On Global Attractors of Nonlinear Hyperbolic PDEs

A.I. Komech

Institute for the Information Transmission Problems RAS

International Conference dedicated to 50-year of IPPI RAS

July 25-29, 2011, Moscow, Russia

Supported partly by the Alexander von Humboldt Research Award,
and grants of FWF and RFBR

We consider Klein-Gordon and Dirac equations coupled to U(1)-invariant nonlinear oscillators. The solitary waves of the coupled nonlinear system form two-dimensional submanifold in the Hilbert phase space of finite energy solutions. Our main results read as follows:

Theorem
Let all the oscillators be strictly nonlinear. Then any finite energy solution converges, in the long time limit, to the solitary manifold in the local energy seminorms.

The investigation is inspired by Bohr's postulates on transitions to quantum stationary states. The results are obtained for:

- 1D KGE coupled to one oscillator [1,2,3], and to finite number of oscillators [4];
- nD KGE and Dirac eqns coupled to one oscillator via mean field interaction [5, 6].

1 Main results

We consider nonlinear Klein-Gordon equation of type

$$\ddot\psi(x,t) = \psi''(x,t) - m^2\psi + \delta(x) f(\psi(0,t)), \quad x \in \mathbb{R} \tag{1}$$

which describes the nonlinear oscillator coupled to the Klein-Gordon field. We suppose that

C1 $$f(\psi) = -\nabla U(\psi), \qquad U(\psi) \in C^2(\mathbb{R}^2, \mathbb{R}) \tag{2}$$ C1

Then equation (1) is Hamiltonian with the Hamilton functional

$$H(t) = \frac{1}{2}\int\limits_{\mathbb{R}} \Big[|\dot\psi(x,t)|^2 + |\psi'(x,t)|^2 + m^2|\psi(x,t)|^2\Big]dx + U(\psi(0,t)) \tag{3}$$ U

We denote the phase space $\mathcal{E} := H^1(\mathbb{R}) \oplus L^2(\mathbb{R})$ and suppose

$$\inf_{\psi\in\mathbb{C}} U(\psi) > -\infty \tag{4}$$ UU

Proposition 1 *(Well posedness) For any* $(\psi(x,0),\dot{\psi}(x,0)\in\mathcal{E}$ *there exists unique solution* $(\psi(x,t),\dot{\psi}(x,$
$C_b(\mathbb{R},\mathcal{E})$ *to (1).*

Proof: Energy conservation $H(t)=\text{const}$ + contraction mapping principle. ■

Further we assume $U(1)$-invariance of the nonlinear oscillator:

C2 U(1)-*invariance:* $$U(\psi)=u(|\psi|) \tag{5}$$

Then

$$f(\psi)=a(|\psi|)\psi \Longrightarrow f(e^{i\theta}\psi)=e^{i\theta}f(\psi), \theta\in\mathbb{R} \tag{6}$$

Solitary waves are the solutions $\psi_\omega(x,t)=\phi(x)e^{i\omega t}$. The solutions exist only for $\omega\in\Omega$ where $\Omega\subset(-m,m)$, and the amplitude is a solution to

Nonlinear Eigenvalue Problem: $$-\omega^2\phi(x)=\phi''(x)-m^2\phi(x)+\delta(x)f(\phi(0)) \tag{7}$$

Definition 2 *Solitary manifold:* $\mathcal{S}=\{e^{i\theta}\phi_\omega(x):\omega\in\Omega,\ \theta\in[0,2\pi]\}$.

