Strong trajectory attractors for 2D Euler equations with dissipation

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The report is based on joint works with M.I. Vishik and S.V. Zelik.

We study the following 2D dissipative Euler system:

$$\partial_t u + (u, \nabla_x)u + Ru + \nabla_x p = g(x),\ \operatorname{div} u = 0,\ x \in \Omega,\ t \geq 0, \tag{1}$$

in the domain $\Omega := [-\pi, \pi]^2$ with periodic boundary conditions. In the system (1), the velocity vector function $u = u(x,t) = (u^1(x,t), u^2(x,t))$ and the scalar pressure function $p = p(x,t)$ are unknowns. The equations contain the known vector function of external force $g = g(x) = (g^1(x), g^2(x))$ and besides the system includes the additional dissipative term $-Ru$ with coefficient $R > 0$ (in the classical conservative Euler system the dissipation is absent, that is the coefficient $R = 0$). We assume that the external force $g(x)$ belongs to the class $W^{1,\infty}(\Omega)$, i.e., the corresponding vorticity function $\operatorname{curl} g := \partial_{x_2} g^1 - \partial_{x_1} g^2$ belongs to the space $L^\infty(\Omega)$.

The system of the form (1) describes the flat motion of inviscid fluid that occupies a vessel with rough bottoms. In particular, such equations are used in mathematical geophysics to describe large-scale processes in atmosphere and ocean. The term $-Ru$ characterizes the main dissipation occurring in the planetary boundary layer (see, for example, the book [1]).

The mathematical features of these and related equations are studied in a number of papers (see, for instance, [2, 3, 4] and references therein), including the analytic properties (which are very similar to the classical Euler equations without dissipative term, see [5]), stability analysis, vanishing viscosity limit and various attractors.

It is well to bear in mind that, in contrast to the 2D Navier-Stokes equations, the considered damped Euler system is *hyperbolic* (in a sense that it is invertible in time), so one cannot expect any smoothing effect for its solutions in a finite time. In addition, up to the moment, the questions related with smoothness of global solutions of that equations are still badly understood. In a fact, to the best of our knowledge, only the modifications of the classical Yudovich result on the global existence of smooth solutions with possible double exponential growth as time $t \to +\infty$ are available in the literature and that is clearly insufficient for the attractors theory in a class of smooth functions. Thus, it seems extremely difficult/impossible to obtain the asymptotic smoothing properties for that equations which are crucial for the classical theory of the (strong) attractors (see [6, 7] and references therein). For this reason, only the existence of the attractor(s) in a *weak* topology have been verified before for the dissipative Euler system (1) (see [2, 4]).

Another essential problem with the Euler equations is related with uniqueness theorem for solutions of the corresponding initial value problem. Indeed, the uniqueness result is

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known only for the solutions with bounded vorticity curl $u := \partial_{x_2} u^1 - \partial_{x_1} u^2 \in L^\infty(\Omega)$ (due to Yudovich, see [8, 9]) and is not known in the natural Sobolev phase spaces $\mathcal{H}$ or $\mathcal{H}^1$. So, for studying the long-time behavior of solutions in the Hilbert space $\mathcal{H}^1$ we can not construct the standard semigroup $\{S(t), t \geq 0\}$ in $\mathcal{H}^1$ using the time shift of initial data along the corresponding solution at time t. Instead, one has either to use the theory of multi-valued semigroups ([2]) or to consider the trajectory space and construct so-called *trajectory* attractors ([4]).

The main aim of the present paper is to prove the existence of the (trajectory) attractor for the dissipative 2D Euler equations in the phase space $\mathcal{H}^1$ in the *strong* topology. The main difficulty here is, of course, to establish the asymptotic compactness. In order to gain it, we first construct the trajectory attractor $\mathcal{A}$ in the *weak* topology of the space $\mathcal{H}^1$, and prove that any solution, belonging to this trajectory attractor, has a bounded vorticity in the space $L^\infty(\Omega)$. Here, we use the maximum principle that we apply to the linear vorticity equation. The main trajectory phase space $\mathcal{K}^+$ is a collection of all weak solutions of the 2D Euler system, that can be obtained as a vanishing viscosity limit $\nu \to 0+$ from the corresponding solutions of the Navier-Stokes equations with the same initial data:

$$\partial_t u_\nu + (u_\nu, \nabla_x) u_\nu + R u_\nu + \nabla_x p_\nu - \nu \Delta_x u_\nu = g(x), \ \operatorname{div} u = 0, \ u|_{t=0} = u_0. \tag{2}$$

We note that, hypothetically, the defined above trajectory space $\mathcal{K}^+$ is more narrow than the trajectory space that was used in [4]. However, the advantage of this construction is that now every weak solution u of the system (1) from the space $\mathcal{K}^+$ can be approximated by regular solutions u_ν of the system (2) and the justification of the maximum principle for such solutions u_ν (for the vorticity equations) becomes immediate and, passing to the limit as $\nu \to 0+$, we obtain the mentioned above smooth property of trajectories from $\mathcal{K}^+$ which form the (weak) trajectory attractor $\mathcal{A}$.

After that, following the Yudovich method (see [8]), we prove the uniqueness theorem on the (weak trajectory) attractor and this allows to establish the *energy identity* for the solutions belonging to the attractor $\mathcal{A}$. This identity is obtained from the corresponding energy inequality (which is true for any weak solution) using the reversing of time.

Finally, we prove the desired asymptotic compactness using the so-called energy identity method which permits to obtain the strong convergence is a Banach space having the weak convergence and the convergence of the appropriate norms. Earlier, this method was successfully applied in the works of many authors in the study of global attractors for dissipative equations in unbounded domains (see, e.g., [10, 11])

Simple, but fruitful observation (in comparison with the previous works) which allows to apply this technique to equation (1) is that, in order to get the strong convergence, we need the energy identity only on the trajectory attractor $\mathcal{A}$, while outside the attractor, it is sufficient to have the energy inequality only.

Having the property of asymptotic compactness in the trajectory space $\mathcal{K}^+$, we prove the attraction to the trajectory attractor $\mathcal{A}$ in the strong topology. We also establish that the set $\mathcal{A}$ is compact in this strong topology.

In conclusion, notice that we consider the periodic boundary conditions only for simplicity. The difference with the case of a general bounded domain is only that one should equip the approximating Navier-Stokes problem with dissipation by the proper boundary conditions in order to avoid the boundary layers.

Similarly, we can also study the dissipative Euler equation on a closed 2D manifold (see [2]), for example, on a sphere, which agree well with mentioned above geophysical model of ocean and atmosphere of the Earth.

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