Comparative renormalization group analysis of the Euclidean and hierarchical models

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Bosonic hierarchical models were introduced in mathematical physics by F. Dyson [8, 5]. In this talk it is convenient to present the formal d-dimensional hierarchical lattice as the lattice of purely fractional d-dimensional p-adic vectors. Every p–adic number $x \in Q_p^d$ can be represented in the form $x = c_{-n}p^{-n} + \cdots + c_{-1}p^{-1} + c_0 + c_1 p + \ldots$, where the coefficients c are integer numbers from 0 to $(p-1)$ and $c_n > 0$, n is some integer number. Then $|x|_p = p^n$ and the fractional part of x is defined as $\{x\} = c_{-n}p^{-n} + \cdots + c_{-1}p^{-1}$. For $x = (x_1, \ldots, x_d) \in Q_p^d$, we set $|x|_p = \max_i |x_i|_p, \qquad \{x\} = (\{x_1\}, \ldots, \{x_d\})$. The discrete set $T_p^d = \{x \in Q_p^d : \; x = \{x\}\}$ can be viewed as a hierarchical lattice with the elementary cell size $n = p^d$ and with the hierarchical distance $d(i,j) = |i-j|_p, \; i, j \in T_p^d$ (see[1]).

Let ξ be a field on the lattice T_p^d. The hierarchical renormalization block-spin transformation is given as

$$r(\alpha)\xi(j) = p^{-\alpha/2} \sum_{l \in T_p^d : |pl-j|_p \le p} \xi(l),$$

α is real-valued renormalization group parameter.

To stress a close similarity between Euclidean and hierarchical models we shall use momentum space representation. We define field $\sigma(k)$ in the unit ball of d-dimensional p-adic space $\{k : |k|_p \le 1\}$ by Fourier series

$$\sigma(k) = \sum_{j \in T_p^d} \exp\{-2\pi i (j,k)\}\xi(j),$$

where $(j,k) = j_1k_1 + \ldots + j_dk_d$. It is easy to find the action of RG - transformation $r(\alpha)$ in terms of the field σ: $r(\alpha)\sigma(k) = p^{-\alpha/2}\sigma(kp)\chi(k)$. Here $\chi(k)$ denotes characteristic function of the unit d-dimensional p-adic ball.

From this point we shall describe the hierarchical and Euclidean models in the similar way. Let $\Omega = \{k : |k| \le 1\}$ denotes the unit ball in the d-dimensional Euclidean or p-adic space Q_p^d (we'll omit index p in the notation of p-adic norm).

Let $\sigma(k)$ be a field defined on this ball, which will denote in this paper bosonic (complex-valued) field or 4-component fermionic field $\sigma(k) = (\overline{\sigma_1}(k), \sigma_1(k), \overline{\sigma_2}(k), \sigma_2(k))$, where the components are generators of the Grassmann algebra.The Wilson's renormalization group transformation (RG) in terms of realizations of the field $\sigma(k)$ is defined as $r_\lambda(\alpha)\sigma(k) = |\lambda|^{-\alpha/2}\sigma(k/\lambda)\chi(k)$, where λ is real or p-adic number, $\chi(k)$ is the characteristic function of the ball Ω. Then Gaussian fields with binary correlation function $\langle\sigma(k_1)\sigma(k_2)\rangle = \delta(k_1+k_2)|k_1|^{d-\alpha}\chi(k_1)$ in the bosonic case and $\langle\sigma_i(k_1)\overline{\sigma_j}(k_2)\rangle = \delta(k_1+k_2)\delta_{ij}|k_1|^{d-\alpha}\chi(k_1)$, $\langle\overline{\sigma_i}(k_1)\overline{\sigma_j}(k_2)\rangle = \langle\sigma_i(k_1)\sigma_j(k_2)\rangle = 0, \quad i,j = 1,2$ in the fermionic one are invariant under transformation $r_\lambda(\alpha)$.

In the Gibbsian form these Gaussian fields are described by the Hamiltonians

$$H_0(\sigma;\alpha) = \frac{1}{2}\int_{\Omega^2} \delta(k_1+k_2)|k_1|^{\alpha-d}\sigma(k_1)\sigma(k_2)\, dk_1 dk_2$$

in the bosonic case and

$$H_0(\sigma;\alpha) = \int\limits_{\Omega^2} \delta(k_1+k_2)|k_1|^{\alpha-d}(\overline{\sigma_1}(k_1)\sigma_1(k_2) + \overline{\sigma_2}(k_1)\sigma_2(k_2))\, dk_1 dk_2$$

in the fermionic one, dk denotes the Haar measure in the p-adic case.