Finally, we assume

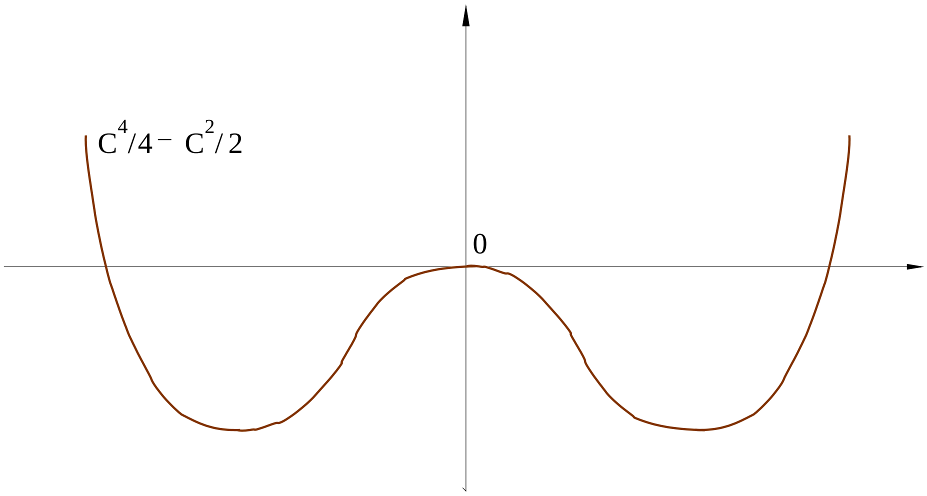
C3 *Equation (1) is strictly nonlinear* (8)

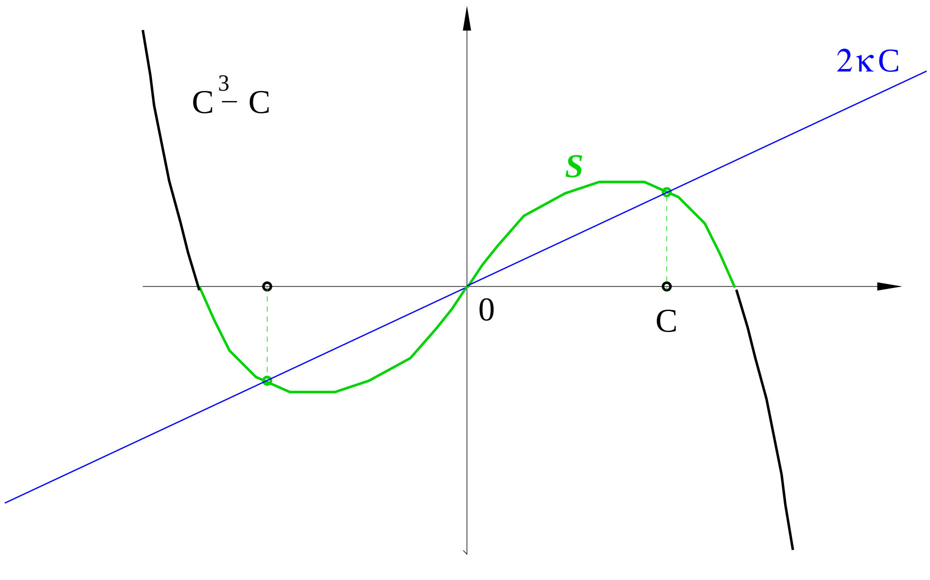
which means the following:

$$U(\psi)=u(|\psi|^2)=\Sigma_0^N u_j|\psi|^{2j},\quad u_N>0,\quad N\geq 2 \tag{9}$$

