Stabilization of statistical solutions for Dirac equations

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Abstract

We consider the Dirac equation in $\mathbb{R}^3$ with a potential, and study the distribution μ_t of the random solution at time $t \in \mathbb{R}$. The initial measure μ_0 has zero mean, a translation-invariant covariance, and a finite mean charge density. We also assume that μ_0 satisfies a mixing condition of Rosenblatt- or Ibragimov-Linnik-type mixing condition. The main result is the convergence of μ_t to a Gaussian measure as $t \to \infty$ which gives the Central Limit Theorem for the Dirac equation.

1 Introduction

We consider the Dirac equation in $\mathbb{R}^3$:

$$\left\{ \begin{array}{l} i\dot\psi(x,t) = H\psi(x,t) := [-i\alpha\cdot\nabla + \beta m + V]\psi(x,t) \\ \psi(x,0) = \psi_0(x) \end{array} \right| \quad x \in \mathbb{R}^3 \tag{1.1}$$

where $m > 0$, β and α_k, $k = 1,2,3$ are hermitian matrices satisfying the following relations:

$$\left\{ \begin{array}{ll} \alpha_k^* = \alpha_k, & \beta^* = \beta, \\ \alpha_k\alpha_l + \alpha_l\alpha_k = 2\delta_{kl}I, & \alpha_k\beta + \beta\alpha_k = 0. \end{array} \right.$$

The standard form of the Dirac matrices α_k and β (in 2×2 blocks) is

$$\beta = \begin{pmatrix} I & 0 \\ 0 & -I \end{pmatrix}, \quad \alpha_k = \begin{pmatrix} 0 & \sigma_k \\ \sigma_k & 0 \end{pmatrix} \quad (k = 1,2,3), \tag{1.2}$$

where I denotes the unit matrix, and

$$\sigma_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \sigma_2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \quad \sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}. \tag{1.3}$$

We assume the following conditions:
E1. The potential $V \in C^\infty(\mathbb{R}^3)$ is a hermitian 4×4 matrix function such that for some $\rho > 5$.

$$|\partial^\alpha V(x)| \le C(\alpha)\langle x\rangle^{-\rho-|\alpha|}, \quad \langle x\rangle^\sigma = (1+|x|^2)^{\sigma/2} \tag{1.4}$$

E2. The operator H presents neither resonance nor eigenvalue at thresholds.
Denote by P_c the projection onto the continuous spectral space of H.

We fix an arbitrary $\delta > 0$ such that $5 + \delta < \rho$ and consider the solutions $\psi(x,t) \in \mathbb{C}^4$ with initial data $\psi_0(x)$ which are supposed a random element of the complex functional space

$\mathcal{H} = P_c L^2_{-5/2-\delta}$. The distribution of ψ_0 is a Borel probability measure μ_0 on $\mathcal{H}$ with zero mean satisfying some additional assumptions, see Conditions **S1-S3** below. Denote by μ_t, $t \in \mathbb{R}$, a Borel probability measure on $\mathcal{H}$, giving the distribution of the random solution $\psi(t)$ to problem (1.1). The correlation functions of the initial measure are supposed to be translation-invariant:

$$Q_0(x,y) := E\Big(\psi_0(x) \otimes \psi_0(y)\Big) = q_0(x-y), \;\; x,y \in \mathbb{R}^3. \tag{1.5}$$

We also assume that the initial mean charge density is finite:

$$e_0 := E|\psi_0(x)|^2 = \operatorname{tr} q_0(0) < \infty, \quad x \in \mathbb{R}^3. \tag{1.6}$$

Finally, we assume that the measure μ_0 satisfies a mixing condition of Ibragimov-Linnik type, which means that

$$\psi_0(x) \;\; \textit{and} \;\; \psi_0(y) \;\; \textit{are asymptotically independent as} \;\; |x-y| \to \infty. \tag{1.7}$$

Our main result is the (weak) convergence of μ_t to a limiting measure μ_∞,

$$\mu_t \rightharpoondown \mu_\infty, \;\; t \to \infty, \tag{1.8}$$

which is an equilibrium Gaussian measure on $\mathcal{H}$. A similar convergence holds for $t \to -\infty$ since our system is time-reversible.

This paper can be considered as a continuation of papers [3]-[5], [7] which concerns the analysis of the long time convergence to equilibrium distribution for partial differential equations of hyperbolic type.

