Thermodynamic Properties of Infinite Non-Negative Matrices and Loaded Graphs

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A classification of non-negative matrices extending the well-known classification of stochastic matrices was suggested by Vere-Jones [1], [2] in the sixties of the last century. In particular, he distinguished the "R-positive" matrices. The number R, for a stochastic matrix, equals 1, and the 1-positivity of such a matrix means that all states of the corresponding Markov chain are positive.

More recently this classification was used in [3], [5] when developing thermodynamic formalism for infinite symbolic Markov chains. To set forth necessary notions of this theory we turn to the geometric language equivalent to the matrix one.

Definition 1. If $G = (V, E)$ is a directed graph and $\mathcal{W}$ a positive function on E, we refer to the pair $(G, \mathcal{W})$ as a *loaded graph* (LG). The number $\mathcal{W}(i, j)$ is the *weight* of the edge (i, j) and $\mathcal{W}$ is the *weight function*.

Let $A = (a_{ij})_{i,j\in\mathbb{N}}$ be an infinite non-negative matrix. We say that a directed graph $G = G_A = (V, E)$ with vertex set $V = \mathbb{N}$ and edge set E is a structural graph of A if $E = \{(i, j) : a_{ij} > 0\}$. Clearly A and the LG $(G_A, \mathcal{W})$ where $\mathcal{W}(i, j) = a_{i,j}$ uniquely determine each other.

From now on we will only consider loaded graphs $(G, \mathcal{W})$, where $G = (V, E)$ is strongly connected. Every sequence $i_0, \dots, i_n$, where $i_r \in V$, $0 \le r \le n$, $(i_r, i_{r+1}) \in E$, $0 \le r \le n-1$, will be referred to as a path of length n in G going from i_0 to i_n.

For $n \ge 1,\ i \in V$, denote by $\Gamma_i^{(n)}$ the set of paths of length n in G going from i to i (i-cycles) and by $\Gamma_{i|i}^{(n))}$ the set of those $\gamma \in \Gamma_i^{(n)}$ containing i only as the first and last vertex. For each path $\gamma = (i_0, i_1, \dots, i_n)$, denote by $\mathcal{W}(\gamma)$ its weight:

$$\mathcal{W}(\gamma) := \prod_{r=0}^{n-1} \mathcal{W}(i_r, i_{r+1}).$$

Let

$$\mathcal{W}_i^{(n)} := \sum_{\gamma\in\Gamma_i^{(n)}} \mathcal{W}(\gamma), \quad \mathcal{W}_{i|i}^{(n)} := \sum_{\gamma\in\Gamma_{i|i}^{(n)}} \mathcal{W}(\gamma), \quad i \in V.$$

(Each sum is assumed to be zero if the summation is performed over the empty set.)

Introduce the generating functions

$$\Phi_i(t) = \sum_{n=1}^{\infty} t^n \sum_{\gamma\in\Gamma_i^{(n)}} \mathcal{W}(\gamma) = \sum_{n=1}^{\infty} \mathcal{W}_i^{(n)} t^n,$$

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$$\Phi_{i|i}(t) = \sum_{n=1}^{\infty} t^n \sum_{\gamma \in \Gamma_{i|i}^{(n)}} \mathcal{W}(\gamma) = \sum_{n=1}^{\infty} \mathcal{W}_{i|i}^{(n)} t^n,$$

and denote by R_i and $R_{i|i}$ the radii of convergence of these power series, respectively. Clearly $R_i \leq R_{i|i}$. Since G is strongly connected, R_i does not depend on i, and we will denote it by R. We only conseder the case $R > 0$.

Definition 2. LG $(G, \mathcal{W})$ is said to be *positive recurrent* if, for some $i \in V$, either $\Phi_{i|i}(R) > 1$, or $\Phi_{i|i}(R) = 1,\ \Phi'_{i|i}(R) < \infty$. In the former case $(G, \mathcal{W})$ is *stable positive recurrent*, in the latter case it is *unstable positive recurrent*.

Both properties in this definition are obeyed or not for all i simultaneously.

All the above-said on infinite loaded graphs is clearly meaningful for finite loaded graphs. These ones are always stable positive recurrent.

Let $X = X(G)$ be the space of two-sided infinite paths in G, i.e.,

$$X = \{x = (x_r)_{r=\mathbb{Z}} : (x_r, x_{r+1}) \in E \text{ for all } r \in \mathbb{Z}\}.$$

Introduce the standard topology on X and remark that X is compact in this topology if and only if G is finite. The shift transformation $S : X \to X$ defined by

$$(Sx)_r = x_{r+1}, \quad x \in X, \quad r \in \mathbb{Z},$$

is clearly a homeomorphism of X.