Example 1. $N=2,\ U(\psi)=|\psi|^4\Longrightarrow f(\psi)=-4|\psi|^2\psi$

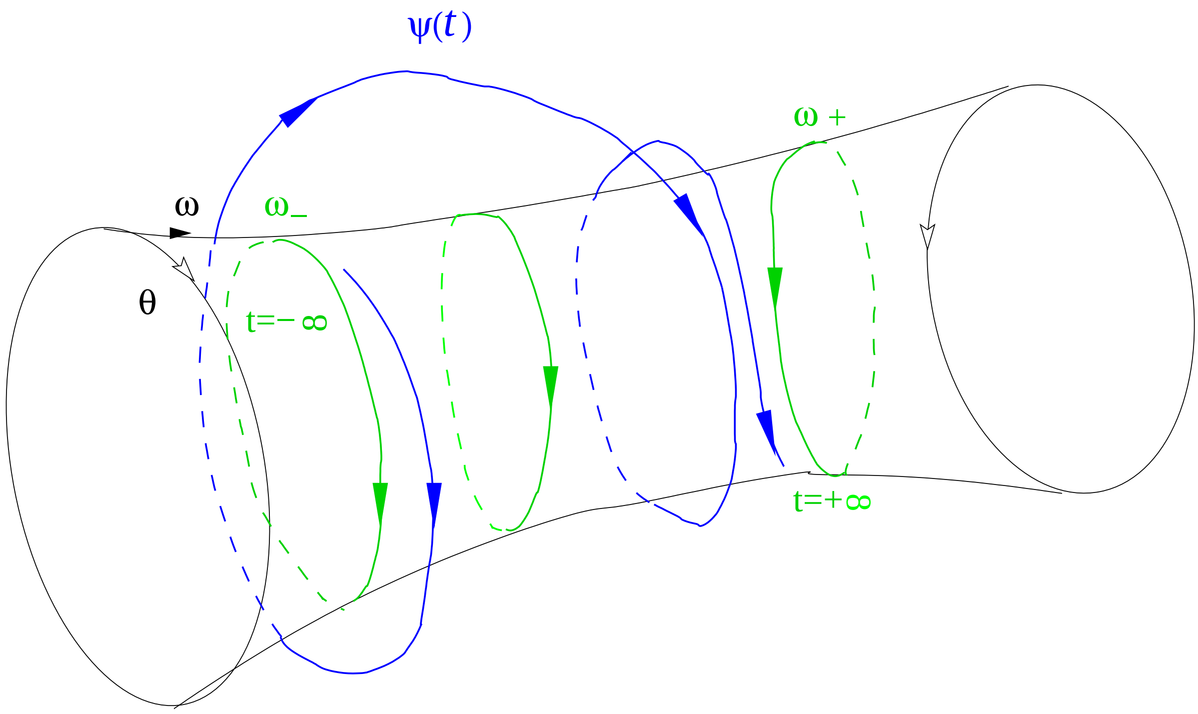
Example 2. *Ginzburg-Landau potential* $N=2,\ U(\psi)=\psi^4/4-\psi^2/2\Longrightarrow f(\psi)=-|\psi|^2\psi+\psi$


C^4/4 − C^2/2
0


C^3 − C
2κC
S
0
C

Main Thm. *Let* **C1 − C3** *hold. Then for any solution* $\psi \in C(\mathbb{R}, H^1(\mathbb{R}))$ *to Eqn (1), the* **global attraction** *holds*

$$\psi(\cdot, t) \xrightarrow{H^1_{\rm loc}(\mathbb{R})} \mathcal{S}, \quad t \to \pm\infty \tag{10}$$



Convergence (10) means that

$$\text{dist}\,(\psi(t), \mathcal{S}) := \inf_{\phi \in \mathcal{S}} \text{dist}\,(\psi(t), \phi) \to 0, \qquad t \to \infty$$

where $\quad \text{dist}\,(\psi, \phi) := \sum_{R=0}^{\infty} 2^{-R}\, \|\psi - \phi\|_{H^1(-R,R)}$

Generalizations: Convergence (10) is extended to eqns

$$\ddot\psi(x,t) = \psi''(x,t) - m^2\psi + \sum_{k=1}^{N} \delta(x - x_k) f_k(\psi(x_k, t)), \qquad x \in \mathbb{R} \tag{2007}$$

$$\ddot\psi(x,t) = \Delta\psi(x,t) - m^2\psi + \rho(x) f(\langle \psi(\cdot,t), \rho \rangle), \qquad x \in \mathbb{R}^n \tag{2008}$$

$$i\dot\psi(x,t) = \big(\boldsymbol{\alpha} \cdot \mathbf{p} + \boldsymbol{\beta} m\big)\psi + \rho(x) f(\langle \psi(\cdot,t), \rho \rangle), \qquad x \in \mathbb{R}^n \tag{2009}$$

For $\rho(x)$ we assume the *Wiener condition*:

$$\hat\rho(k) \neq 0, \quad k \in \mathbb{R}^n \tag{W}$$

which is analogue of the *Fermi Golden Rule.*

Open Questions: I. The proving of

$$\psi(x,t) \sim \phi_\pm(x)e^{i\omega_\pm t}, \quad t \to \pm\infty \tag{11}$$

with some fixed $\omega_\pm$ and $\phi_\pm$.

