Averaging and regularization mechanism for the periodic Korteweg-de Vries equation

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We consider the space-periodic Korteweg–de Vries equation (KdV)

$$\partial_t u = u\partial_x u + \partial_x^3 u, \qquad u(0,x) = u^0(x), \tag{1}$$

where $x \in \mathbb{S}^1 = [0, 2\pi]$ and $u(t,0) = u(t,2\pi)$. The linear term with third-order derivative generates rapid oscillations for higher Fourier modes. In this work we reveal connection between the smoothness properties of the solutions and the algebraic structure of the nonlinear resonances between the high-frequency oscillations.

Our main goal is to make the relations between time-averaging effects and smoothness issues more explicit, rather than to obtain global regularity results under minimal restrictions (see, e.g., [2], [3], [4], [5] and the references therein). In particular, our approach and aim are completely different than the machinery and harmonic analysis tools that were developed over the past decade and half for investigating dispersive partial differential equations. Moreover, we also remark that our tools and ideas can be easily applied to multi-dimensional equations and multi-component systems.

Our approach is as follows. First, we rewrite the problem as a system of ODE for time-dependent Fourier coefficients. Second, to make the effects of time averaging explicit we single out oscillating factors and do several integration by parts, with respect to time, to obtain several generations of equations for slowly varying coefficients. Resonances reveal themselves as obstacles to the integrations by parts and produce resonant terms in the equations, integrated terms become more and more regular. Higher generations of equations allow solutions with less regularity. To show their regularity we use straightforward estimates of multilinear operators, energy estimates and the contraction principle. To use the contraction principle in low-order Sobolev spaces we use splitting to high and low Fourier modes and exploit averaging-induced squeezing of higher modes. In order to justify our estimates we use a Galerkin approximations procedure. We do not use in our analysis the specific properties of the KdV equation such as complete integrability or special conserved quantities.

We now describe our method in greater detail. One can easily see that smooth solutions conserve the L_2-norm and, in addition, $\int_0^{2\pi} u(t,x)dx = \int_0^{2\pi} u(0,x)dx$. Therefore we assume that the solution has mean value zero. Using the Fourier series in x

$$u(t,x) = \sum_{k\in\mathbb{Z}_0} u_k(t)e^{ikx}, \quad u_k \in \mathbb{C}, \quad u_k(t) = \frac{1}{2\pi}\int_0^{2\pi} u(t,x)e^{-ikx}dx, \tag{2}$$

we define the homogeneous Sobolev spaces

$$\|u\|^2_{\dot{H}^s} = \|\{|k|^s u_k\}\|^2_{l_2} = \sum_{k\in\mathbb{Z}_0} |k|^{2s}|u_k|^2, \quad s \in \mathbb{R},$$

and obtain for the coefficients $u_k(t)$ the infinite coupled system of ODE's

$$\partial_t u_k = \frac{1}{2}ik \sum_{k_1+k_2=k} u_{k_1}u_{k_2} - ik^3 u_k, \quad u_k(0) = u_k^0, \quad k \in \mathbb{Z}_0 = \mathbb{Z}\setminus\{0\}. \tag{3}$$

Transformation of variables. Next, we use the following transformation of variables:

$$u_k(t) = e^{-ik^3t} v_k(t), \qquad k \in \mathbb{Z}_0. \tag{4}$$

Substituting (4) into (3), multiplying by e^{ik^3t}, and using the identity

$$(k_1 + k_2)^3 - k_1^3 - k_2^3 = 3(k_1 + k_2)k_1k_2, \tag{5}$$

we obtain the equivalent system

$$\partial_t v_k = \frac{1}{2} ik \sum_{k_1+k_2=k} e^{i3kk_1k_2t} v_{k_1} v_{k_2}, \quad v_k(0) = v_k^0 = u_k^0, \quad k \in \mathbb{Z}_0. \tag{6}$$

In the first place, the substitution (4) eliminates the linear term ik^3u_k in (3) with highest growth as $|k| \to \infty$; secondly, (and most importantly) introduces oscillating exponentials into the nonlinear term.

