THE THEORY OF FUNDAMENTAL OPERATOR-FUNCTIONS OF DEGENERATIVE INTEGRAL-DIFFERENTIAL OPERATORS IN BANACH SPACES

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Introduction. In this report we present the main ideas of a new approach to the study of degenerate linear differential equations in Banach spaces. Interest in these equations as an independent object of research, has been initiated in mathematical periodicals observed since 1950–1960s. In particular, the problem of constructing a general theory of differential equations Banach spaces with Noetherian operator in the main part was formulated by L.A. Lyusternik during his seminars in Moscow State University. Studies of solvability of the Cauchy problem for these equations in the classes of finitely smooth functions have shown that such problems have smooth (classical) solutions only for certain relations between the input data of the problem, i.e., between initial conditions and right-hand side (of free function) equation. The search for these sufficient conditions, as well as formulas for the solution itself, usually is the goal of such studies. In general case the absence of classic solution naturally leads (in linear case) to the formulation of problems in the class of distributions (generalized functions), since in this case there is no need to match the input data of the problem. Therefore, for linear equations the three problems have been formulated. First we need to allocate classes of generalized functions in Banach spaces in which solutions are unique. Second, we need to develop the technology of the generalized solutions construction. And finally we have to study the relationship between the classic generalized solutions. Such triple problem we study in terms of fundamental operator-functions of degenerate integral-differential operators. In order to find the solutions of differential equations in distributions spaces we employ the fundamental operator function which appears to be the most natural tool.

In order to present the essence of this approach we use the following example of the Cauchy problem for integral-differential equation of the second kind

$$Bu^{(2)}(t) = Au(t) + \int_0^t g(t-s)Au(s)ds + f(t), \tag{1}$$

$$u(0) = u_0, \quad u'(0) = u_1, \tag{2}$$

where A, B are closed linear operators from E_1 to E_2, with dense с плотными областями определения, $D(B) \subset D(A)$, E_1 and E_2 are banach spaces, $g(t)$ is continuous function, $f(t)$ is sufficiently smooth function B is fredholm operator.

Let us introduce the main terminology from [3], which use below.

Generalized functions in Banach spaces. Let E be banach space, let E^* be – conjugate Banach space. We call the set of finite infinitly differentiable functions $s(t)$ with значениями в $K(E^*)$ as the main space $K(E^*)$. The convergence in $K(E^*)$ we introduce as follows. The sequence of functions $s_n(t)$ converge to $s(t)$ in $K(E^*)$ if:

a) $\exists R > 0$ such that $\forall n \in N \; supp s_n(t) \subset [-R, R]$;

b) $\forall \alpha \in N$ for $n \to +\infty \sup\limits_{[-R,R]} \parallel s_n^{(\alpha)}(t) - s^{(\alpha)}(t) \parallel \to 0$.

Generalized function (distribution) with values in Banach space E we call any linear continuous functional defined on $K(E^*)$. The set of all generalized functions with values in E we note as $K'(E)$. Convergence in $K'(E)$ is defined as week (point-wise). Here we follow the classic monograph of V.S.Vladimirov and define the set of generalized functions as $\mathcal{D}'$. The equality of two generalized functions, support of generalized function, multiplication of generalized function on infinitly differentiable function are defined as for classic generalized functions. Any locally Bohner integrable function $f(t)$ with values in E derive the following regular generalized function

$$\Big(f(t), s(t)\Big) = \int\limits_{-\infty}^{+\infty} \langle f(t), s(t)\rangle dt, \quad \forall s(t) \in K(E).$$

All the generalized functions, which operations can be defined using that rule are called as regular generalized functions. The rest of the generalized functions are called as singular. The classic example of singular generalized function is the Dirac delta-function:

$$\Big(a\delta(t), s(t)\Big) = \langle a, s(0)\rangle dt, \quad \forall s(t) \in K(E), \quad \forall a \in E.$$

The distribution set with left-bounded support $(K'_+(E) \subset K'(E))$ we denote as $K'_+(E)$. This class is the most conventional in our studies.