II. The proving of (10) and (11) for general $f(x,\psi) = -\nabla_\psi U(x,\psi)$

$$\ddot{\psi}(x,t) = \psi''(x,t) - m^2\psi(x,t) + f(x,\psi(x,t)) \tag{12}$$

Methods: The proof of global attraction (10) is based on a novel strategy:

I. Omega-limit trajectories

II. The Fourier integral representation

$$\psi(x,t) = \int e^{i\omega t}\tilde{\psi}(x,\omega)d\omega \tag{13}$$

Example: for solitary wave $\psi(x,t) = \phi(x)e^{i\omega_+ t}$

$$\tilde{\psi}(x,\omega) = \delta(\omega - \omega_+)\phi(x) \tag{14}$$

III. Dispersive radiation in continuous spectrum

IV. Titchmarsh Convolution Theorem: Nonlinear *inflation of spectrum*

2 Radiative mechanism

Our approach relies on two crucial observations on linear and nonlinear radiative mechanism: *linear dispersion* and *nonlinear inflation of spectrum.*

I. Linear dispersion Let us consider linear Klein-Gordon equation with a harmonic source

$$\ddot{\psi}(x,t) = \psi''(x,t) - m^2\psi + C\delta(x)e^{i\omega_0 t}$$

Then *Principle of Limiting Amplitude* holds:

$$\psi(x,t) \sim a(x)e^{i\omega_0 t}, \quad t \to \infty$$

where the *limiting amplitude* $a(x)$ is a solution to *stationary Helmholtz equation*

$$-\omega_0^2 a(x) = a''(x) - m^2 a(x) + C\delta(x)$$

In the Fourier transform

$$\hat{a}(k) = \frac{C}{k^2 - (\omega_0^2 - m^2)}$$

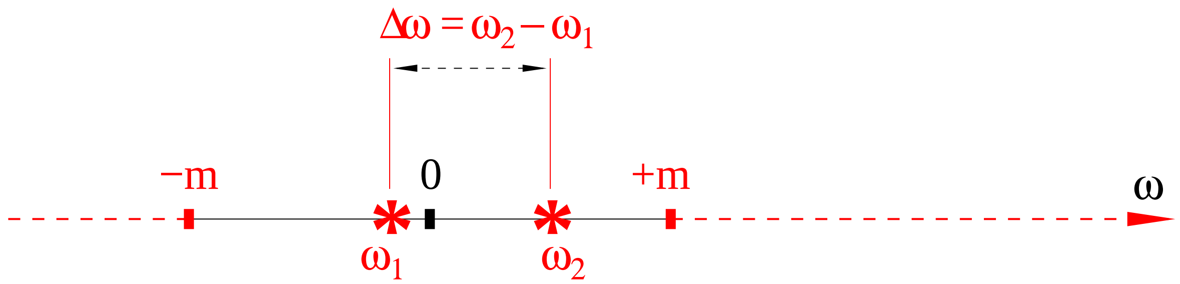
For $|\omega_0| > m$**:** $\hat{a}(k) \notin L^2(\mathbb{R})$, hence

$$\|\psi(t)\|_{H^1} \to \infty, \qquad t \to \infty$$

Conclusion: *For* $|\omega_0| > m$ *the source radiates the energy to the field.*

