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ON ESTIMATION OF A GAUSSIAN RANDOM WALK FIRST-PASSAGE TIME FROM CORRELATED OBSERVATIONS [3]

Given a Gaussian random walk X with drift, we consider estimating its first-passage time τ, of a given level A, with a stopping time η defined over an observation process Y that is either a noisy version of X, or a delayed by d version of X. For a given loss function $f(x)$, for both cases, we provide lower bounds on expectations $\mathbf{E}f(\eta-\tau)$, for any stopping rule η, and exhibit simple stopping rules that achieve these bounds in the large threshold A regime and in the large threshold A large delay d regime, respectively. The results immediately extend to the corresponding continuous time settings where X and Y are Brownian motions with drift.

1. Problem statement. Consider the discrete-time process

$$X: \quad X_0 = 0, \qquad X_n = \sum_{i=1}^{n} V_i + sn, \qquad n \geq 1,$$

where $s > 0$ is a given constant and where $V_1, V_2, \ldots$ are independent $\mathcal{N}(0,1)$–Gaussian random variables. For a given threshold level $A > 0$ consider the first-passage time

$$\tau_A = \min\{n \geq 0 : X_n \geq A\}.$$

We assume that the loss function $f(x)$ satisfies the following conditions:

$\mathbf{A_1}$) $f(x), x \in \mathbf{R}^1$ is a continuous nonnegative function such that $f(0) = 0$;

$\mathbf{A_2}$) $f(x)$ monotone decreases for $x < 0$ and monotone increases for $x > 0$;

$\mathbf{A_3}$) for some $\alpha \geq \beta > 0$ and some constant C the function $f(x)$ satisfies the bound

$$f(x) \leq C\left(|x|^{\alpha} + |x|^{\beta}\right), \qquad x \in \mathbf{R}^1;$$

$\mathbf{A_4}$) for some $a_2 \geq 0$ and some constant C the function $f(x)$ satisfies the condition

$$|f(x+\varepsilon) - f(x)| \leq Cf(x)(|\varepsilon| + |\varepsilon|^{a_2}) + Cf(\varepsilon), \qquad x, \varepsilon \in \mathbf{R}^1;$$

$\mathbf{A_5}$) $f(x)$ satisfies the conditions

$$\lim_{x\to\infty} f(x) > 0 \quad \text{and} \quad \lim_{x\to-\infty} f(x) > 0.$$

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The condition $\mathbf{A_5}$ is not fulfilled if, for example, $f(x) = 0$ for all $x \le 0$.

The function $f(x)$ may be nonsymmetric. In particular, the function $f(x) = (-x)^{p_1}$, $x \le 0$, $f(x) = x^{p_2}, x \ge 0$ with $\min\{p_1, p_2\} > 0$ satisfies conditions $\mathbf{A_1} - \mathbf{A_5}$.

Observing sequentially the process $Y = \{Y_n, n = 0, 1, \ldots\}$ correlated to X, it is desirable to estimate the moment τ_A in a best way with respect to the loss function $f(x)$.

Concerning the observation process $Y = \{Y_n, n = 0, 1, \ldots\}$, we consider two cases: *Noisy observations* and *Delayed observations*.

Noisy observations. In that case the observation process Y has the form

$$Y: \quad Y_0 = 0, \qquad Y_n = X_n + \varepsilon \sum_{i=1}^{n} W_i, \qquad n \ge 1, \tag{1}$$

where $W_1, W_2, \ldots$ are independent $\mathcal{N}(0,1)$–Gaussian random variables (independent of $\{V_i\}$), and where $\varepsilon > 0$ is known.

For given A and an estimate η for τ_A, introduce the function

$$q(A, \eta) = \mathbf{E} f\left(\frac{\eta - \tau_A}{r}\right), \qquad r = \varepsilon \sqrt{\frac{A}{s^3(1+\varepsilon^2)}}. \tag{2}$$

We are interested in the minimal possible function $q(A, \eta)$

$$q(A) = \inf_{\eta} q(A, \eta), \tag{3}$$

where the infimum is taken over all stopping times η with respect to the process Y from (1). We use the normalization by r in (2) because for good estimates η and large A the normalized difference $(\eta - \tau_A)/r$ will be approximately $\mathcal{N}(0,1)$–Gaussian, and such normalization will allow us to avoid some bulky coefficients. For simplicity, we consider only the case when the positive values s, ε are fixed and $A \to \infty$.

Delayed observations. In that case we are given some fixed delay $d = d(A) > 0$ and the process Y has the form

$$Y: \quad Y_0 = Y_1 = \ldots = Y_d = 0; \qquad Y_n = X_{n-d}, \qquad n \ge d+1. \tag{4}$$

Similarly to (2)–(3), for given d and an estimate η for τ_A, introduce the functions

$$q(d, \eta) = \mathbf{E} f\left(\frac{\eta - \tau_A}{r_d}\right), \qquad r_d = \sqrt{\frac{d}{s^2}}, \tag{5}$$

and

$$q(d) = \inf_{\eta} q(d, \eta), \tag{6}$$

where the infimum is taken over all stopping times η defined with respect to the process Y from (4).

2. Main results. Introduce the value

$$m(f) = \inf_a \mathbf{E} f(\xi + a), \qquad \xi \sim \mathcal{N}(0,1). \tag{7}$$

Since $f(x)$ satisfies the conditions $\mathbf{A_1}, \mathbf{A_3}$ and $\mathbf{A_5}$, we have $0 < m(f) < \infty$.

Theorem 1 (Noisy observations). *If* $f(x)$ *satisfies conditions* $\mathbf{A_1}$-$\mathbf{A_5}$ *then*

$$q(A) = m(f) + o(1), \qquad A \to \infty. \tag{8}$$

Theorem 1 generalizes [1, Theorem 2.3] which considers the case $f(x) = |x|$.

Theorem 2 (Delayed observations). *If* $f(x)$ *satisfies conditions* $\mathbf{A_1}$-$\mathbf{A_5}$ *then*

$$q(d) = m(f) + o(1), \qquad A, d \to \infty. \tag{9}$$

Remark. Theorems 1 and 2 remain valid if we replace X and Y by their continuous time counterparts; i.e., $X_t = st + B_t$ and $Y_t = X_t + \varepsilon W_t$ for the noisy case, and $Y_t = X_{t-d}$ for the delayed case, where $\{B_t\}_{t \geq 0}$ and $\{W_t\}_{t \geq 0}$ are independent standard Brownian motions.

REFERENCES

1. *Burnashev M. V., Tchamkerten A.* Tracking a threshold crossing time of a gaussian random walk through correlated observations. – $http://arxiv4.library.cornell.edu/abs/$1005.0616, 2010.