Let $E_1, \; E_2$ are the Banach spaces, $A(t) \in \mathcal{C}^\infty$ is operator-function with values in $\mathcal{L}(E_1, E_2), \; h(t) \in \mathcal{D}'$ is classic generalized function [1]. Then the following multiplication (formal expression) $A(t)h(t)$ is called as generalized operator-function. The following generalized operator-function will correspond to integral-differential operator (1)

$$\mathcal{L}_2(\delta(t)) = B\delta''(t) - A(\delta(t) + g(t)\theta(t)).$$

Let $f(t) \in K'_+(E_1), \; h(t) \in \mathcal{D}'_+$, then the generalized function $A(t)h(t) * f(t) \in K'_+(E_2)$ defined as follows

$$\Big(A(t)h(t) * f(t), s(t)\Big) = \Big(h(t), \Big(f(\tau), A^* s(t+\tau)\Big)\Big), \quad \forall s(t) \in K(E_2)$$

is called as convolution of generalized operator-function $A(t)h(t)$ and generalized function$f(t)$.

This definition is correct since supports of the functions $h(t) \in \mathcal{D}'_+$ и $f(t) \in K'_+(E_1)$ are left bounded. It's proofed using the same scheme as proof of the convolution existence in algebra $\mathcal{D}'_+$ in classical theory of generalized functions [1]. It is to be noted that convolution exists in the distributions space with left bounded support and it has associativity property which we employ to proof the principal statements here.

Let us introduce the key concept. The fundamental operator-function of integral-differential operator $\mathcal{L}_2(\delta(t))$ is called generalized operator-function $\mathcal{E}_2(t)$, which satisfies the following equalities:

$$\mathcal{E}_2(t) * \mathcal{L}_2(\delta(t)) * u(t) = u(t), \quad \forall u(t) \in K'_+(E_1),$$

$$\mathcal{L}_2(\delta(t)) * \mathcal{E}_2(t) * v(t) = v(t), \quad \forall v(t) \in K'_+(E_2).$$

The reason for such construction introduction is as follows. If the fundamental operator-function $\mathcal{E}_2(t)$ is known for integral-differential operator $\mathcal{L}_2(\delta(t))$, then in class $K'_+(E_1)$ exists the unique generalized solution

$$u(t) = \mathcal{E}_2(t) * f(t) \in K'_+(E_1)$$

of

$$\mathcal{L}_2(\delta(t)) * u(t) = f(t), \quad f(t) \in K'_+(E_2).$$

Indeed, if $v(t) \neq u(t)$ is other solution of convolution equation then

$$v(t) = \mathcal{E}_2(t) * \mathcal{L}_2(\delta(t)) * v(t) = \mathcal{E}_2(t) * f(t) = u(t).$$

Fundamental operator-functions of degenerative integral-differential operators.

Theorem. *If A, B are closed linear operators from E_1 into $E_2, D(B) \subset D(A), \overline{D(A)} = \overline{D(B)} = E_1$, B is Fredholm operator, $\overline{R(B)} = R(B)$, B has complete A-Jordan set $\{\varphi_i^{(j)},\ i = \overline{1,n},\ j = \overline{1,p_i}\}$ [2], then*

a) 2nd order differential operator $\big(B\delta''(t) - A\delta(t)\big)$ on the class $K'_+(E_2)$ has fundamental operator-function

$$\mathcal{E}_1(t) = \Gamma\frac{\sinh(\sqrt{A\Gamma}t)}{\sqrt{A\Gamma}}\left[I - \sum_{i=1}^{n}\sum_{j=1}^{p_i}\langle\cdot,\psi_i^{(j)}\rangle A\varphi_i^{(p_i+1-j)}\right]\theta(t) -$$

$$-\sum_{i=1}^{n}\left[\sum_{k=0}^{p_i-1}\left\{\sum_{j=1}^{p_i-k}\langle\cdot,\psi_i^{(j)}\rangle\varphi_i^{(p_i-k+1-j)}\right\}\delta^{(k)}(t)\right];$$