Denote by $\mathcal{E} = \mathcal{E}(X, S)$ the set of S-invariant ergodic Borel probability measures on X and define a function $f = f_{G,\mathcal{W}} : X \to \mathbb{R}$ by

$$f(x) := -\ln \mathcal{W}(x_0, x_1), \quad x = (x_r)_{r \in \mathbb{Z}} \in X(G).$$

Let

$$P_f(\mu) := h_\mu(S) + \mu(f)$$

for all $\mu \in \mathcal{E}$ such that $f \in L^1_\mu$ (here, $\mu(f) := \int_X f d\mu$ and $h_\mu(S)$ is the entropy of the dynamical system (X, μ, S)).

In [3] it is shown that the just defined functional can be extended to the whole set $\mathcal{E}$ with values in $\mathbb{R} \cup \{+\infty\}$.

The quantity

$$P_f := \sup_{\mu \in \mathcal{E}} P_f(\mu)$$

is called the *pressure* (or *topological pressure*) of f. (The supremum can be evaluated over those $\mu \in \mathcal{E}$ for which the right-hand side is well defined). A measure μ is called *f-equilibrium*, if $P_f(\mu) = P(f)$.

Theorem 1 (see [3]). *An f-equilibrium measure μ exists if and only if $(G, \mathcal{W})$ is positive recurrent, such μ is unique and is a Markov measure.*

Thus, stable and unstable positive recurrent LGs are indistinguishable with respect to the existence of an equilibrium measure. There is, however, a more subtle difference between them related to this measure. To describe this difference we need the following definition.

Definition 3. A sequence of finite subgraphs $G_n = (V_n, E_n)$ of an infinite graph $G = (V, E)$ is called an *exhaustive* sequence in G if G_n is a subgraph of G_{n+1} for all n and $\bigcup_n V_n = V$, $\bigcup_n E_n = E$.

Below we assume that all exhaustive sequences under consideration consist of strongly connected graphs (such sequences in a strongly connected infinite graph always exist).

Let $(G, \mathcal{W})$ be an LG and $\{G_n\}$ an exhaustive sequence in G. For each n, there exists a unique $f_{G_n,\mathcal{W}}$- equilibrium measure $\mu_{G_n,\mathcal{W}}$ defined on the space $X(G_n)$. Since $X(G_n) \subset X(G)$, this measure can be viewed as a measure on $X(G)$.

Theorem 2 (see [4], [5]). *If $(G, \mathcal{W})$ is a stable positive recurrent LG, then, for every exhaustive sequence $\{G_n\}$ in G, the corresponding sequence of the $f_{G_n,\mathcal{W}}$-equilibrium measures $\mu_{G_n,\mathcal{W}}$ converges weakly to the $f_{G,\mathcal{W}}$-equilibrium measure $\mu_{G,\mathcal{W}}$.*

Examples constructed in [5] show that, for an unstable positive recurrent LG, this theorem is not true. These examples suggest the following conjecture: for an unstable positive recurrent LG $(G, \mathcal{W})$, there exist two exhaustive sequences in G: for one of them (we call such a sequence *regular*) the conclusion of Theorem 2 is true, for the other (we call it *irregular*) is not. Let us now describe a class of LGs for which the conjecture can be proved.

Definition 4. Let Γ be a collection of paths in a graph G. We say that G is generated by Γ if every edge of G belongs to one of the paths in Γ. We call G *a cascade graph* with input $i \in V$ and output $j \in V$ (i and j need not be different) if there are paths $\gamma_1, \gamma_2, \dots$ of length ≥ 2 in G all going from i to j and such that G is generated by the set $\{\gamma_1, \gamma_2, \dots\}$, and γ_s, $\gamma_{s'}$ for $s \neq s'$ have no joint vertices different from i and j.

We also recall that the *union* of graphs $G' = (V', E')$ and $G'' = (V'', E'')$ is the graph $G = (V, E)$ with $V = V' \cup V''$ and $E = E' \cup E''$.

Definition 5. We say that G is a *bunch* of cascade graphs $G_1, \dots, G_k$ if it is a union of a finite connected graph $G_0 = (V_0, E_0)$ (the *basis* of the bunch) and the cascade graphs G_r, $1 \leq r \leq k$, such that the inputs and outputs of all G_r are vertices of G_0, and for each pair of graphs $G_r, G_{r'}$, $0 \leq r, r' \leq k$, $r \neq r'$, only these inputs and outputs can be joint vertices.

Certainly, the notion of a bunch of cascade graphs is meaningful only in the case when these cascade graphs are infinite, because finite ones can be included in the basis.

The following result was obtain together with O.R. Novokreschenova.

Theorem 3. *If $(G, \mathcal{W})$ is an unstable positive recurrent LG where G is a bunch of cascade graphs, then there exists both regular an irregular sequences in G.*

References

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