Covering a sphere with caps: Rogers bound revisited

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Abstract

We consider coverings of a sphere S_r^n of radius r with the balls of radius one in an n-dimensional Euclidean space $\mathbb{R}^n$. Our goal is to minimize the *covering density*, which defines the average number of the balls covering a point in S_r^n. For a growing dimension n, we obtain the covering density at most $(n \ln n)/2$ for any sphere S_r^n and the entire space $\mathbb{R}^n$. This new upper bound reduces two times the density $n \ln n$ established in the classical Rogers bound.

1 Introduction

Spherical coverings. Let $B_\varepsilon^n(\mathbf{x})$ be a *ball* (solid sphere) of radius ε centered at some point $\mathbf{x} = (x_1, \ldots, x_n)$ of an n-dimensional Euclidean space $\mathbb{R}^n$. For any subset $A \subseteq \mathbb{R}^n$, we consider any subset $\mathsf{Cov}(A, \varepsilon)$ in $\mathbb{R}^n$ that forms an ε-*covering* (an ε-net) of the set A :

$$\mathsf{Cov}(A, \varepsilon): \ A \subseteq \bigcup_{\mathbf{x} \in \mathsf{Cov}(A,\varepsilon)} B_\varepsilon^n(\mathbf{x}).$$

By changing the scale in $\mathbb{R}^n$, we can always consider the rescaled set $A' = A/\varepsilon$ and its covering $\mathsf{Cov}(A', 1)$ with unit balls. Without loss of generality, below we consider these (unit) coverings. We then define the n-dimensional volume $\mathsf{vol}(A)$ of the set A and consider the minimum *covering density*

$$\vartheta(A) = \min_{\mathsf{Cov}(A,1)} \sum_{\mathbf{x} \in \mathsf{Cov}(A,1)} \frac{\mathsf{vol}(B_1^n(\mathbf{x}) \cap A)}{\mathsf{vol}(A)}.$$

One of the classical problems is to derive the minimum covering density $\vartheta_{n,r}$ for an ($n-1$ dimensional) sphere of radius r

$$S_r^n \stackrel{\text{def}}{=} \left\{ \mathbf{z} \in \mathbb{R}^n : \sum_{i=1}^n z_i^2 = r^2 \right\}.$$

Equivalently, we can replace any ball $B_1^n(\mathbf{x})$ with its spherical cap

$$C_r^n(\rho, \mathbf{y}) = S_r^n \cap B_1^n(\mathbf{x}),$$

where $\mathbf{y} \in S_r^n$ is the center of the cap and $\rho \leq 1$ is its half-chord. Here we obtain the biggest caps with $\rho = 1$ if all centers $\mathbf{x}$ have Euclidean weight $\sqrt{r^2 - 1}$.

We now proceed with the bounds for the minimum covering density $\vartheta_{n,r}$. The celebrated Coxeter-Few-Rogers lower bound [1] shows that for a sufficiently large radius r,

$$\vartheta_{n,r} \geq c_0 n.$$

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Here and below c_i denote some universal constants. Classical papers [2] and [3] also give the upper bounds on the minimum covering density . In particular, for a sufficiently large radius r, Rogers proved that

$$\vartheta_{n,r} \leq \left(1 + \tfrac{\ln\ln n}{\ln n} + \tfrac{5}{\ln n}\right) n \ln n. \tag{1}$$

The sphere-covering problem has raised substantial interest since 1950s. Recently, the Rogers bound was improved in [4] for small $r < 1 + \frac{1}{\sqrt{n}}$. Another result of [4] extends the Rogers bound to any dimension $n \geq 3$ and any radius $r > 1$:

$$\vartheta_{n,r} \leq \left(1 + \tfrac{2}{\ln n}\right)\left(1 + \tfrac{\ln\ln n}{\ln n} + \tfrac{\sqrt{e}}{n \ln n}\right) n \ln n. \tag{2}$$

The asymptotic Rogers bound $\vartheta_{n,r} \lesssim n \ln n$ also holds for solid spheres and entire Euclidean spaces $\mathbb{R}^n$ of growing dimension n. Our result (see [7]) is presented below in Theorem 1, which reduces about two times the present upper bounds (1) and (2) for $n \to \infty$.