Definition 1. A function v is called a solution of (6) on $[0, T]$, if $v \in L_\infty([0, T]; \dot{H}^0)$ and the integrated equation (6)

$$v_k(t) - v_k(0) = \frac{1}{2} ik \int_0^t \sum_{k_1+k_2=k} e^{i3kk_1k_2\tau} v_{k_1}(\tau) v_{k_2}(\tau) d\tau, \quad k \in \mathbb{Z}_0 \tag{7}$$

is satisfied for all $k \in \mathbb{Z}_0$. Accordingly, $u(x, t)$ with $u_k(t) = e^{-ik^3t} v_k(t)$ is then called a weak solution of the KdV.

Averaging. Differentiation by parts in time I. We apply the elementary formula $e^{at}w(t) = (e^{at}w(t)/a)' - (e^{at}/a)w(t)'$ to the right-hand side of (6):

$$\begin{gathered}\partial_t v_k = \partial_t \left(\frac{1}{2} ik \sum_{k_1+k_2=k} \frac{e^{i3kk_1k_2t} v_{k_1} v_{k_2}}{i3kk_1k_2} \right) - \frac{1}{2} ik \sum_{k_1+k_2=k} \frac{e^{i3kk_1k_2t}}{i3kk_1k_2} \partial_t (v_{k_1} v_{k_2}) = \\ \frac{1}{6} \partial_t \left(\sum_{k_1+k_2=k} \frac{e^{i3kk_1k_2t} v_{k_1} v_{k_2}}{k_1k_2} \right) - \frac{1}{6} \sum_{k_1+k_2=k} \frac{e^{i3kk_1k_2t}}{k_1k_2} (v_{k_2} \partial_t v_{k_1} + v_{k_1} \partial_t v_{k_2}).\end{gathered} \tag{8}$$

Expressing $\partial_t v_{k_1}$ and $\partial_t v_{k_2}$ from the original equation (6) and using the identity

$$(k_1 + k_2 + k_3)^3 - k_1^3 - k_2^3 - k_3^3 = 3(k_1 + k_2)(k_2 + k_3)(k_3 + k_1), \tag{9}$$

we obtain the KdV in the following *first form*:

$$\partial_t \left(v_k - \frac{1}{6} B_2(v, v)_k \right) = \frac{i}{6} R_3(v, v, v)_k, \qquad k \in \mathbb{Z}_0, \tag{10}$$

where

$$B_2(u, v)_k = B_2(u, v, t)_k = \sum_{k_1+k_2=k} \frac{e^{i3kk_1k_2t} u_{k_1} v_{k_2}}{k_1k_2}, \qquad k \in \mathbb{Z}_0, \tag{11}$$

and

$$R_3(u, v, w)_k = \sum_{k_1+k_2+k_3=k} \frac{e^{i3(k_1+k_2)(k_2+k_3)(k_3+k_1)t}}{k_1} u_{k_1} v_{k_2} w_{k_3}, \qquad k \in \mathbb{Z}_0. \tag{12}$$

As a result, we obtain the equation (10) with *bounded* operators:

Lemma 1. *The operator R_3 is bounded: $R_3 : \dot{H}^s \times \dot{H}^s \times \dot{H}^s \to \dot{H}^s$ for $s > 1/2$, and the operator B_2 is smoothing: $B_2 : \dot{H}^s \times \dot{H}^s \to \dot{H}^{s+1}$ for $s > -1/2$.*

Averaging. Differentiation by parts in time II. The trilinear operator R_3 is bounded only in sufficiently smooth Sobolev spaces $\dot{H}^s$, $s > 1/2$, therefore we are unable to use (10) to establish a priori estimates for all $s \geq 0$. For this reason we again use the idea of differentiation by parts and represent the exponential as a time derivative. But before doing that we have to take care of the resonances that are the obstruction to this procedure.