2 Well posedness

Definition 2.1. *For $s, \gamma \in \mathbb{R}$, let us denote by $H^s_\nu = H^s_\nu(\mathbb{R}^3, \mathbb{C}^4)$ the weighted Sobolev with the finite norms*

$$\|\psi\|_{H^s_\nu} = \|\langle x\rangle^\nu \langle\nabla\rangle^s \psi\|_{L^2} < \infty.$$

We set $L^2_\nu = H^0_\nu$. The finite speed of propagation for equation (1.1) implies

Lemma 2.2. *i) For any $\psi_0 \in L^2_{-\nu}$ with $0 \le \nu \le \rho$ there exists a unique solution $\psi(\cdot,t) \in C(\mathbb{R}, L^2_{-\nu})$ to the Cauchy problem (1.1).*
ii) For any $t \in \mathbb{R}$, the operator $U(t) : \psi_0 \mapsto \psi(\cdot,t)$ is continuous in $L^2_{-\nu}$.

Proof. Fist, consider the solution $\chi(x,t)$ of the free Dirac equation with $V(x) \equiv 0$. In the Fourier space we have: $\hat\chi(k,t) = e^{i(\alpha\cdot k - \beta m)t}\hat\psi_0(k)$. Then

$$\|\chi(\cdot,t)\|_{L^2_{-\nu}} = C\|\hat\chi(\cdot,t)\|_{H^{-\nu}} \le C_1(t)\|\hat\psi_0(\cdot,t)\|_{H^{-\nu}} \le C_2(t)\|\psi_0(\cdot,t)\|_{L^2_{-\nu}} \tag{2.1}$$

Now we represent the solution of the perturbed equation (1.1) as $\psi(t) = \chi(t) + \phi(t)$, where

$$\dot\phi(t) = H\phi(t) + V\chi(t), \quad \phi(0) = 0.$$

Applying the Duhamel representation, we obtain

$$\phi(x,t) = \int_0^t U(t-\tau)V\chi(\tau)d\tau.$$

It remains to prove that $\phi(\cdot,t) \in L^2_{-\nu}$. By charge conservation for the Dirac equation we have

$$\|U(t-s)V\chi(s)\|_{L^2_{-\nu}} \le \|U(t-s)V\chi(s)\|_{L^2} = \|V\chi(s)\|_{L^2} \le C\|\chi(s)\|_{L^2_{-\rho}} \le C\|\chi(s)\|_{L^2_{-\nu}} < \infty$$

□

3 Random solution

Let (Ω, Σ, P) be a probability space with expectation E and $\mathcal{B}(\mathcal{H})$ denote the Borel σ-algebra in $\mathcal{H}$. We assume that $\psi_0 = \psi_0(\omega, \cdot)$ in (1.1) is a measurable random function with values in $(\mathcal{H}, \mathcal{B}(\mathcal{H}))$. In other words, $(\omega, x) \mapsto \psi_0(\omega, x)$ is a measurable map $\Omega \times \mathbb{R}^3 \to \mathbb{C}^4$ with respect to the (completed) σ-algebras $\Sigma \times \mathcal{B}(\mathbb{R}^3)$ and $\mathcal{B}(\mathbb{C}^4)$. Then, owing to Lemma 2.2, $\psi(t) = U(t)\psi_0$ is again a measurable random function with values in $(\mathcal{H}, \mathcal{B}(\mathcal{H}))$. We denote by $\mu_0(d\psi_0)$ a Borel probability measure in $\mathcal{H}$ giving the distribution of the random function ψ_0. Without loss of generality, we assume $(\Omega, \Sigma, P) = (\mathcal{H}, \mathcal{B}(\mathcal{H}), \mu_,)$ and $\psi_0(\omega, x) = \omega(x)$ for $\mu_0(d\omega) \times dx$-almost all $(\omega, x) \in \mathcal{H} \times \mathbb{R}^3$.

Definition 3.1. *μ_t is a probability measure on $\mathcal{H}$ which gives the distribution of $\psi(t)$:*

$$\mu_t(B) = \mu_0(U(-t)B), \quad \forall B \in \mathcal{B}(\mathcal{H}), \ \ t \ge 0. \tag{3.1}$$

Our main goal is to derive the weak convergence of the measures μ_t in the Hilbert space $H^{-\varepsilon}_{-\nu}$ for any $\varepsilon > 0$, and $\nu > 5/2 + \delta$

$$\mu_t \xrightarrow{H^{-\varepsilon}_{-\nu}} \mu_\infty \ \text{ as } \ t \to \infty, \tag{3.2}$$

where μ_∞ is a Borel probability measure in the space $H^{-\varepsilon}_{-\nu}$. By definition, this means the convergence

$$\int f(\psi)\mu_t(d\psi) \to \int f(\psi)\mu_\infty(d\psi) \ \text{ as } \ t \to \infty. \tag{3.3}$$

for any bounded and continuous functional $f(\psi)$ in $H^{-\varepsilon}_{-\nu}$.

