3, \tag{1.7}$$

we see, by a heuristic argument, that the leading term of the asymptotics (1.4), as $t \to \infty$, behaves roughly as κ_1^t if $\Phi_1 \neq 0$ ("one-particle case") and as κ_2^t if $\Phi_1 = 0$, and $\Phi_2 \neq 0$ ("two-particle case"). The exact asymptotics is exponential with power-law factors, and is usually called "Ornstein-Zernike" (O.Z.), after the pioneering work of those authors [2].

For many models the operator S_1 is reduced to the standard form (1.1) by choosing an appropriate basis $\{v_x : x \in \mathbb{Z}^d\}$ in $\mathcal{H}_1$, and, similarly, by choosing a basis $\{v_{x_1,x_2} : x_1, x_2 \in \mathbb{Z}^d\}$ in $\mathcal{H}_2$ the operator S_2 is reduced to the form

$$(\mathcal{T} f)(x_1, x_2) = \sum_{y_1, y_2 \in \mathbb{Z}^d} \left(a(y_1 - x_1, y_2 - x_2) + c(x_1, x_2; y_1, y_2)\right) f(y_1, y_2). \tag{1.8}$$

The function c is usually translation invariant, i.e.,

$$c(x_1, x_2; y_1, y_2) = c(x_1 + u, x_2 + u; y_1 + u, y_2 + u), \qquad \forall u \in \mathbb{Z}^d \tag{1.9}$$

and the operator defined by (1.8) differs from the general case of the problem (1.1) in $\mathbb{Z}^{2d}$. The correlations are then given by the scalar product (1.2).

The O.Z. asymptotics was studied for concrete models in many mathematical and physical papers [2-12]. Rigorous results, based on the spectral analysis as above, were obtained in the papers [3-8]. All such results rely on the particular features of the models.

Quite recently [12] we were able to give a general answer for the decay of the correlations (1.2) in the two-particle case, under the condition that the "interaction" term c in (1.8) is small. Our analysis is based on techniques of analytic functions and requires an exponential decay of the quantities in (1.8). If the decay is only power-law, the problem looks much more difficult. To be precise we assume that the functions a and c in (1.8) are real and satisfy, for some constants C_1, C_2 and $q \in (0, 1)$, the inequalities

$$|a(x_1, x_2)| \leq C_1 \, q^{|x_1| + |x_2|}, \qquad |c(x_1, x_2; y_1, y_2)| \leq C_2 \, q^{\min_\tau d(\tau)}. \tag{1.10}$$

Here τ is a connected graph with vertices at x_1, x_2, y_1, y_2, $d(\tau)$ is its length, and the minimum is taken over all such graphs. We also assume exponential decay for the functions $f^{(1)}, f^{(2)} \in \mathcal{H} := \ell_2(\mathbb{Z}^d \times \mathbb{Z}^d)$ as for a in (1.10), exchange symmetry for a, i.e., $a(x_1, x_2) = a(x_2, x_1)$, and that a and c are even (so that the Fourier transforms are real):

$$a(x_1, x_2) = a(-x_1, -x_2), \qquad c(x_1, x_2; y_1, y_2) = c(-x_1, -x_2; -y_1, -y_2). \tag{1.11}$$

The crucial assumption on the spectrum is that the Fourier transform

$$\tilde{a}(\lambda_1, \lambda_2) = \sum_{y_1, y_2} e^{i(\lambda_1, y_1) + i(\lambda_2, y_2)} a(y_1, y_2) \tag{1.12}$$

has a unique absolute positive maximum at some point $(\bar{\lambda}_1, \bar{\lambda}_2)$, with a negative-definite hessian matrix. We take for definiteness $\bar{\lambda}_1 = \bar{\lambda}_2 = 0$, so that

$$\max_{(\lambda_1, \lambda_2) \in T^d \times T^d} \tilde{a}(\lambda_1, \lambda_2) = \tilde{a}(0, 0) := \kappa > 0. \tag{1.13}$$

As c is supposed to be small, we write αc instead of c, and we understand that c is fixed and α is a positive parameter which is as small as required.

It turns out that the power-law prefactor of the correlation, which is t^{-d} for $d \geq 3$, for $d = 1, 2$ may be "anomalous", as first discovered by Polyakov [11], due to the fact that the correlations are dominated by the interaction c.

The main results of [12]) are the following.

Theorem 1.1 *For $d \geq 3$ there is a constant $\mathcal{M}_d$, depending on $f^{(1)}, f^{(2)}$ such that the following asymptotics holds, as $t \to \infty$:*

$$\left(\mathcal{T}^t f^{(1)}, f^{(2)}\right) = \mathcal{M}_d \frac{\kappa^t}{t^d} \left(1 + r_d(t)\right)), \tag{1.14}$$

where $r_d(t) = \mathcal{O}(\frac{\ln t}{t})$ for $d = 4$ and $r_d(t) = \mathcal{O}(\frac{1}{t})$ for $d > 4$.