We single out in (12)the terms for which

$$(k_1+k_2)(k_2+k_3)(k_3+k_1) = 0, \qquad k_1+k_2+k_3 = k \in \mathbb{Z}_0. \tag{13}$$

Accordingly,

$$\begin{aligned} &R_3(v,v,v)_k = R_{3\mathrm{res}}(v^3)_k + R_{3\mathrm{nres}}(v^3)_k, \\ &R_{3\mathrm{res}}(v^3)_k = \sum_{k_1+k_2+k_3=k}^{\mathrm{res}} \frac{v_{k_1}v_{k_2}v_{k_3}}{k_1}, \\ &R_{3\mathrm{nres}}(v^3)_k = \sum_{k_1+k_2+k_3=k}^{\mathrm{nonres}} \frac{e^{i3(k_1+k_2)(k_2+k_3)(k_3+k_1)t}}{k_1} v_{k_1}v_{k_2}v_{k_3}, \end{aligned} \tag{14}$$

where the first summation is carried out over k_1, k_2, k_3, satisfying (13) (the resonance), while in the second summation $(k_1+k_2)(k_2+k_3)(k_3+k_1) \neq 0$ (the nonresonant terms).

For the resonant operator we obtain using (13) the expression

$$R_{3\mathrm{res}}(v^3)_k = \frac{v_k}{k}(\|v\|^2_{L_2} - |v_k|^2) =: A_{\mathrm{res}}(v)_k = \frac{v_k}{k}(\|v^0\|^2_{L_2} - |v_k|^2), \tag{15}$$

where the last equality holds if the energy is conserved. Equation (10) goes over into

$$\partial_t\left(v_k - \frac{1}{6}B_2(v,v)_k\right) = \frac{i}{6}A_{\mathrm{res}}(v)_k + \frac{i}{6}R_{3\mathrm{nres}}(v^3). \tag{16}$$

We express the last term on the right-hand side as a time derivative:

$$\begin{aligned} R_{3\mathrm{nres}}(v^3)_k = \sum_{k_1+k_2+k_3=k}^{\mathrm{nonres}} \frac{e^{i3(k_1+k_2)(k_2+k_3)(k_3+k_1)t}}{k_1} v_{k_1}v_{k_2}v_{k_3} = \frac{1}{3i}\partial_t B_3(v,v,v)_k - \\ \frac{1}{3i}\sum_{k_1+k_2+k_3=k}^{\mathrm{nonres}} \frac{e^{i3(k_1+k_2)(k_2+k_3)(k_3+k_1)t}}{k_1(k_1+k_2)(k_2+k_3)(k_3+k_1)}(\partial_t v_{k_1}v_{k_2}v_{k_3} + v_{k_1}\partial_t v_{k_2}v_{k_3} + v_{k_1}v_{k_2}\partial_t v_{k_3}), \end{aligned} \tag{17}$$

where

$$B_3(u,v,w)_k = \sum_{k_1+k_2+k_3=k}^{\mathrm{nonres}} \frac{e^{i3(k_1+k_2)(k_2+k_3)(k_3+k_1)t}}{k_1(k_1+k_2)(k_2+k_3)(k_3+k_1)} u_{k_1}v_{k_2}w_{k_3}. \tag{18}$$

As before, the $\partial_t v_{k_1}, \partial_t v_{k_2}, \partial_t v_{k_3}$ in (17) are expressed from the equation (6), and after straight forward calculations we obtain

$$\begin{aligned} \sum_{k_1+k_2+k_3=k}^{\mathrm{nonres}} \frac{e^{i3(k_1+k_2)(k_2+k_3)(k_3+k_1)t}}{k_1(k_1+k_2)(k_2+k_3)(k_3+k_1)}(\partial_t v_{k_1}v_{k_2}v_{k_3} + v_{k_1}\partial_t v_{k_2}v_{k_3} + v_{k_1}v_{k_2}\partial_t v_{k_3}) = \\ = iB_4(v,v,v,v)_k, \end{aligned}$$

where

$$B_4(u,v,w,\varphi)_k = \frac{1}{2}B_4^1(u,v,w,\varphi)_k + B_4^2(u,v,w,\varphi)_k$$

and

$$B_4^1(u,v,w,\varphi)_k = \sum_{k_1+k_2+k_3+k_4=k}^{\mathrm{nonres}} \frac{e^{i\Phi(k_1,k_2,k_3,k_4)t}}{(k_1+k_2)(k_1+k_3+k_4)(k_2+k_3+k_4)} u_{k_1}v_{k_2}w_{k_3}\varphi_{k_4},$$

$$B_4^2(u,v,w,\varphi)_k = \sum_{k_1+k_2+k_3+k_4=k}^{\text{nonres}} \frac{e^{i\Phi(k_1,k_2,k_3,k_4)t}\ (k_3+k_4)}{k_1(k_1+k_2)(k_1+k_3+k_4)(k_2+k_3+k_4)} u_{k_1}v_{k_2}w_{k_3}\varphi_{k_4}.$$