Set $\mathcal{R}\psi \equiv (\operatorname{Re}\psi, \operatorname{Im}\psi) = \{\operatorname{Re}\psi_1, \dots, \operatorname{Re}\psi_4, \operatorname{Im}\psi_1, \dots, \operatorname{Im}\psi_4\}$ for $\psi = (\psi_1, \dots \psi_4) \in \mathbb{C}^4$ and denote by $\mathcal{R}^j\psi$ the j-th component of the vector $\mathcal{R}\psi$, $j = 1, ..., 8$. The brackets $(\cdot, \cdot)$ mean the inner product in the real Hilbert spaces $L^2 \equiv L^2(\mathbb{R}^3)$, in $L^2 \otimes \mathbb{R}^N$, or in some their different extensions. For $\psi(x), \phi(x) \in L^2(\mathbb{R}^3, \mathbb{C}^4)$, write

$$\langle \psi, \phi \rangle := (\mathcal{R}\psi, \mathcal{R}\phi) = \sum_{j=1}^{8} (\mathcal{R}^j\psi, \mathcal{R}^j\phi). \tag{3.4}$$

Definition 3.2. *The correlation functions of the measure* μ_0 *are defined by*

$$Q_0^{ij}(x,y) \equiv E\Big(\mathcal{R}^i\psi_0(x)\mathcal{R}^j\psi_0(y)\Big) \quad \text{for almost all } \ x,y \in \mathbb{R}^3, \quad i,j=1,...,8, \tag{3.5}$$

Denote by D the space of complex- valued functions in $C_0^\infty(\mathbb{R}^3)$ and write $\mathcal{D} := [D]^4$. For a Borel probability measure μ on $\mathcal{H}$ denote by $\hat{\mu}$ the characteristic functional (the Fourier transform)

$$\hat{\mu}(\phi) \equiv \int \exp(i\langle\psi,\phi\rangle)\,\mu(d\psi), \quad \phi \in \mathcal{D}.$$

A measure μ is said to be Gaussian (with zero expectation) if its characteristic functional is of the form

$$\hat{\mu}(\phi) = \exp\{-\frac{1}{2}\mathcal{Q}(\phi,\phi)\}, \quad \phi \in \mathcal{D},$$

where $\mathcal{Q}$ is a real nonnegative quadratic form on $\mathcal{D}$. A measure μ is said to be translation-invariant if

$$\mu(T_hB) = \mu(B), \quad B \in \mathcal{B}(\mathcal{H}), \quad h \in \mathbb{R}^3,$$

where $T_h\psi(x) = \psi(x-h)$, $x \in \mathbb{R}^3$.

4 Mixing condition

Let $O(r)$ denote the set of all pairs of open bounded subsets $\mathcal{A},\mathcal{B} \subset \mathbb{R}^3$ at distance $\mathrm{dist}(\mathcal{A},\,\mathcal{B}) \geq r$ and $\sigma(\mathcal{A})$ the σ-algebra in $\mathcal{H}$ generated by the linear functionals $\psi \mapsto \langle\psi,\phi\rangle$, where $\phi \in \mathcal{D}$ with $\mathrm{supp}\,\phi \subset \mathcal{A}$. Define the Ibragimov-Linnik mixing coefficient of a probability measure μ_0 on $\mathcal{H}$ by (cf. [6, Def. 17.2.2])

$$\varphi(r) \equiv \sup_{(\mathcal{A},\mathcal{B})\in O(r)} \ \sup_{\substack{A \in \sigma(\mathcal{A}),\, B \in \sigma(\mathcal{B}) \\ \mu_0(B) > 0}} \frac{|\mu_0(A\cap B) - \mu_0(A)\mu_0(B)|}{\mu_0(B)}. \tag{4.1}$$

Definition 4.1. *The measure* μ_0 *satisfies the strong, uniform Ibragimov-Linnik mixing condition if*

$$\varphi(r) \to 0 \quad \text{as} \quad r \to \infty. \tag{4.2}$$

Below, we specify the rate of decay of φ (see Condition **S3**).