Theorem 1. *[7] Unit balls can cover a sphere S_r^n of any radius $r > 1$ and any dimension $n \geq 3$ with density*

$$\vartheta_{n,r} \leq \left(\tfrac{1}{2} + \tfrac{2\ln\ln n}{\ln n} + \tfrac{5}{\ln n}\right) n \ln n. \tag{3}$$

For $n \to \infty$, there exists $o(1) \to 0$ such that

$$\vartheta_{n,r} \leq \tfrac{1}{2} n \ln n + \left(\tfrac{3}{2} + o(1)\right) n \ln\ln n. \tag{4}$$

The following corollary to Theorem 1 shows that a similar reduction also holds for general Euclidean spaces $\mathbb{R}^n$.

Corollary 2. *For $n \to \infty$, unit balls can cover the entire Euclidean space $\mathbb{R}^n$ with density*

$$\vartheta_n \leq (\tfrac{1}{2} + o(1)) n \ln n. \tag{5}$$

2 Preliminaries: embedded coverings

The preliminary bounds obtained in this section are similar to (1) and (2). However, we employ a slightly different, two-step technique. First, we will cover some dense discrete subset on S_r^n and then proceed with the whole sphere. These embedded coverings will further be extended in Section 3 to improve the former bounds. Let $C(\rho, \mathbf{y})$ denote a cap $C_r^n(\rho, \mathbf{y})$ whenever parameters n and r are fixed and $C(\rho)$ denote such a cap when its center $\mathbf{y}$ is of no importance. In this case, $\mathsf{Cov}(\rho)$ will denote any covering of S_r^n with spherical caps $C(\rho)$. Let

$$\delta_\rho = \frac{\mathsf{vol}(C(\rho))}{\mathsf{vol}(S_r^n)}$$

be the fraction of the surface of the sphere S_r^n covered by a cap $C(\rho)$. For any $\tau < \rho \leq 1$, we extensively use inequality [4]:

$$\delta_\tau \geq \delta_\rho \left(\tfrac{\tau}{\rho}\right)^n$$

and its particular version $\delta_\tau \geq \delta_1 \tau^n$ obtained for $\rho = 1$. We will also choose parameters

$$\varepsilon = \tfrac{1}{n \ln n}, \quad \lambda = 1 + \tfrac{\ln\ln n}{\ln n} + \tfrac{2}{n}$$

For any $n \geq 4$, we also use inequality

$$(1 - \varepsilon)^{-n} < 1 + {}^1\!/\!\ln n + {}^1\!/\!\ln^2 n. \tag{6}$$

An embedded algorithm. To design a covering $\mathsf{Cov}(1)$, we will perform 3 steps.

1. Consider any covering of S_r^n with small caps $C(\varepsilon, \mathbf{u})$:

$$\mathsf{Cov}(\varepsilon):\ S_r^n \subseteq \bigcup_{\mathbf{u}\in\mathsf{Cov}(\varepsilon)} C(\varepsilon,\mathbf{u})$$

2. Randomly choose N large caps $C(\rho, \mathbf{y})$ of radius $\rho = 1-\varepsilon$. Specifically, the number N is chosen to obtain the covering density

$$N\delta_\rho \lesssim \lambda n \ln n.$$

This choice of N will leave a small average fraction $\exp\{-N\delta_\rho\}$ of non-covered centers $\mathbf{u}'$ of $\mathsf{Cov}(\varepsilon)$.

3. Consider the extended set of points

$$\{\mathbf{x}\} = \{\mathbf{y}\} \cup \{\mathbf{u}'\}.$$

Then the extended set of caps $C(\rho, \mathbf{x})$ leaves only the holes of size ε or less on the sphere S_r^n. Thus, we can expand the caps $C(\rho, \mathbf{x})$ to he required radius 1 and obtain the unit covering

$$\mathsf{Cov}(1):\ S_r^n \subseteq \bigcup_{\mathbf{x}\in\{\mathbf{x}\}} C(1,\mathbf{x}).$$

Lemma 3. *For any $n \geq 8$, covering $\{\mathbf{x}\}$ has density*

$$\vartheta_* \leq \left(1 + \tfrac{\ln\ln n}{\ln n} + \tfrac{2}{\ln n}\right) n \ln n. \tag{7}$$