The phase function Φ here is

$$\Phi(k_1,k_2,k_3,k_4) = (k_1+k_2+k_3+k_4)^3 - k_1^3 - k_2^3 - k_3^3 - k_4^3,$$

and, unlike (5), (9), does not have a nice factorization, however, its particular analytic expression is not used below. As a result, we obtain the *second form of the KdV.*

$$\partial_t \left(v_k - \frac{1}{6}B_2(v,v)_k - \frac{1}{18}B_3(v,v,v)_k \right) = \frac{i}{6}A_{\text{res}}(v)_k + \frac{i}{18}B_4(v,v,v,v)_k, \quad k \in \mathbb{Z}_0. \tag{19}$$

Lemma 2. *The operators B_2,B_3, B_4 and A_{res} are smoothing in $\dot H^s$ for $s \ge 0$:*

$$\begin{aligned} B_2 &: \dot H^s \times \dot H^s \to \dot H^{s+1} \text{ for } s > -1/2, \\ B_3 &: \dot H^s \times \dot H^s \times \dot H^s \to \dot H^{s+2} \text{ for } s \ge 0, \\ B_4 &: \dot H^s \times \dot H^s \times \dot H^s \times \dot H^s \to \dot H^{s+\varepsilon} \text{ for } s \ge 0,\ \varepsilon < 1/2, \\ A_{\text{res}} &: \dot H^s \to \dot H^{s+1} \text{ for } s \ge 0. \end{aligned}$$

The conservation of the $\dot H^0$-norm, this lemma, and equation (19) make it possible to obtain a priori estimates in the Sobolev spaces $\dot H^s$ for the solution v (more precisely, uniform in m a priori estimates for the Galerkin approximations $v^{(m)}$) for any $s \ge 0$. Based on this we have

Theorem 1. *Let $s_0 > 0$, $v(0) \in \dot H^{s_0}$ and $T > 0$ be fixed, and let $0 < \sigma < s_0$. Then the exists a subsequence $v^{(m)}(t)$ of the Galerkin solutions, which converges strongly in $L_p([0,T];H^\sigma)$, for any $1 < p < \infty$ and $*$-weak in $L_\infty([0,T];\dot H^{s_0})$ to $v^\infty(t)$, which is a solution of the KdV equation in the sense of Definition 1 and*

$$\|v^\infty\|_{L_\infty([0,T],\dot H^{s_0})} \le M_{s_0} = M_{s_0}(T, s_0, \|v(0)\|_{\dot H^{s_0}}).$$

In addition, $v^\infty(t)$ conserves energy:

$$\|v^\infty(t)\|_{\dot H^0} = \|v^0\|_{\dot H^0} \quad a.\,e.\ on \quad [0,T].$$

Uniqueness and Lipschitz continuity (regular case** $s > 1/2$**). The uniqueness and Lipschitz continuity with respect to the initial data is proved in terms of the KdV equation in the first form (10). We observe that the equation is not resolved with respect to the time derivative, and the next lemma on the invertibility of the linearized operator on the left-hand side in (10) plays the the key role.

Lemma 3. *Let $\varphi \in \dot H^0$. Then the linear operator L_φ, $L_\varphi v = v - B_2(\varphi, v)$ is bounded and invertible in $\dot H^\theta$ for $\theta > -1/2$, for any fixed t. For $f \in \dot H^\theta$ the equation $L_\varphi v = f$ has a unique solution $v = L_\varphi^{-1} f \in \dot H^\theta$, and*

$$\|v\|_{\dot H^\theta} \le \|L_\varphi^{-1}\|_{\mathcal{L}(\dot H^\theta)} \|f\|_{\dot H^\theta}, \quad \|L_\varphi^{-1}\|_{\mathcal{L}(\dot H^\theta)} \le F(\|\varphi\|_{\dot H^0}),$$

where F is monotone increasing and independent of t.

