How heavy-tailed distributions appear in different models in probability

Dmitry Korshunov
Sobolev Institute of Mathematics

In this talk we discuss three interesting probabilistic models where light-tailed input distributions generate heavy-tailed output. We say that a distribution F is light-tailed if it possesses some positive exponential moment. The distribution F is called heavy-tailed if all positive exponential moments are infinite.

The first model is about Lamperti problem for Markov chains with asymptotically zero drift. Let X_n, $n = 0, 1, 2, \ldots$, be a time homogeneous Markov chain on $\mathbb{Z}^+$ of Lamperti type, that is, with asymptotically zero drift, so that $m(i) := \mathbb{E}\{X_1 - X_0 | X_0 = i\} \to 0$ as $i \to \infty$. We assume that $m(i)$ is ultimately negative and such that X_n is stable with stationary distribution π. The problem is to describe asymptotic behaviour of $\pi(i)$ as $i \to \infty$. It is known from [2] that for asymptotically zero drifted Markov chain one should expect heavy-tailed π even if X has bounded jumps, in contrast to more classical case of asymptotically negative drift where the stationary distribution is usually light-tailed.

We propose a novel method of investigating such a Markov chains. The main idea is to apply a suitable non-exponential change of measure and to study limit distributional properties of the process obtained which drifts to infinity. Then the inverse transformation allows to compute the required approximation for the large deviation probabilities of π. In this way we prove that, if $m(i) \sim -\mu/x$, $\mathbb{E}\{(X_1 - X_0)^2 | X_0 = i\} \to b > 0$ as $i \to \infty$, and $\mu > b/2$, then $\pi(i) \sim c/i^{2\mu/b}$, $c > 0$. Also, in the note [1] we found what moments of π exist and what are infinite.

The second model is Gaussian chaos, that is, a polynomial of standard normal variables. The simplest case is given by a product of components of normal random vector with general covariance matrix. The distribution of Gaussian chaos is usually heavy-tailed. The problem is (see e.g [3, 4, 5]) how to compute the tail behaviour given (i) the polynomial for Gaussian chaos and (ii) covariance matrix in the case of product.

The third model is the following perpetuity:

$$e^{S_1} + e^{S_2} + e^{S_3} + \dots,$$

where $S_{n+1} = S_n + f(X_n, \xi_{n+1})$ is a Markov modulated random walk, that is, X_n is an ergodic Markov chain, and ξ_n are independent identically distributed random variables. Assume S_n drifts to minus infinity, so that the perpetuity is well defined. The distribution of the perpetuity is always heavy-tailed. We explain the tail behaviour of this perpetuity under general conditions on distributions of X and ξ.

References

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