Markov processes of infinitely many nonintersecting random walks

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Abstract

We will discuss multidimensional Markov processes which can be viewed as independent random walks conditioned never to collide. Our main topic is construction of such Feller Markov processes in the infinite-dimensional space of point configurations in $\mathbb{Z}$ with finitely many particles to the left of the origin. These processes turn out to be (indirectly) related to totally asymmetric simple exclusion processes with jump rates depending on particles.

In the talk we are going to discuss multidimensional Markov processes $\{\mathcal{Z}_i(t)\}_{i\geq 1}$ in $\mathbb{Z}$ whose trajectories form *non-intersecting paths* in $\mathbb{Z}\times[0,\infty]$. These process are obtained from collections of i.i.d. random walks by imposing certain non-intersecting conditions.

An example of such *finite-dimensional* Markov process can be constructed by means of a classical theorem due to Karlin and McGregor [KM]. Let $X(t) = (X_1(t),\dots,X_N(t)) \in \mathbb{Z}^N$ be $N\geq 1$ independent Poisson processes started at $X(0) = (0,1,\dots,N-1)$ and conditioned to finish at $X(T) = Y$ while taking mutually distinct values for all $0\leq t\leq T$. A result of [KOR] says that as $T\to\infty$ with Y being asymptotically linear, $Y\sim T\xi$ with a collection of asymptotic speeds $\xi = (\xi_1\leq\dots\leq\xi_N)\in\mathbb{R}^N_{>0}$, the process $(X(t), 0\leq t\leq T)$ has a limit $(\mathcal{X}(t), t\geq 0)$, which is a homogeneous Markov process on $\mathbb{Z}^N$ with initial condition $X(0)$ and transition probabilities

$$P_t\big((x_1,\dots,x_N)\to(y_1,\dots,y_N)\big) = const\cdot\frac{\Xi_N(y)}{\Xi_N(x)}\det\left[\frac{t^{y_i-x_j}}{(y_i-x_j)!}\right]_{i,j=1}^N$$

*This talk is based on the joint work with Alexei Borodin with $\Xi_N(u) = \det\bigl[\xi_i^{u_j}\bigr]_{i,j=1}^N$. (If some of ξ_j's coincide, for the formula to make sense one needs to perform a limit transition from the case of distinct speeds.) The process $\mathcal{X}(t)$ is the Doob h–transform of N independent Poisson processes with respect to the harmonic function Ξ_N.

The situation becomes more complicated when one tries to construct a Markovian infinite-dimensional version of such process, i.e. to send N to ∞. Properties (and even the existence) of the limit process depends on the sequence $\{\xi_i\}$. During the talk we will mostly deal with the case $\xi_i = q^{1-i}$, $0 < q < 1$. We will see that the limit of $\mathcal{X}(t)$ (and other similar processes) as $N \to \infty$ can be seen as a Feller Markov process on all point configurations in $\mathbb{Z}_{\geq 0}$, started from the densely packed initial condition. Note that no scaling is needed in the limiting procedure. As all Feller processes, our limiting process can be started from any initial condition, and it has a modification with càdlàg sample paths.

The procedure of passing to a limit by increasing the number of particles and scaling the space appropriately in models similar to the ones we study is very well developed. In many cases such a limit can be successfully performed for joint distributions at finitely many time moments, if the original model undergoes a Markov dynamics. The most common approach is to control of the limiting behavior of local *correlation functions* for the model at hand.

However, although it is a common belief in many models, but there are no *a priori* reasons for the Markov property of the dynamics to be preserved under such limit transitions. Thus, to understand the Markovian structure of the limit we adopt a different approach, similar to that of [BO]: We use the fact (proved earlier by the author, see [Gor]) that infinite point configurations in $\mathbb{Z}$ with finitely many particles to the left of the origin can be identified with ergodic q-Gibbs measures on infinite Gelfand-Tsetlin schemes. We further show that the Markov processes $\mathcal{X}(t)$ for different N's are consistent with respect to natural projections from the N-particle space to the $(N-1)$-particle one; the projections are uniquely determined by the q-Gibbs property. Together with certain (nontrivial) estimates, this leads to the existence of the limiting Feller Markov process $\mathfrak{X}(t)$. One interesting feature of the construction is that we need to add Gelfand-Tsetlin schemes with infinite entries in order to make the space of the ergodic q-Gibbs measures locally compact.