Theorem 2. *Let $s > 1/2$, $v(0) = v^0 \in \dot H^s$, and $T > 0$ be fixed. Then the volution $v = v^\infty$ of the KdV equation constructed in Theorem 1 is unique, is of class $C([0,T];\dot H^s)$, is Lipschitz continuous with respect to the initial data, and conserves energy.*

Proof. The solution $v = v^\infty$, from Theorem 1 satisfies the integrated first form of the KdV

$$v_k(t) - \frac{1}{6}B_2(v(t), v(t))_k = v_k(0) - \frac{1}{6}B_2(v^0, v^0)_k + \frac{i}{6}\int_0^t R_3((v(\tau))^3)_k d\tau, \quad v_k(0) = v_k^0,$$

for all $t \in [0, T]$. Setting $y(t) = v(t) - v^0, \quad y(0) = 0$, we have $y \in L_\infty([0,T]; \dot{H}^s)$. The symmetry of B_2, $B_2(u, v) = B_2(v, u)$, gives

$$y(t) - \frac{1}{3}B_2(v^0, y(t)) = \frac{1}{6}B_2(y(t), y(t)) + \frac{i}{6}\int_0^t R_3(\big(y(\tau) + v^0\big)^3)d\tau.$$

Setting $L_{v^0}\, y = y - \frac{1}{3}B_2(v^0, y)$, by Lemma 3 we have

$$y(t) = L_{v^0}^{-1}(t)\left(\frac{1}{6}B_2(y(t), y(t)) + \frac{i}{6}\int_0^t R_3(\big(y(\tau) + v^0\big)^3)d\tau\right) =: \mathcal{F}(y)(t).$$

Since $y(0) = 0$, $B_2(y, y)$ is "quadratically" small and the mapping $\mathcal{F}$ is a contraction in $C([0, T^*]; \dot{H}^s)$ for small T^*. Therefore this equation has a unique small solution on a short time interval. Step by step we reach T. This follows from the established a priori estimates of the solution in $\dot{H}^s$, $s \geq 0$. $\square$

Remark 1. R_3 is bounded in $\dot{H}^s$, $s > 1/2$, hence the restriction $s > 1/2$.

Uniqueness in the non-regular case ($0 \leq s \leq 1/2$***).*** We first point out that for the proof of uniqueness for $0 \leq s \leq 1/2$ we cannot use the equation (10) because R_3 is unbounded for $s \leq 1/2$. The equation (19) has all terms bounded, however, it is not clear how to prove the invertibility of the linearized operator under the time derivative in (19).

Let v, w be two solutions in $\dot{H}^\theta$ with $v(0) = v^0$, $w(0) = w^0$. Similarly to Theorem 2 we want to transform the problem to the equation in $L_\infty([T_1, T_1 + \tau], \dot{H}^\theta)$ for $y(t)$, where $v(t) = v^0 + y(t)$, of the form

$$y = \mathcal{F}_{\tau,n}(y, v^0). \tag{20}$$

Here $\mathcal{F}_{\tau,n}$ is a Lipschitz mapping in $L_\infty([T_1, T_1 + \tau], \dot{H}^\theta)$ with Lipschitz constant < 1, so that this equation has a unique small solution y in $L_\infty([T_1, T_1 + \tau], \dot{H}^\theta)$. The parameter n describes the construction of the operator $\mathcal{F}_{\tau,n}$, which involves the splitting of the Fourier modes y_k of the solution y into high modes (with $|k| > n$) and low modes (with $|k| \leq n$). The Lipschitz estimate for $y(t) = v(t) - v^0$ and $z(t) = w(t) - w^0$ will have the form

$$\begin{aligned} &\|\mathcal{F}_{\tau,n}(y, v^0) - \mathcal{F}_{\tau,n}(z, w^0)\|_{L_\infty([T_1,T_1+\tau],\dot{H}^\theta)} \leq \\ &(F_1(n) + \tau F_2(n))\|y - z\|_{L_\infty([T_1,T_1+\tau],\dot{H}^\theta)} + C\|v^0 - w^0\|_{\dot{H}^\theta}, \end{aligned} \tag{21}$$

where $F_1(n) \to 0$ as $n \to \infty$. Once (21) is constructed, we first choose n large enough and then τ small so that

$$\|\mathcal{F}_{\tau,n}(y) - \mathcal{F}_{\tau,n}(z)\|_{L_\infty([T_1,T_1+\tau],\dot{H}^\theta)} \leq \frac{1}{2}\|y - z\|_{L_\infty([T_1,T_1+\tau],\dot{H}^\theta)} + C\|v^0 - w^0\|_{\dot{H}^\theta}.$$