b) 2nd order integral-differential operator $\big(B\delta''(t) - A(\delta(t) + g(t)\theta(t))\big)$ *in class* $K'_+(E_2)$ *has the following fundamental operator-function*

$$\mathcal{E}_2(t) = \Gamma \sum_{k=1}^{\infty} \Big(\delta(t) + g(t)\theta(t)\Big)^{k-1} * \frac{t^{2k-1}}{(2k-1)!}\theta(t)(A\Gamma)^{k-1} \times$$

$$\times \left[I - \sum_{i=1}^{n} \sum_{j=1}^{p_i} \langle \cdot, \psi_i^{(j)} \rangle A \varphi_i^{(p_i+1-j)} \right] -$$

$$- \sum_{i=1}^{n} \left[\sum_{k=0}^{p_i-1} \left\{ \sum_{j=1}^{p_i-k} \langle \cdot, \psi_i^{(j)} \rangle \varphi_i^{(p_i-k+1-j)} \right\} \delta^{(2k)}(t) * \Big(\delta(t) + \mathcal{R}(t)\theta(t)\Big)^{k+1} \right],$$

where $\{\psi_i^{(j)},\ i = \overline{1,n},\ j = \overline{1,p_i}\}$ *–* A^**-Jordan set of the operator* B^**,* $\Gamma-$ *is the Trenogin-Schmidt [2] operator,* $\mathcal{R}(t)$ *is resolvent of the kernel* $(-g(t)\theta(t))$*.*

The Cauchy problem (1)-(2) in terms of generalized functions can be presented as following convolution equation

$$\mathcal{L}_2(\delta(t)) * \tilde{u}(t) = f(t)\theta(t) + Bu_1\delta(t) + Bu_0\delta'(t),$$

which is class of distributions with left bounded support $K'_+(E_1)$ has the following unique solution

$$\tilde{u}(t) = \mathcal{E}_2(t) * \Big(f(t)\theta(t) + Bu_1\delta(t) + Bu_0\delta'(t)\Big). \tag{3}$$

Further analysis of the singular and regular components of the expression (3) for generalized solution allows us to obtain the theorems on classic solutions of the problem (1)-(2).

Let us demonstrate that based on the following examples.

Example 1. (Boussinesk-Love Equation) For equations which model (in 1D case) longitudinal oscillations in thin elastic bar with taking into account the lateral inertia [4],

$$(\lambda - \Delta)v_{tt}(t, \bar{x}) = \alpha^2 \Delta v(t, \bar{x}) + f(\bar{x}), \quad \lambda, \alpha \neq 0,$$

where $\bar{x} \in \Omega \subset R^m$, Ω is bounded area with boundary $\partial\Omega$ of the class C^∞, we study the Cauchy-Dirichlet problem in the cylinder $\Omega \times R_+$

$$v\Big|_{t=0} = v_0(\bar{x}), \quad \frac{\partial v}{\partial t}\Big|_{t=0} = v_1(\bar{x}) \quad x \in \Omega$$

$$v\Big|_{\partial\Omega} \equiv 0 \quad (x, t) \in \partial\Omega \times R_+.$$

We can reduce that problem to Cauchy problem (1)-(2) with $g(t) \equiv 0$, if the spaces E_1 and E_2 can be selected as follows

$$E_1 \equiv \overset{\circ}{H}{}^{k+2}[\Omega] \equiv \left\{ u \in W_2^{k+2} : \ u(\bar{x}) = 0, \, \bar{x} \in \partial\Omega \right\}, \quad E_2 \equiv H^k \equiv W_2^k$$

where $W_p^k \equiv W_p^k(\Omega)$ is Sobolev space $1 < p < \infty$, and let

$$B = \lambda - \Delta, \quad A = \alpha^2 \Delta, \quad \lambda \in \sigma(\Delta).$$