Sketch of the proof. According to inequality (6), the caps $C(1, \mathbf{x})$ and $C(\rho, \mathbf{y})$ have similar size

$$\delta_1/\delta_\rho \leq (1-\varepsilon)^{-n} = 1 + o(1).$$

Thus, the expansion of Step 3 leaves almost the same covering density. Next, note that any $\mathsf{Cov}(\varepsilon)$ with density ϑ_ε has huge size

$$|\mathsf{Cov}(\varepsilon)| = \vartheta_\varepsilon/\delta_\varepsilon \leq (n\ln n)^n\, \vartheta_\varepsilon/\delta_\rho$$

that exceeds our N by an (approximate) factor of $(n \ln n)^n$. However, a random set of caps $C(\rho, \mathbf{y})$ with covering density $\vartheta = N\delta_\rho$ fails to cover only the small average fraction $\exp\{-\vartheta\}$ of any set $\{\mathbf{u}\}$. Namely, N random caps fail to cover an average number

$$N' = (1-\delta_\rho)^N \cdot |\mathsf{Cov}(\varepsilon)| \leq e^{-N\delta_\rho}\, |\mathsf{Cov}(\varepsilon)| \leq \vartheta_\varepsilon/n^2\delta_\rho$$

of centers $\mathbf{u}'$. Then a new covering $\{\mathbf{x}\}$ with $N + N'$ caps $C(\rho, \mathbf{x})$ has average density

$$\vartheta_\rho = \delta_\rho(N + N') \lesssim (\lambda n \ln n) + \vartheta_\varepsilon/n^2$$

Thus, the new density ϑ_ρ only fractionally (with coefficient n^{-2}) depends on ϑ_ε. If ϑ_ε is bounded away from $\lambda n \ln n$, our algorithm reduces ϑ_ε to ϑ_ρ. Thanks to rescaling of $\mathbb{R}^n$, we can then repeat the algorithm using the new ϑ_ρ in place of the former ϑ_ε. Namely, we take a new covering $\mathsf{Cov}(1)$ of a sphere $S_{r/\varepsilon}^n$ and convert it into the covering $\mathsf{Cov}(\varepsilon)$ of the sphere S_r^n. This rescaling shows that ϑ_ρ will converge to $\lambda n \ln n$. More precise analysis gives estimate (7).

Remark. By reducing the density $N\delta_\rho \sim n \ln n$, we exponentially increase the number N' of non-capped centers $\mathbf{u}$. In particular, N' reaches the huge order of $(n \ln n)^{n/2}\, N$ if $N\delta_\rho = (n \ln n)/2$.

3 New covering algorithm for a sphere S_r^n

Covering design. In this section, we obtain a covering of the sphere S_r^n with asymptotic density $(n \ln n)/2$. We use two different unrelated coverings of a sphere S_r^n :

$$\mathsf{Cov}(\varepsilon): \; S_r^n \subseteq \bigcup_{\mathbf{u} \in \mathsf{Cov}(\varepsilon)} C(\varepsilon, \mathbf{u})$$

$$\mathsf{Cov}(\mu): \; S_r^n \subseteq \bigcup_{\mathbf{z} \in \mathsf{Cov}(\mu)} C(\mu, \mathbf{z})$$

that employ "small" caps of size ε and "medium-sized" caps of radius μ, where

$$\varepsilon = \tfrac{1}{2n \ln n}, \quad \mu \sim \tfrac{1}{2\sqrt{3n} \ln^2 n}$$

Here we assume that both coverings have the former density ϑ_* of (7) of order $n \ln n$. We will then randomly choose N large caps of size $\rho = 1 - \varepsilon$. Here N is chosen to obtain half the former density:

$$N \delta_\rho \sim \tfrac{n \ln n}{2}.$$

Our large caps will fall short of covering the set $\mathsf{Cov}(\varepsilon)$. Instead, they will achieve three other tasks:

A. The center of a typical μ-cap $C(\mu, \mathbf{z})$ belongs to about $\frac{n \ln n}{2}$ ρ-caps. For

$$s \sim n/(3 \ln \ln n),$$

only $N' \ll N$ centers $\mathbf{z}'$ (in $\sim n^{n/2} N$ caps) will have less than s intersections with ρ-caps.