Iterating then last estimate over short time intervals we obtain

$$\|v - w\|_{L_\infty([0,T],\dot{H}^\theta)} \leq C''(2C + 1)^T\|v^0 - w^0\|_{\dot{H}^\theta}.$$

Thus, we need to construct (20) with property (21). This is achieved by the following representation of the nonresonant operator $R_{3\text{nres}}$. We denote by Π_n the spectral projection onto the Fourier modes with wave numbers m with $|m| \leq n$, and set $\Pi_{-n} = I - \Pi_n$. Using this splitting we have

$$R_{3\text{nres}}(u, v, w) = R^{(n)}_{3\text{nres}0}(u, v, w) + R^{(n)}_{3\text{nres}1}(u, v, w)$$

where

$$R^{(n)}_{3\mathrm{nres}0}(u,v,w)_k=\sum_{k_1+k_2+k_3=k}^{\mathrm{nonres}}\frac{e^{i3(k_1+k_2)(k_2+k_3)(k_3+k_1)t}}{k_1}u_{k_1}\Pi_{-n}v_{k_2}\Pi_{-n}w_{k_3}.$$

The operator $R^{(n)}_{3\mathrm{nres}1}$ has better continuity properties than $R_{3\mathrm{nres}}$, however, the corresponding constant grows as $n\to\infty$ and plays the role of $F_2(n)$ in (21).

Lemma 4. *Let $0\le s\le 1$. Then $R^{(n)}_{3\mathrm{nres}1}$ is bounded in $\dot H^s$ and satisfies*

$$\|R^{(n)}_{3\mathrm{nres}1}(u,v,w)\|_{\dot H^s}\le c_4 n^{s+1}\|u\|_{\dot H^0}\|v\|_{\dot H^0}\|w\|_{\dot H^0}+c_4 n\|u\|_{\dot H^0}\|v\|_{\dot H^0}\|w\|_{\dot H^s}.$$

We represent the remaining operator $R^{(n)}_{3\mathrm{nres}0}$ as the time derivative as we did before for the entire $R_{3\mathrm{nres}}$ in (17). As a result we the *third* form of the KdV:

$$\partial_t\left(v_k-\frac{1}{6}B_2(v^2)_k-\frac{1}{18}B^{(n)}_{30}(v^3)_k\right)=\frac{i}{6}R_{3\mathrm{res}}(v^3)_k+\frac{i}{6}R^{(n)}_{3\mathrm{nres}1}(v^3)_k+\frac{i}{18}B^{(n)}_{40}(v^4)_k. \quad (22)$$

It can be shown that $B^{(n)}_{30}$ has a small norm as $n\to\infty$; the corresponding constant is the $F_1(n)$ in (21).

Lemma 5. *If $0<s\le 1$, then*

$$\|B^{(n)}_{30}(v,v,v)\|_{\dot H^s}\le\frac{C}{n^s}\|v\|^2_{\dot H^0}\|v\|_{\dot H^s}.$$

For $s\le 0$

$$\|B^{(n)}_{30}(v,v,v)\|_{\dot H^s}\le\frac{C(p,\alpha)}{n^{2\alpha}}\|v\|^3_{\dot H^s},$$

where $p=-s\ge 0$, $\alpha>0$ and $p+\alpha<5/6$.

This shows that (22) can be written in the form (20), (21), and we obtain the main result:

Theorem 3. *Let $s\in[0,1/2]$ and $T>0$. For any $v(0)=v^0\in\dot H^s$ the solution $v=v^\infty$ of the KdV equation constructed in Theorem 1 is unique, belongs to $C([0,T],\dot H^s)$ and Lipschitz continuously depends on the initial data.*

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