5 Main assumptions

We assume that measure μ_0 has the following properties:

S0 μ_0 has zero expectation value,

$$E\psi_0(x) \equiv 0, \quad x \in \mathbb{R}^3.$$

. **S1** μ_0 has translation invariant correlation functions,

$$Q_0^{ij}(x,y) \equiv E\Big(\mathcal{R}^i\psi_0(x)\mathcal{R}^j\psi_0(y)\Big) = q_0^{ij}(x-y), \quad i,j=1,...,8 \tag{5.1}$$

for almost all $x, y \in \mathbb{R}^3$.
S2 μ_0 has a finite mean charge density, i.e. Eqn (1.6) holds.
S3 μ_0 satisfies the strong uniform Ibragimov-Linnik mixing condition, with

$$\int_0^\infty r^2 \varphi^{1/2}(r) dr < \infty. \tag{5.2}$$

6 Convergence to equilibrium distribution

Introduce the following 8×8 real valued matrices (in 4×4 blocks)

$$\Lambda_1 = \begin{pmatrix} \alpha_1 & 0 \\ 0 & \alpha_1 \end{pmatrix}, \quad \Lambda_2 = \begin{pmatrix} 0 & i\alpha_2 \\ -i\alpha_2 & 0 \end{pmatrix}, \quad \Lambda_3 = \begin{pmatrix} \alpha_3 & 0 \\ 0 & \alpha_3 \end{pmatrix}, \quad \Lambda_0 = \begin{pmatrix} 0 & -\beta \\ \beta & 0 \end{pmatrix}. \tag{6.1}$$

Denote

$$\Lambda = (\Lambda_1, \Lambda_2, \Lambda_3), \quad P = \Lambda \cdot \nabla + m\Lambda_0. \tag{6.2}$$

For almost all $x, y \in \mathbb{R}^3$, introduce the correlation matrix

$$Q_\infty(x, y) \equiv \Big(Q_\infty^{ij}(x, y)\Big)_{i,j=1,\ldots,8} = \Big(q_\infty^{ij}(x - y)\Big)_{i,j=1,\ldots,8}. \tag{6.3}$$

Here

$$q_\infty(z) = \frac{1}{2} q_0(z) + \frac{1}{2} \mathcal{P} * P q_0(z) P, \tag{6.4}$$

where $\mathcal{P}(z) = e^{-m|z|}/(4\pi|z|)$ is the fundamental solution for the operator $-\Delta + m^2$, and $*$ stands for the convolution of distributions. Our main result is the following:

Theorem 6.1. *Let $m > 0$, and* **E1–E2**, **S0–S3** *hold. Then*
i) the convergence in (3.2) holds for any $\varepsilon > 0$ and $\nu > 5/2 + \delta$.
ii) the limiting measure μ_∞ is a Gaussian equilibrium measure on $\mathcal{H}$.
iii) the limiting characteristic functional of μ_∞ is of the form

$$\hat{\mu}_\infty(\phi) = \exp\{-\frac{1}{2}\mathcal{Q}_\infty(W\phi, W\phi)\}, \quad \phi \in \mathcal{D},$$

where $W : \mathcal{D} \to L^2$ is a wave operator.

Theorem 6.1 can be derived from Propositions 6.2-6.3 below by using the same arguments as in [10, Theorem XII.5.2].

Proposition 6.2. *The family of the measures $\{\mu_t, t \in \mathbb{R}\}$ is weakly compact in $H_{-\nu}^{-\varepsilon}$ for any $\varepsilon > 0$ and $\nu > 5/2 + \delta$.*

Proposition 6.3. *For any $\phi \in \mathcal{D}$*

$$\hat{\mu}_t(\phi) \equiv \int \exp(i\langle\psi, \phi\rangle)\mu_t(d\psi) \to \exp\{-\frac{1}{2}\mathcal{Q}_\infty(W\phi, W\phi)\}, \quad t \to \infty. \tag{6.5}$$

Proposition 6.2 provides the existence of the limiting measures of the family μ_t, and Proposition 6.3 provides the uniqueness of the limiting measure, and hence the convergence (3.3).

The similar result for the free Dirac equation with $V(x) \equiv 0$ has been proved in [5]. The case of the perturbed equation requires new constructions due to the absence an explicit formula for the solution. To reduce the case of perturbed equation to the case of free equation we formally need a scattering theory for the solutions of infinite global charge. We construct the dual scattering theory for the finite energy solutions to avoid the infinite charge scattering theory. This version of scattering theory is based on the weighted time - decay established in [1].

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