For $d = 1, 2$ we have different behaviors, depending, for small α on the quantity

$$C = \sum_{x_1 y_1, y_2} c(x_1, 0; y_1, y_2).$$

Using a physical terminology, we can say that the interaction is "repulsive", for $C > 0$, "attractive", for $C < 0$, and "neutral", for $C = 0$.

Theorem 1.2 *For $d = 1, 2$, the following asymptotics hold, as $t \to \infty$.*

i) If $C > 0$ ("repulsive case"), there are constants $\mathcal{M}_d^{(+)}$ such that

$$\left(\mathcal{T}^t f^{(1)}, f^{(2)}\right) = \frac{\mathcal{M}_1^{(+)} \kappa^t}{t^2} \left(1 + \mathcal{O}(\frac{1}{t})\right), \qquad d = 1\,; \tag{1.15}$$

$$\left(\mathcal{T}^t f^{(1)}, f^{(2)}\right) = \frac{\mathcal{M}_2^{(+)} \kappa^t}{t^2 \ln^2 t} \left(1 + \mathcal{O}(\frac{1}{\ln t})\right), \qquad d = 2\,; \tag{1.16}$$

ii) If $C < 0$ ("attractive case"), there are constants $\bar{\kappa}_d > \kappa$ and $\mathcal{M}_d^{(-)}$, such that

$$\left(\mathcal{T}^t f^{(1)}, f^{(2)}\right) = \frac{\mathcal{M}_1^{(-)} \bar{\kappa}_1^t}{\sqrt{t}} \left(1 + \mathcal{O}(\frac{1}{t})\right), \qquad d = 1\,; \tag{1.17}$$

$$\left(\mathcal{T}^t f^{(1)}, f^{(2)}\right) = \frac{\mathcal{M}_2^{(-)} \bar{\kappa}_2^t}{t} \left(1 + \mathcal{O}(\frac{1}{t})\right), \qquad d = 2\,; \tag{1.18}$$

iii) If $C = 0$ ("neutral case"), there are constants $\mathcal{M}_d^{(0)}$ such that

$$\left(\mathcal{T}^t f^{(1)}, f^{(2)}\right) = \frac{\mathcal{M}_1^{(0)} \kappa^t}{t} \left(1 + \mathcal{O}(\frac{1}{\sqrt{t}})\right), \qquad d = 1\,; \tag{1.19}$$

$$\left(\mathcal{T}^t f^{(1)}, f^{(2)}\right) = \frac{\mathcal{M}_2^{(0)} \kappa^t}{t^2} \left(1 + \mathcal{O}(\frac{\ln t}{t})\right), \qquad d = 2\,. \tag{1.20}$$

For the constants $\mathcal{M}_d^{(0)}, \mathcal{M}_d^{(\pm)}$, *if* $\tilde{f}^{(1)}(0,0)\tilde{f}^{(2)}(0,0) \neq 0$ *we have*

$$\mathcal{M}_d = c_d(\alpha)\ \tilde{f}^{(1)}(0,0)\overline{\tilde{f}^{(2)}(0,0)}\ [1+\mathcal{O}(\alpha)]\,, \qquad d \geq 3, \tag{1.21}$$

$$\mathcal{M}_d^{(\Theta)}(f^{(1)}, f^{(2)}) = c_d^{(\Theta)}(\alpha)\tilde{f}^{(1)}(0,0)\overline{\tilde{f}^{(2)}(0,0)}\ [1+\mathcal{O}(\alpha)]\,, \qquad d=1,2,\ \Theta=+,-,0. \tag{1.22}$$

Remark 1.3 *The constants* c_d, $c_d^{(\Theta)}$, *are non-vanishing for* $\alpha>0$. *As* $\alpha \to 0$, $c_d(\alpha)$ *has a finite limit for* $d \geq 3$, *and for* $d=1,2$ $c_d^{(\Theta)}(\alpha)$ *diverges for* $\Theta=+$, *vanishes for* $\Theta=-$, *and tends to a finite limit for* $\Theta=0$.

