ON ESTIMATION OF A GAUSSIAN RANDOM WALK 
FIRST-PASSAGE TIME FROM CORRELATED 
OBSERVATIONS

Given a Gaussian random walk $X$ with drift, we consider estimating its first-passage time $\tau$, of a given level $A$, with a stopping time defined over an observation process $Y$ that is either a noisy version of $X$, or a delayed by $d$ version of $X$. For a given loss function $f(x)$, for both cases, we provide lower bounds on expectations $E f(\eta - \tau)$, for any stopping rule $\eta$, and exhibit simple stopping rules that achieve these bounds in the large threshold $A$ regime and in the large threshold $A$ large delay $d$ regime, respectively. The results immediately extend to the corresponding continuous time settings where $X$ and $Y$ are Brownian motions with drift.

1. Problem statement. Consider the discrete-time process

$$ X : \quad X_0 = 0, \quad X_n = \sum_{i=1}^{n} V_i + sn, \quad n \geq 1, $$

where $s > 0$ is a given constant and where $V_1, V_2, \ldots$ are independent $\mathcal{N}(0, 1)$-Gaussian random variables. For a given threshold level $A > 0$ consider the first-passage time

$$ \tau_A = \min\{n \geq 0 : X_n \geq A\}. $$

We assume that the loss function $f(x)$ satisfies the following conditions:

$A_1$) $f(x), x \in \mathbb{R}^1$ is a continuous nonnegative function such that $f(0) = 0$;

$A_2$) $f(x)$ monotone decreases for $x < 0$ and monotone increases for $x > 0$;

$A_3$) for some $\alpha \geq \beta > 0$ and some constant $C$ the function $f(x)$ satisfies the bound

$$ f(x) \leq C \left( |x|^\alpha + |x|^\beta \right), \quad x \in \mathbb{R}^1; $$

$A_4$) for some $a_2 \geq 0$ and some constant $C$ the function $f(x)$ satisfies the condition

$$ |f(x + \varepsilon) - f(x)| \leq C f(x)(|\varepsilon| + |\varepsilon|^{a_2}) + Cf(\varepsilon), \quad x, \varepsilon \in \mathbb{R}^1; $$

$A_5$) $f(x)$ satisfies the conditions

$$ \lim_{x \to -\infty} f(x) > 0 \quad \text{and} \quad \lim_{x \to -\infty} f(x) > 0. $$

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The condition $A_5$ is not fulfilled if, for example, $f(x) = 0$ for all $x \leq 0$.

The function $f(x)$ may be nonsymmetric. In particular, the function $f(x) = (-x)^{p_1}$, $x \leq 0$, $f(x) = x^{p_2}$, $x \geq 0$ with $\min\{p_1, p_2\} > 0$ satisfies conditions $A_1 - A_5$.

Observing sequentially the process $Y = \{Y_n, n = 0, 1, \ldots\}$ correlated to $X$, it is desirable to estimate the moment $\tau_A$ in a best way with respect to the loss function $f(x)$.

Concerning the observation process $Y = \{Y_n, n = 0, 1, \ldots\}$, we consider two cases: 

Noisy observations. In that case the observation process $Y$ has the form

$$Y : \quad Y_0 = 0, \quad Y_n = X_n + \varepsilon \sum_{i=1}^{n} W_i, \quad n \geq 1,$$

where $W_1, W_2, \ldots$ are independent $\mathcal{N}(0, 1)$–Gaussian random variables (independent of \{Vi\}), and where $\varepsilon > 0$ is known.

For given $A$ and an estimate $\eta$ for $\tau_A$, introduce the function

$$q(A, \eta) = \mathbb{E} f \left( \frac{\eta - \tau_A}{r} \right), \quad r = \varepsilon \sqrt{\frac{A}{s^3(1 + \varepsilon^2)}}.$$

We are interested in the minimal possible function $q(A, \eta)$

$$q(A) = \inf_{\eta} q(A, \eta),$$

where the infimum is taken over all stopping times $\eta$ with respect to the process $Y$ from (1). We use the normalization by $r$ in (2) because for good estimates $\eta$ and large $A$ the normalized difference $(\eta - \tau_A)/r$ will be approximately $\mathcal{N}(0, 1)$–Gaussian, and such normalization will allow us to avoid some bulky coefficients. For simplicity, we consider only the case when the positive values $s, \varepsilon$ are fixed and $A \to \infty$.

Delayed observations. In that case we are given some fixed delay $d = d(A) > 0$ and the process $Y$ has the form

$$Y : \quad Y_0 = Y_1 = \ldots = Y_d = 0; \quad Y_n = X_{n-d}, \quad n \geq d + 1.$$

Similarly to (2)–(3), for given $d$ and an estimate $\eta$ for $\tau_A$, introduce the functions

$$q(d, \eta) = \mathbb{E} f \left( \frac{\eta - \tau_A}{r_d} \right), \quad r_d = \sqrt{\frac{d}{s^2}},$$

and

$$q(d) = \inf_{\eta} q(d, \eta),$$

where the infimum is taken over all stopping times $\eta$ defined with respect to the process $Y$ from (4).

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2. Main results. Introduce the value

\[ m(f) = \inf_a \mathbb{E} f(\xi + a), \quad \xi \sim \mathcal{N}(0, 1). \] (7)

Since \( f(x) \) satisfies the conditions \( A_1, A_3 \) and \( A_5 \), we have \( 0 < m(f) < \infty \).

**Theorem 1 (Noisy observations).** If \( f(x) \) satisfies conditions \( A_1-A_5 \) then

\[ q(A) = m(f) + o(1), \quad A \to \infty. \] (8)

Theorem 1 generalizes [1, Theorem 2.3] which considers the case \( f(x) = |x| \).

**Theorem 2 (Delayed observations).** If \( f(x) \) satisfies conditions \( A_1-A_5 \) then

\[ q(d) = m(f) + o(1), \quad A, d \to \infty. \] (9)

**Remark.** Theorems 1 and 2 remain valid if we replace \( X \) and \( Y \) by their continuous time counterparts; i.e., \( X_t = st + B_t \) and \( Y_t = X_t + \varepsilon W_t \) for the noisy case, and \( Y_t = X_{t-d} \) for the delayed case, where \( \{B_t\}_{t \geq 0} \) and \( \{W_t\}_{t \geq 0} \) are independent standard Brownian motions.

REFERENCES

1. Burnashev M. V., Tchamkerten A. Tracking a threshold crossing time of a gaussian random walk through correlated observations. –