B. We count only those ρ-caps that leave a small uncapped fraction $\exp\left(-\ln^2 n\right)$ of a μ-cap.

C. All s-times covered μ-caps will typically have very few uncapped holes of size greater than ε. Namely, only $N'' \ll N$ centers $\mathbf{u}''$ (of $\sim n^n N$ centers $\mathbf{u}$) will be left uncapped.

We then form the joint set $\{\mathbf{x}\} = \{\mathbf{y}\} \cup \{\mathbf{z}'\} \cup \{\mathbf{u}''\}$, by adding all s-deficient centers $\mathbf{z}'$ and uncapped centers $\mathbf{u}''$. This set covers $\mathsf{Cov}(\varepsilon)$ with ρ-caps and can be expanded to the unit covering

$$\mathsf{Cov}(1): \; S_r^n \subseteq \bigcup_{\mathbf{x} \in \{\mathbf{x}\}} C(1, \mathbf{x}).$$

Remark. Note that we can still use the caps $C(\rho, \mathbf{y})$ to cover the centers of the caps $C(\mu, \mathbf{z})$. We can even reduce our covering density, thanks to a smaller size of $\mathsf{Cov}(\mu)$ relative to $\mathsf{Cov}(\varepsilon)$. However, we can no longer expand the caps $C(\rho, \mathbf{y})$ to cover the whole μ-caps without *an exponential increase in the covering density*. Indeed, straightforward calculations show that

$$\delta_1/\delta_{1-\varepsilon} \to 1, \quad \delta_1/\delta_{1-\mu} = \exp\{n^{1/2}\}, \quad n \to \infty.$$

To circumvent this problem, we change our design as follows.

1. Given any cap $C(\mu, \mathbf{z})$, we say that a cap $C(\rho, \mathbf{y})$ is d-close if $d(\mathbf{y}, \mathbf{z}) = d$. We will see that any ρ-close cap $C(\rho, \mathbf{y})$ covers only about a half of $C(\mu, \mathbf{z})$. For this reason, we will count only d-close caps $C(\rho, \mathbf{y})$ located at a slightly smaller distance

$$d = \rho - \varepsilon - \mu^2$$

The main idea of our design is the observation that a μ-cap is almost completely covered by a d-close cap $C(\rho, \mathbf{y})$ but is not by a ρ-close cap. This is illustrated in Fig. 1, where we show how substantially the covered fraction of a cap $C(\mu, \mathbf{z})$ depends on the distance $d(\mathbf{z}, \mathbf{y})$.

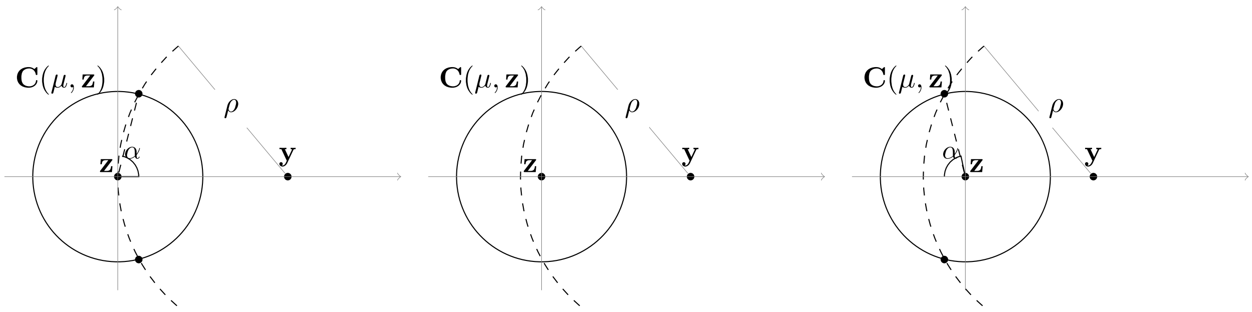

Fig. 1.1 Fig 1.2 Fig. 1.3

Fig. 1: Intersection of two caps $C(\mu, \mathbf{z})$ and $C(\rho, \mathbf{y})$ with a slight variation of the distances $d(\mathbf{z}, \mathbf{y})$. Parameters $\varepsilon = 1/(2n \ln n)$, $\mu = 1/\left(2\sqrt{3n} \ln^2 n\right)$, $\rho = 1 - \varepsilon$.