We also show that the dynamical correlation functions of $\mathfrak{X}(t)$ started from the packed initial condition are determinantal, and they are the limits of the corresponding correlation functions for $\mathcal{X}(t)$.

It is worth noting what happens in the limit $q \to 1$. In that case, the distribution of $\mathcal{X}(t)$ for a fixed $t > 0$ is known as the *Charlier orthogonal poly-*

nomial ensemble, which is the basic discrete probabilistic model of random matrix type. If one considers its limiting behavior as $N \to \infty$, in different parts of the state space via suitable scaling limits one uncovers discrete sine, sine, and Airy determinantal point processes which play a fundamental role in Random Matrix Theory, cf. [J], [BO2].

As for our constructions, the space of ergodic 1-Gibbs measures has countably many continuous parameters (as opposed to discrete ones for $q < 1$), and it is naturally isomorphic to the space of indecomposable characters of the infinite-dimensional unitary group, and to the space of totally positive doubly infinite Toeplitz matrices (see e.g. [VK], [OO], [Olsh] and references therein) . The $q \to 1$ limit of our infinite–dimensional Feller Markov process ends up being a *deterministic* (in fact, linear) flow on this space.

It is interesting to note that the process $\mathfrak{X}(t)$ is related to other well known Markov process with infinite number of particles and non-intersecting trajectories — totally asymmetric simple exclusion process. This connection is based on the construction of 'local' Markov process on the infinite Gelfand-Tsetlin schemes introduced in [BF]. This interpretation implies, in particular, that the distribution of the smallest particle of $\mathfrak{X}(t)$ coincides with the asymptotic displacement of the mth particle, as $m \to \infty$, in TASEP with jump rates depending on particles, particle j has rate ξ_j, and step initial condition (the step initial condition means that we place the particle j at location $-j$ at $t = 0$.) Inspired by this connection, we will also discuss large time asymptotics for TASEP, and more general PushASEP of [BF-Push], with finitely many particles that have arbitrary jumps rates.

References

[BO] A. Borodin, G. Olshanski, Markov processes on the path space of the Gelfand-Tsetlin graph and on its boundary. arXiv:1009.2029

[BO2] A. Borodin, G. Olshanski, Z-Measures on partitions, Robinson-Schensted-Knuth correspondence, and beta=2 random matrix ensembles, Random matrix models and their applications (P.M.Bleher and R.A.Its, eds), Math. Sci. Res. Inst. Publ., vol. 40, Cambridge Univ. Press, Cambridge, 2001, 71–94

[BF] A. Borodin, P. Ferrari, Anisotropic growth of random surfaces in $2+1$ dimensions. arXiv:0804.3035.

[BF-Push] A. Borodin, P. Ferrari, Large time asymptotics of growth models on space-like paths I: PushASEP, Electron. J. Probab. 13 (2008), 1380-1418.

[Gor] V.Gorin, The q–Gelfand–Tsetlin graph, Gibbs measures and q–Toeplitz matrices. arXiv:1011.1769

[J] K. Johansson, Discrete orthogonal polynomial ensembles and the Plancherel measure, Ann. Math.,(2), 153, no.1, 259–296, 2001.

[KM] S. P. Karlin and G. MacGregor, Coincidence probabilities, Pacif. J. Math. 9, 1141-1164, 1959.

[KOR] W. Konig, N. O'Connell, S. Roch, Non-colliding random walks, tandem queues and discrete orthogonal polynomial ensembles. Electronic Journal of Probability, vol.7 (2002), paper no. 1, 1–24.

[Olsh] G. Olshanski, The problem of harmonic analysis on the infinite-dimensional unitary group, Journal of Functional Analysis, 205 (2003), 464–524

[OO] A. Okounkov, G. Olshansky, Asymptotics of Jack Polynomials as the Number of Variables Goes to Infinity, International Mathematics Research Notices 13 (1998), pp. 641–682.

[VK] A. M. Vershik, S. ,V. Kerov, Characters and factor representations of the inifinite unitary group, Sov. Math. Dokl. 26 (1982), pp. 570–574.