II. Nonlinear inflation of spectrum

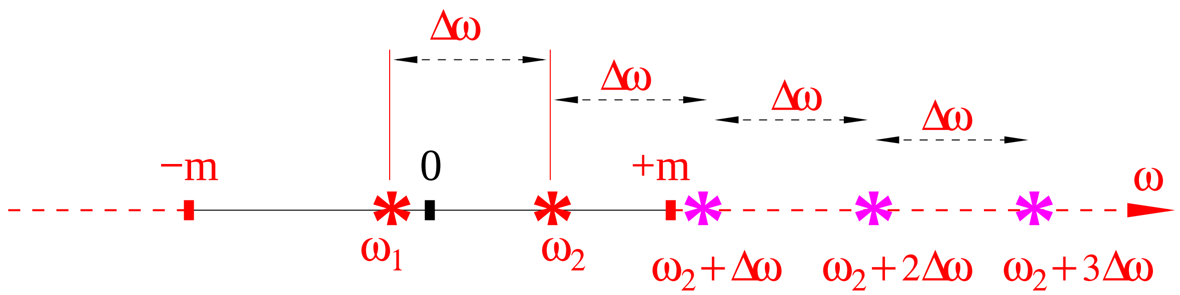
Let us consider for example $U(|\psi|^2) = |\psi|^4$. Then $f(\psi) = -\nabla_\psi U(|\psi|^2) = -4|\psi|^2\psi$
Let us sibstitute $\psi(0,t) = e^{i\omega_1 t} + e^{i\omega_2 t}$ with two point spectrum



Then we obtain the *inflation of spectrum.* Namely,

$$f(\psi(0,t)) \sim \psi\overline{\psi}\psi = e^{i\omega_2 t}e^{-i\omega_1 t}e^{i\omega_2 t} + \ldots = e^{i(\omega_2+\Delta\omega)t} + \ldots$$

This means that $\omega_2 + \Delta\omega$ also belongs to the spectrum of the solution $\psi(x,t)$ by the equation (KG1 1) ! And similarly we obtain that $\omega_2 + 2\Delta\omega, \ldots$ also belong to the spectrum.



Omega-limit trajectories

Theorem 1. (Compactness) Let $\psi(t) \in C(\mathbb{R}, H^1(\mathbb{R}))$ be a solution to (1). Then $\forall\ s_k \to \infty$ there exists a subsequence $s'_k \to \infty$ such that there exists the limit

$$\beta(x,t) = \lim_{k\to\infty} \psi(x, s_{k'} + t) \quad \text{in} \quad C(\mathbb{R}, H^1_{\text{loc}}(\mathbb{R})) \tag{15}$$

Crucial riole in our approach plays the following notion.
Definition. $\beta(x,t)$ is *omega-limit trajectory.*
Example: $\psi(x,t) \sim \phi(x)e^{i\omega_+ t} \Longrightarrow \beta(x,t) = e^{i\theta}\phi(x)e^{i\omega_+ t}$

Lemma 3 *Global attraction (10) is equivalent to the fact that* **any omega-limit trajectory** $\beta(x,t)$ **is a solitary wave:**

$$\beta(x,t) = \phi(x)e^{i\omega_+ t} \tag{16}$$

In terms of the Fourier representation,

$$\tilde\beta(x,\omega) = \delta(\omega - \omega_+)\phi(x) \tag{17}$$

We prove (17) in two steps:

$$\textbf{Step A} \qquad \text{supp}\ \tilde\beta(x,\cdot) \subset [-m, m], \qquad x \in \mathbb{R} \tag{18}$$

$$\textbf{Step B} \qquad \text{supp}\ \tilde\beta(x,\cdot) \subset \{\omega_+\}, \qquad x \in \mathbb{R} \tag{19}$$

A. Nonlinear Kato's theorem

Continuous spectrum of the Klein-Gordon equation

$$\ddot\psi(x,t) = \psi''(x,t) - m^2\psi(x,t), \qquad \psi(x,t) = e^{i(kx-\omega t)}$$

$$\omega^2 = k^2 + m^2 \quad \Longrightarrow \quad \omega \in \Sigma := (-\infty, -m] \cup [m, \infty)$$

Nonlinear version of Kato's Theorem on absence of *embedded eigenvalues*:
Theorem 2. $\tilde\psi(x,\cdot) \in L^2(\Sigma)$ for $x \in \mathbb{R}$.