Fig. 1.1: $d(\mathbf{z}, \mathbf{y}) = \rho$. Then $\cos\alpha \simeq \mu/2$ and $\sin^n \alpha \sim 1$. Then the ρ-cap covers $\sim$ half of the μ-cap.

Fig. 1.2: $d(\mathbf{z}, \mathbf{y}) = (\rho^2 - \mu^2)^{1/2} \simeq \rho - \mu^2/2$. Then the ρ-cap covers half of the μ-cap.

Fig. 1.3: $d(\mathbf{z}, \mathbf{y}) = \rho - \varepsilon - \mu^2$. Then $\cos\alpha > \varepsilon/\mu$ and $\sin^n \alpha \leq \exp\left(-\ln^2 n\right)$. Then the ρ-cap covers all but the fraction $\exp\{-\ln^2 n\}$ of the μ-cap.

Formally, we have

Lemma 4. *For any cap $C(\mu, \mathbf{Z})$, a randomly chosen d-close cap $C(\rho, \mathbf{Y})$ fails to cover any given point $\mathbf{x}$ of $C(\mu, \mathbf{Z})$ with probability $p(\mathbf{x}) \leq \omega$, where*

$$\omega \leq \tfrac{1}{4 \ln n} \exp\left(-\tfrac{3}{2} \ln^2 n\right)$$

Remark. The above choice of d is central to Lemma 4, and even a marginal increase in d will completely change our setting. Namely, it can be proven that about half the base of the μ-cap is uncovered if a ρ-cap is $(d + \varepsilon)$-close.

2. It is also easy to verify that the above distance d is so close to $\rho = 1 - \varepsilon$ that

$$\delta_\rho / \delta_d \to 1, \quad n \to \infty.$$

For this reason, counting only d-close caps will carry no overhead to covering density.

3. The uncapped fraction ω is still too large given the huge order $n^{n/2} N$ of μ-caps. To reduce this fraction, we use the fact that a typical μ-cap is covered about $(n \ln n)/2$ times by d-close caps $C(\rho, \mathbf{y})$. We then consider the s-deficient caps $C(\mu, \mathbf{z}')$ covered by s or fewer d-close caps. We then prove that there exist only very few such caps even when compared to the number N :

Lemma 5. *For large n, the expected number N' of s-deficient caps $C(\mu, \mathbf{z}')$ is*

$$N' < 2^{-n/4} N.$$

4. Next, we proceed with the remaining, s-saturated μ-caps and count all centers $\mathbf{u}'' \in \mathsf{Cov}(\varepsilon)$ covered by none of s (or more) overlapping ρ-caps. For a given s, we then prove

Lemma 6. *For large n, the total number of centers $\mathbf{u}'' \in \mathsf{Cov}(\varepsilon)$ left uncovered in all s-saturated caps $C(\mu, \mathbf{z})$ has expectation*

$$N'' < 2^{-n/2} N.$$

Lemma 6 shows that the set $\{\mathbf{u}''\}$ forms a very small portion of not only $\mathsf{Cov}(\varepsilon)$, but also of a much smaller set $\mathsf{Cov}(\rho)$.

5. Finally, we consider the set $\{\mathbf{x}\} = \{\mathbf{y}, \mathbf{z}', \mathbf{u}''\}$ that includes the centers $\mathbf{z}'$ of s-deficient μ-caps and the centers $\mathbf{u}''$ of uncapped ε-caps in all s-saturated μ-caps. This set $\{\mathbf{x}\}$ completely covers the set $\mathsf{Cov}(\varepsilon)$ with the caps $C(\rho, \mathbf{x})$. Therefore, $\{\mathbf{x}\}$ also covers S_r^n with unit caps. Then the straightforward but lengthy calculations show that we obtain the required bounds (3) and (4).

Finally, Theorem 1 directly leads to Corollary 2. Indeed, here we can use the well known fact (see [5] or [6]) that

$$\vartheta_n = \lim_{r \to \infty} \vartheta_{n,r}$$

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