Here is a brief outline of the proof. By translation invariance the Hilbert space $\mathcal{H}$ is decomposed as a direct integral $\mathcal{H} = \oint_{T^d} \mathcal{H}_\Lambda d\Lambda$ of Hilbert spaces $\mathcal{H}_\Lambda$, $\Lambda \in T^d$, which reduce $\mathcal{T}$: $\mathcal{T} = \oint_{T^d} \mathcal{T}_\Lambda d\Lambda$, and $\mathcal{T}_\Lambda$ is unitarily equivalent to an operator $\tilde{\mathcal{T}}_\Lambda$ acting on $L^2(T^d, dm)(dm(\Lambda) = \frac{d^d\Lambda}{(2\pi)^d}$ is the Haar measure on T^d) as

$$\left(\mathcal{T}_\Lambda \phi\right)(\lambda) = \tilde{a}_\Lambda(\lambda)\phi(\lambda) + \alpha \int_{T^d} K_\Lambda(\lambda,\mu)\phi(\mu)dm(\mu), \tag{1.23}$$

where $\tilde{a}_\Lambda(\lambda) = \tilde{a}(\lambda, \Lambda-\lambda)$ and $K_\Lambda(\lambda,\mu) = \tilde{c}(\lambda;\mu,\Lambda-\mu)$. If now γ is a clockwise contour in the complex z-plane going around the spectrum of $\mathcal{T}_\Lambda$, the resolvent formula gives

$$\tilde{\mathcal{T}}_\Lambda^t = \frac{1}{2\pi i}\int_\gamma \left(\tilde{\mathcal{T}}_\Lambda - zE\right)^{-1} z^t dz, \tag{1.24}$$

where E is the unit operator. By the Fredholm theory, the resolvent has a kernel

$$\left(\tilde{\mathcal{T}}_\Lambda - zE\right)^{-1}(\lambda,\mu) = \frac{\delta_{\lambda,\mu}}{\tilde{a}_\Lambda(\lambda)-z} - \frac{1}{\Delta_\Lambda(z)}\frac{D_\Lambda(\lambda,\mu;z)}{(\tilde{a}_\Lambda(\lambda)-z)(\tilde{a}_\Lambda(\mu)-z)}, \tag{1.25}$$

where the functions $\Delta_\Lambda(z)$, $D_\Lambda(\lambda,\mu;z)$ are expressed by converging power series in α the terms of which are multiple integrals of functions of the type

$$\frac{\det\{K_\Lambda(\lambda_i,\lambda_j)\}_{i,j=1,\ldots,n}}{\prod_{i=1}^n(\tilde{a}_\Lambda(\lambda_i)-z)}.$$

It is not hard to see that the spectrum of $\tilde{\mathcal{T}}_\Lambda$ is made of the cut $I_\Lambda = [\kappa_1(\Lambda), \kappa_0(\Lambda)]$, where $\kappa_1(\Lambda) = \min_\lambda \tilde{a}_\Lambda(\lambda)$, $\kappa_0(\Lambda) = \max_\lambda \tilde{a}_\Lambda(\lambda)$, and of possible zeroes of $\Delta_\Lambda(z)$ which, for small α, lie near I_Λ. The leading contribution to the asymptotics comes from the region of z near $\kappa_0(\Lambda)$, for Λ near the origin, as the maximum of $\tilde{a}$ is $\kappa = \kappa_0(0)$.

Our main technical tool is a representation for the basic integrals which appear in the resolvent (1.24). If Λ near the origin and $f(\lambda)$ can be extended to a complex neighborhood of the torus T^d then for $z \in U \setminus I_\Lambda$, where U is a complex neighborhood of κ, we have

$$\int_{T^d} \frac{f(\lambda)\ dm(\lambda)}{\tilde{a}_\Lambda(\lambda)-z} = \begin{cases} h_f(z;\Lambda)\ \zeta_\Lambda^{s-\frac{1}{2}} + H_f(z;\Lambda) & d=2s+1, \\ h_f(z;\Lambda)\zeta_\Lambda^s\ \log\frac{1}{\zeta_\Lambda} + H_f(z;\Lambda) & d=2s+2, \end{cases} \tag{1.26}$$

where $\zeta_\Lambda = z - \kappa_0(\Lambda)$, $s=0,1,\ldots$, and $h_f(z;\Lambda), H_f(z;\Lambda)$ are analytic functions for $z \in U$.

Such representation can be iterated to multiple integrals and leads to a manageable representation for the resolvent (1.24).

2 Local Limit Theorems for locally inhomogeneous random Walks

The local limit theorem for a locally inhomogeneus random walk appears at first glance as a particular case of O.Z. asymptotics.

In fact, if in (1.1) we set $a(x) = P_0(x)$, where P_0 is the transition probability of a homogeneous random walk, and $c(x;y)$ is such that $P_0(y-x) + c(x;y) \in [0,1)$ with $\sum_y c(x;y) = 0$ for all $x \in \mathbb{Z}^d$, then $\mathcal{T}$ is the stochastic operator of a locally inhomogeneous random walk. By Fourier transform $\mathcal{T}$ goes over into $\tilde{\mathcal{T}}$ which acts on $L_2(T^d, dm)$ as

$$\left(\tilde{\mathcal{T}} f\right)(\lambda) = \tilde{p}_0(\lambda)\tilde{f}(\lambda) + \int_{T^d} \tilde{c}(\lambda;\mu)\tilde{f}(\mu)dm(\mu), \qquad \lambda \in T^d \tag{2.1}$$

where $\tilde{p}_0(\lambda) = \sum_x P_0(x)e^{i(\lambda,x)}$, and $\tilde{c}(\lambda;\mu) = \sum_{x,y} c(x;y)e^{i(\lambda,x)-i(\lambda,y)}$.