Corollary. supp $\tilde\beta(x,\cdot) \subset [-m, m],\ x \in \mathbb{R}$
Proof: By definition of omega limiting trajectory

$$\tilde\beta(x,\omega) = \lim_{k\to\infty} e^{is_k\omega}\tilde\psi(x,\omega) \Rightarrow \tilde\beta(x,\omega) = 0,\ |\omega| > m$$

by the Riemann-Lebesgue Theorem. ■

B. Nonlinear spectral analysis

Next we should prove supp $\beta(x,\cdot)=\{\omega_+\}$:

Theorem 4 $\tilde\beta(x,\omega)=\delta(\omega-\omega_+)\phi(x)\quad$ *i.e.* $\beta(x,t)=\phi(x)e^{i\omega_+t}$

For the proof, we need some **equation** for β. Namely, (1) and (15) imply that

$$\ddot\beta(x,t)=\beta''(x,t)-m^2\beta+\delta(x)f(\beta(0,t))$$

In the Fourier transform:

$$-\omega^2\tilde\beta(x,\omega)=\tilde\beta''(x,\omega)-m^2\tilde\beta+\delta(x)\tilde F(\omega),\tag{20}$$

where $F(t):=f(\beta(0,t))=f(\gamma(t))$, $\gamma(t):=\beta(0,t)$.

(6) $\Rightarrow F(t)=a(|\gamma(t)|^2)\gamma(t)=A(t)\gamma(t),\ \ A(t)=a(|\gamma(t)|^2)$

Then $\tilde F(\omega)=\tilde A*\tilde\gamma$, hence (20) reads

$$-\omega^2\tilde\beta(x,\omega)=\tilde\beta''(x,\omega)-m^2\tilde\beta+\delta(x)\tilde A*\tilde\gamma\tag{21}$$

Lemma supp $\tilde\beta(x,\cdot)=$ supp $\tilde\gamma$, $\forall x\in\mathbb{R}$

Corollary (21) $\Rightarrow$ *Spectral Incluzion:*

$$\text{supp } \tilde A*\tilde\gamma\subset\text{supp } \tilde\gamma\tag{22}$$

$$\textbf{Spectral Incluzion}:\quad \text{supp } \tilde A*\tilde\gamma\subset\text{supp } \tilde\gamma\tag{23}$$

It is well known: supp $\tilde A*\tilde\gamma\subset$ supp $\tilde A+$ supp $\tilde\gamma$

The Titchmarsh Convolution Theorem:

$$[\text{supp } \tilde A*\tilde\gamma]=[\text{supp } \tilde A]+[\text{supp } \tilde\gamma]\tag{24}$$

where $[X]$ is the *convex hull* of the set X.

Now the **Spectral Incluzion** reads

$$[\text{supp } \tilde A]+[\text{supp } \tilde\gamma]\subset\text{supp } \tilde\gamma\tag{25}$$

Corollary I. $[\text{supp } \tilde A]=\{0\}\Longrightarrow\tilde A=C\delta(\omega)$.
Hence, $A(t):=a(|\gamma(t)|^2)=C$

Corollary II. CIV $\Longrightarrow\gamma(t)\overline\gamma(t)=$ const. Hence,
$\tilde\gamma*\check{\tilde\gamma}=C_1\delta(\omega)$, where $\check{\tilde\gamma}(\omega):=\tilde\gamma(-\omega)$.

Now **TCT** implies $[\text{supp } \tilde\gamma]+[-\text{supp } \tilde\gamma]=\{0\}$. Hence,

$$[\text{supp } \tilde\gamma]=\{\omega_+\}\Longrightarrow\tilde\gamma=C\delta(\omega-\omega_+)$$ ■