Here B is Fedholm operator and lengths of all the A-Jordan chains are 1s, i.e. in the formula for fundamental operator-function $\mathcal{E}_1(t)$ from the theorem $p_i = 1$. Which means that generalized solution (3) do not contains the singular component. The remaining regular component will be classic solution of this problem if the following conditions are fuilfilled

$$(f(\bar{x}) + \alpha^2 \lambda v_0(\bar{x}), \varphi_k) = 0, \ \ (v_1(\bar{x}), \varphi_k) = 0 \ \forall \varphi_k : \ \ \lambda = \lambda_k,$$

here φ_k are eigen functions of the Laplace operator, which correspond eigen value $\lambda \in \sigma(\Delta)$.

Example 2. (Equation of viscoelastic plates with memory) Let us address the following equation

$$(\gamma - \Delta) v_{tt}(t, \bar{x}) = -\Delta^2 v(t, \bar{x}) + \int_0^t g(t-s) \Delta^2 v(s, \bar{x}) ds + f(t, \bar{x}),$$

where $\bar{x} \in \Omega \subset R^m$, Ω is bounded area with boundary $\partial\Omega$ of the class C^∞, for $m = 2$ и $f(t, \bar{x}) = 0$ such equation describes the oscillation of viscoelastic plates with memory [5]. We follow here the last example and study the Cauchy-Dirichlet problem on cylibder $\Omega \times R_+$

$$v\Big|_{t=0} = v_0(\bar{x}), \quad \frac{\partial v}{\partial t}\Big|_{t=0} = v_1(\bar{x}) \quad x \in \Omega$$

$$v\Big|_{\partial\Omega} \equiv 0 \quad (x, t) \in \partial\Omega \times R_+.$$

Such problem we can reduce to the Cauchy problem (1)–(2), if we select spaces and operators as follows

$$E_1 \equiv \overset{\circ}{H}{}^{k+4}[\Omega] \equiv \left\{ u \in W_2^{k+4} : \ u(\bar{x}) = 0, \, \bar{x} \in \partial\Omega \right\}, \quad E_2 \equiv H^k \equiv W_2^k$$

$$B = \gamma - \Delta, \quad A = -\Delta^2, \quad \gamma \in \sigma(\Delta).$$

Here (similar with example 1) B is Fredholm operator and lengths of all the A-Jordan chains are equal to 1, i.e. in the formula for fundamental operator-function

$\mathcal{E}_2(t)$ from the theorem all $p_i = 1$, i.e. generalized solution (3) does not contain singular component. Hence the remaining component will be the classic solution if the following conditions are fuilfilled

$$\Big(f(0,\bar{x}) - \gamma^2 v_0(\bar{x}), \varphi_k\Big) = 0,$$

$$\left(\frac{\partial f(0,\bar{x})}{\partial t} - \gamma^2 v_1(\bar{x}) + g(0)\gamma^2 v_0(\bar{x}), \varphi_k\right) = 0 \;\forall \varphi_k : \quad \lambda = \lambda_k,$$

here φ_k are eigen functions of Laplace operator which correspond to eigen value $\lambda \in \sigma(\Delta)$.

References

1. Vladimirov V.S. *Generalized functions in mathematical physics* – M.: Nauka, 1979.

2. Vainberg M.M., Trenogin V.A. *Branching theory of nonlinear equations* – M.: Nauka, 1969.

3. N. Sidorov, B. Loginov, A. Sinithin and M. Falaleev *Lyapunov-Schmidt Methods in Nonlinear Analysis and Applications* – Kluwer Academic Publishers, 2002. – 548 p.

4. Whitham G.B. *Linear and Non-Linear Wales* – M.: Mir, 1977.

5. Cavalcanti M.M., Domingos Cavalcanti V.N., Ferreira J. *Existence and Uniform Decay for a Non-Linear Viscoelastic Equation with Strong Damping* – Math. Meth. Appl. Sci., 2001, Vol. 24, P. 1043–1053.