On geometric meaning of entropy in symbolic dynamics

S. Komech *
IITP RAS

komech@iitp.ru

The conjecture that Kolmogorov entropy relates to the boundary distortion rate for regions in the phase space of a dynamical system is based on some notes in the book by G. Zaslavsky [1]. Exact result was first established by B.Gurevich [2] for subshift of finite type. In [3] such a relation was established by the author for classical dynamical systems of toral automorphisms. It was shown in [4] that for smooth systems in the general case (diffeomorphism of a compact Riemannian manifold) boundary distortion rate relates to the sum of all the positive Lyapunov exponents. Such a sum, in turn, will coincide with Kolmogorov entropy if invariant measure is a SRB measure.

In this paper we show that the conjecture is true for the wider class of symbolic dynamical systems. Precisely, we establish it, with additional technical assumption, for synchronized systems introduced by F. Blanchard and G. Hansel [5]. Synchronized systems contain all subshifts of finite type and sofic systems introduced by B. Weiss [6].

Let S be homeomorphism of compact metric space X and μ a S-invariant Borel probability measures on X. For every $D \subset X$ and $\varepsilon > 0$, we denote the ε-neighborhood of D by $\mathcal{O}^\varepsilon(D)$. For $r > 0$ we denote by $B(x, r)$ the ball of radius r centered at $x \in X$.

We estimate the boundary distortion rate as asymptotic of fraction

$$\frac{1}{n} \ln \frac{\mu(O^\varepsilon(S^n B(x,\varepsilon)))}{\mu(B(x,\varepsilon))} \tag{1}$$

*This research was partially supported by Russian Foundation for Basic Research RFBR 11-01-00485-a

for $n \to \infty$ and $\varepsilon \to 0$.

It is necessary to assume some relation between n and ε. Otherwise $S^n B(x, \varepsilon)$ could be ε-net for the whole X, in which case the ratio in (1) would be independent of S, since the numerator would be equal 1 for every sufficiently small ε.

Let $X \subseteq A^{\mathbb{Z}}$, A is a finite alphabet, is compact and S-invariant, S is a shift transformation, and $\mu(X) = 1$. The metric on X is defined as follows

$$\rho(x, y) = \theta^{k(x,y)},$$

where $\theta \in (0, 1)$ and

$$k(x, y) = \min\{k \in \mathbb{Z}_+ : x(-k) \neq y(-k) \text{ or } x(k) \neq y(k)\}$$

if $x \neq y$, and $k(x, x) = \infty$.

We recall the definition of synchronized system.

Definition. Let (X, S) be a transitive subshift. A word in the alphabet A is an X-block if it is a subblock in some $x \in X$. The pair (X, S) is a synchronized system if there exists an X-block w (a 'magic' word) such that if uw and wv (and hence u, v) are X-blocks, then uwv is also an X-block.

The following Theorem is the main result of this paper.

Theorem. *Let (X, S) be a synchronized system with a S-invariant ergodic probability measure μ and let a function $n : \mathbb{R}^+ \to \mathbb{Z}^+$ satisfies*

$$\lim_{\varepsilon \to 0} (|\ln \varepsilon| - n(\varepsilon)) \to \infty \ .$$

We assume there exists a 'magic' word w such that

$$\mu(\{y \in X : y_1, \dots, y_{|w|} = w\}) > 0 \ ,$$

where $|w|$ is a length of the word w. Then

$$\lim_{\varepsilon \to 0} \frac{1}{n(\varepsilon)} \ln \frac{\mu(\mathcal{O}^{\varepsilon}(S^{n(\varepsilon)}(B(x, \varepsilon) \cap X)))}{\mu(B(x, \varepsilon))} = h_\mu(T) \ ,$$

the convergence holds in L_1.

References

[1] G. M. Zaslavsky, *Chaos in dynamic systems*, Harwood, Chur, 1985.

[2] B. M. Gurevich, Geometric Interpretation of Entropy for Random Processes., *AMS Transl.*, Vol 171, 1996, pp. 81–87.

[3] S. Komech, Entropy and the boundary distortion rate for hyperbolic systems,*Conference of young scientists, MSU "Lomonosov 2005" Proceedings*, pp.76–80.

[4] B. M. Gurevich, S. A. Komech, Lyapunov exponents and evolution of boundaries in the phase space of a hyperbolic dynamical system, *Proceedings of Topology, Geometry, and Dynamics: Rokhlin Memorial Conference*, 2010, pp.41–43, Saint Petersburg, Russia.

[5] F. Blanchard and G. Hansel, Syste'mes code's, Theoret. Comput. Sci. 44, 1986, pp. 17–49.

[6] B. Weiss, Subshifts of finite type and sofic systems, Monats. Math. 77, 1973, pp. 462–474.