3 Physical motivations

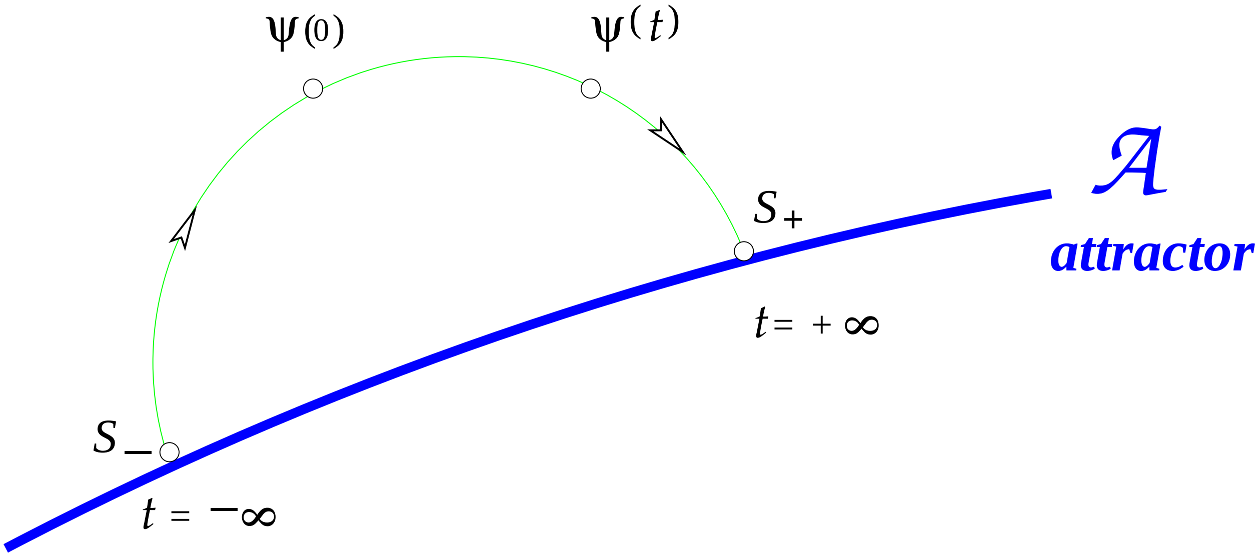
1) N. Bohr in 1913 postulate the *Quantum Transitions*

$$(\mathbf{QT}) \qquad |E_-\rangle \mapsto |E_+\rangle$$

$|E_\pm\rangle$ are *Quantum Stationary Orbits*

Possible *mathematical interpretation* of the posulate: Long-time asymptotics

$$(\mathbf{A}) \qquad \psi(x,t) \sim S_\pm(x), \qquad t \to \pm\infty$$



For **dissipative systems** convergence $(\mathbf{A})$ is known since 1975 **for all finite energy solutions** to Navier-Stokes, reaction-diffusion, nonlinear parabolic eqns: Foias, Temam, Henry, Hale, Babin, Vishik, Chepyzhov, and others.

i) In bounded regions; **ii)** In **global energy norm**; **iii)** For $t \to +\infty$.

2) Schrödinger in 1926 identified the quantum stationary orbits with waves of type

$$|E\rangle = \phi(x)e^{i\omega t}, \qquad \omega = E/h$$

Then asymptotics $(\mathbf{QT})$ reads,

$$(\mathbf{AS}) \qquad \psi(x,t) \sim \phi_\pm(x)e^{i\omega_\pm t}, \qquad t \to \pm\infty$$

3) Nonlinear coupled Maxwell-Schrödinger eqns appears in the first Schrödinger papers 1926:

$$[i\partial_t - V(x,t) - V_{\rm ext}(x)]\psi = [-i\nabla - \mathbf{A}(x,t) - \mathbf{A}_{\rm ext}(x)]^2\psi$$

$$\Box V(x,t) = \rho = |\psi(x,t)|^2, \qquad x \in \mathbb{R}^3$$

$$\Box \mathbf{A}(x,t) = \mathbf{j} = \frac{1}{m}\mathrm{Im}\,\{\overline{\psi(x,t)}[-i\nabla - \mathbf{A}(x,t) - \mathbf{A}_{\rm ext}(x)]\psi\}$$

4) Nonlinear coupled Maxwell-Dirac Equations were introduced in 1927:

$$\left.\begin{array}{l}\gamma^\mu[i\nabla_\mu - A_\mu(x) - A^{\text{ext}}_\mu(x)]\psi(x) = m\psi(x) \\ \Box A^\mu(x) = \overline{\psi(x)}\gamma^0\gamma^\mu\psi(x)\end{array}\right| \quad x \in \mathbb{R}^4$$