The two-particle operator (1.8) can also describe, under suitable assumptions, the random walk of two particles with local interaction.

In actual fact, we are interested in the asymptotics of $P(X_t = y|X_0 = x)$, where X_t is the position of the random walk at time t, which reduces to a usual a O.Z. asymptotics (in the "neutral" case, as $\sum_y c(x;y) = 0$) only if y is fixed. But in the local limit theorem asymptotics y can grow with t, and the previous approach runs into difficulties.

In fact, in terms of the operator (2.1) we have

$$P(X_t = y|X_0 = x) = \left(\mathcal{T}^t\delta_y\right)(x) = \int_{T^d} (\tilde{\mathcal{T}}^t\tilde{\delta}_y)(\lambda)\, e^{-i(\lambda,x)}dm(\lambda), \qquad \delta_y(u) = \delta_{y,u},$$

where $\tilde{\delta}_y(\lambda) = e^{i(\lambda,y)}$ is the Fourier transform of δ_y. When we express $\tilde{\mathcal{T}}^t$ in terms of the resolvent we run into integrals of the form

$$\int_{T^d} \frac{g(\lambda)\, e^{i(\lambda,y)}}{\tilde{p}_0(\lambda) - z} dm(\lambda), \tag{2.2}$$

where g is an analytic function, which have a representation of the type (1.26) with ζ_Λ replaced by $z-1$, but if y grows the bounds for the analogues of h_f, H_f diverge.

Such difficulties were overcome in the papers by Minlos and Zhizhina [13] [14] , which, to my knowledge, are the only results of a general kind for the local limit theorem of locally inhomogenous random walks. We briefly present here a refinement of the representation (1.26), for $d = 1$, which allows a better control of the dependence on y and can lead to an improvement of the results in [MZh].

The integral (2.2) defines an analytic function of z except for the cut on the real interval $I = [\kappa_1, 1]$, where $\kappa_1 = \min_\lambda \tilde{p}_0(\lambda)$. We are interested in the values of z near the edge of the cut $z = 1$, which give the leading contribution to the asymptotics.

Suppose that $\beta = 1 - z = |\beta|e^{i\theta}$ is small enough, with $\theta \neq 0, \pi$, and let $\sqrt{\beta} = |\beta|^{\frac{1}{2}} e^{i\frac{\theta}{2}}$. Then, by simple Cauchy integrals in the complex λ-plane one sees that the integral (2.2), for $y \neq 0$ and $\mathrm{Im}(\beta) > 0$ can be represented as

$$-\int_T \frac{g(\lambda)e^{i\lambda y}}{1 - \tilde{p}_0(\lambda) - \beta} dm(\lambda) = e^{-\kappa|y|} \int_T \frac{-e^{i\lambda y} g(\lambda \pm i\kappa)}{1 - \tilde{p}_0(\lambda \pm i\kappa) - \beta} dm(\lambda) + \tag{2.3}$$

$$-i\frac{\mathcal{J}(\beta)}{\sqrt{\beta}}g(\pm\lambda(\beta))\exp\{\pm i\lambda(\beta)y\},$$

where $\lambda(\beta) = \sqrt{2\beta}u(\beta)$, $\mathcal{J}$ and u are analytic functions which are real for real β, κ is a positive constant depending only on P_0, and we take $\pm$ according to the sign of y.

If $\mathrm{Im}(\beta) < 0$ the second term on the right of (2.3) changes sign.

By(2.3) we can improve a result of [13] in the simple case when the perturbation is located at the origin, i.e., $P(X_{t+1} = y|X_t = x) = P_0(y-x) + \delta_{x,0}c(y)$, where $P_0(y) + c(y), y \in \mathbb{Z}$ is the transition probability of a nondegenerate random walk. More precisely, if P_0 is even and c is odd, with $P_0(y) + |c(y)| \leq Cq^{|y|}$, for some $C > 0$, $q \in (0,1)$, and $\sum_y P_0(y)y^2 = 1$, $\sum_y c(y)y = b$, the following result holds:

Theorem 2.1 *There are a positive constant κ, and a function Φ, bounded with its derivatives, such that the following asymptotics holds as $t \to \infty$, uniformly in $y = o(t^{3/4})$*

$$P(X_t = y|X_0 = 0) = \frac{1}{\sqrt{2\pi t}}\left[(1 + b\ sign(y))\ e^{-\frac{y^2}{2t}} + e^{-\kappa|y|}\Phi(y)\right](1 + o(1))\,.$$

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