(**A**) reads for the M-S and M-D equations

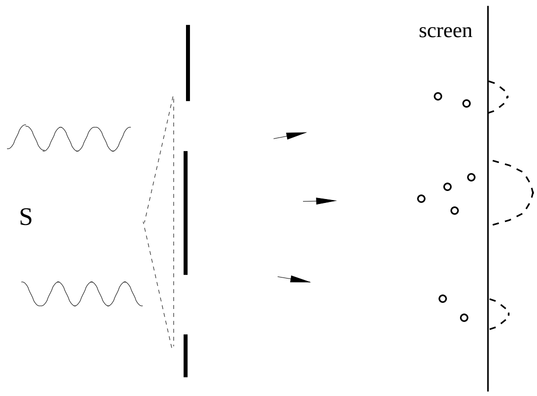
$$(\mathbf{AS'})\ (\psi(\mathbf{x},t), A(\mathbf{x},t)) \sim (e^{-i\omega t}\psi_\pm(\mathbf{x}), A_\pm(\mathbf{x})),\ t \to \pm\infty$$

corresponding to the symmetry *gauge group*

$$(\psi(\mathbf{x},t), A(\mathbf{x},t)) \mapsto (e^{i\theta}\psi(\mathbf{x},t), A(\mathbf{x},t))$$

Open Problem: Proof of (**AS′**) for M-S and M-D.

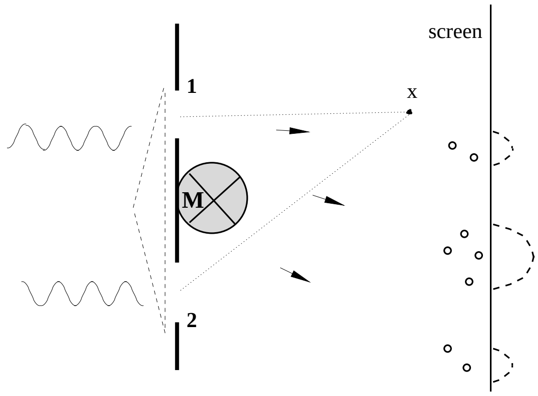
5) Diffraction "double slit" experiment:
Davisson and Germer 1924-1927, Tonomura et al. 1989



Mathematical treatment: long time *soliton asymptotics*

$$(\psi(\mathbf{x},t), A(\mathbf{x},t)) \sim \sum_k (\psi^k_\pm(\mathbf{x} - \mathbf{v}^k_\pm t)e^{i\Phi(\mathbf{v}^k_\pm,\mathbf{x},t)}, A^k_\pm(\mathbf{x} - \mathbf{v}^k_\pm t))t \to \pm\infty. \qquad (26)$$

6) Aharonov-Bohm effect: *shift in magnetic field*



1 [1] A.I.Komech, On attractor of a singular nonlinear U(1)-invariant Klein-Gordon equation, pp 599-611 in: Proc. of the 3rd ISAAC Congress, Freie Universität Berlin, Berlin, 2003.

2 [2] A.I.Komech, A.A.Komech, On global attraction to solitary waves for the Klein-Gordon equation coupled to nonlinear oscillator, C. R., Math., Acad. Sci. Paris 343, Issue 2, 15 July 2006, 111-114.

3 [3] A.I.Komech, A.A.Komech, Global attractor for a nonlinear oscillator coupled to the Klein-Gordon field, Arch. Rat. Mech. Anal. 185 (2007), 105-142.

4 [4] A.I.Komech, A.A.Komech, On global attraction to solitary waves for the Klein-Gordon field coupled to several nonlinear oscillators, *J. des Mathematiques Pures et App.*, **93** (2010), 91-111. arXiv:math/0702660

5 [5] A.I.Komech, A.A.Komech, Global attraction to solitary waves for Klein-Gordon equation with mean field interaction, *Annales de l'IHP-ANL* **26** (2009), no. 3, 855-868. arXiv:math-ph/0711.1131

6 [6] A.I. Komech, A.A. Komech, Global attraction to solitary waves for nonlinear Dirac equation with mean field interaction, *SIAM J. Math. Analysis*, **42** (2010), no. 6, 2944-2964.