Let us consider non-Gaussian Hamiltonians $H_0(\sigma;\alpha) + H(\sigma)$, where

$$H(\sigma) = \sum_{m=1}^{\infty} \int\limits_{\Omega^m} h_m(k_1,\ldots,k_m)\delta(k_1+\ldots+k_m)\sigma^*(k_1)\ldots\sigma^*(k_m)\, dk_1\ldots dk_m.$$

Here $\sigma^*(k)$ denotes $\sigma(k)$ in the bosonic case and any one of the components $\overline{\sigma_1}(k), \sigma_1(k)$, $\overline{\sigma_2}(k), \sigma_2(k)$ in the fermionic case. Then RG-transformation $r_\lambda(\alpha)$ induces RG-transformation in the space of the Hamiltonians , which transforms the Hamiltonian $H_0(\sigma;\alpha) + H$ to the Hamiltonian $H_0(\sigma;\alpha) + R_\lambda(\alpha)H$, where $R_\lambda(\alpha)H = P_\lambda(\alpha)S_\lambda(\alpha)H$. Scaling operator $S_\lambda(\alpha)$ transforms the coefficient functions of the Hamiltonian H:

$$h_m(k_1,\ldots,k_m) \to |\lambda|^{\frac{m\alpha}{2}-md+d} h_m\left(k_1/\lambda,\cdots,k_m/\lambda\right).$$

Operator $P_\lambda(\alpha)$ is induced by the restriction of the field $\sigma(k)$ from the ball $\lambda\Omega$ onto the ball Ω: $(P_\lambda(\alpha)H)(\sigma) = -\ln\langle\exp\{-H(\sigma+\eta)\}\rangle_{\mu(d\eta)}$. Here the averaging is taken over the Gaussian field in the in the annulus $\lambda\Omega\setminus\Omega$ with zero mean and binary correlation function $\langle\eta(k_1)\eta(k_2)\rangle = \delta(k_1+k_2)|k_1|^{d-\alpha}\left(\chi(k_1/\lambda) - \chi(k_1)\right)$ in the bosonic case and $\langle\eta_i(k_1)\overline{\eta_j}(k_2)\rangle = \delta(k_1+k_2)\delta_{ij}|k_1|^{d-\alpha}\left(\chi(k_1/\lambda) - \chi(k_1)\right)$ in the fermionic one.

Let us define the functional Fourier transformation:

$$(FH)(\theta) = -\ln\frac{\int\exp\{(\theta,\sigma) - H(\sigma)\}D\sigma}{\int\exp\{-H(\sigma)\}D\sigma}. \tag{1}$$

Here $\theta(k)$ denotes the dual field on the ball Ω, the integral is treated as the formal functional integral over the field σ, the form (θ,σ) denotes

$$(\theta,\sigma) = i\int\theta(k)\sigma(-k)\, dk,$$

in the bosonic case and

$$(\theta,\sigma) = \int(\overline{\theta_1}(k)\sigma(-k) + \theta_1(k)\overline{\sigma_1}(-k) + \overline{\theta_2}(k)\sigma_2(-k) + \theta_2(k)\overline{\sigma_2}(-k)\, dk. \tag{2}$$

in the fermionic case.

Theorem 1 *There is formal (in the terms of functional integrals) commutative relation between the transformations $R(\alpha)$ and F:*

$$R_\lambda(\alpha)F = FR_\lambda(2d-\alpha). \tag{3}$$

In the p-adic case this relation can be derived rigorously.

As was shown in [6, 7] for the Euclidean bosonic case the non-Gaussian branch of fixed points can be constructed in the form of ε-expansion in the neighborhood of the Gaussian fixed point, where ε is a deviation of the parameter α from its bifurcation value

$3d/2$. In the p-adic bosonic case the same construction was realized in [11]. We seek this fixed point as the Hamiltonian of the projection of the φ^4-theory in the whole space onto the ball: $P(\alpha)(H(\sigma; u, v)) = -\ln\langle\exp\{-H(\sigma + \eta; u, v)\}\rangle_{\mu(d\eta)}$, where σ is defined in the ball Ω, the averaging is taken over the Gaussian field η with binary correlation function $\langle\eta(k_1)\eta(k_2)\rangle = \delta(k_1 + k_2)|k_1|^{d-\alpha}(1 - \chi(k_1))$ in the bosonic case and $\langle\eta_i(k_1)\overline{\eta_j}(k_2)\rangle = \delta(k_1 + k_2)\delta_{ij}|k_1|^{d-\alpha}(1 - \chi(k_1))$ in the fermionic one. Here $H(\sigma; u, v) = uH_2(\sigma) + vH_4(\sigma)$, where

$$H_2(\sigma) = \int \delta(k_1 + k_2)\sigma(k_1)\sigma(k_2)\, dk_1 dk_2,$$

$$H_4(\sigma) = \int \delta(k_1 + \ldots + k_4)\sigma(k_1)\ldots\sigma(k_4)\, dk_1 \ldots dk_4$$

in the bosonic case and

$$H_2(\sigma) = \int \delta(k_1 + k_2)(\overline{\sigma_1}(k_1)\sigma_1(k_2) + \overline{\sigma_2}(k_1)\sigma_2(k_2))\, dk_1 dk_2,$$

$$H_4(\sigma) = \int \delta(k_1 + \ldots + k_4)\overline{\sigma_1}(k_1)\sigma_1(k_2)\overline{\sigma_2}(k_3)\sigma_2(k_4)\, dk_1 \ldots dk_4$$

in the fermionic case.

The scaling transformation $S_\lambda(\alpha)$ acts linearly on the Hamiltonians $H(\sigma; u, v)$ in the whole space:$S_\lambda(\alpha)H(\sigma; u, v) = H(\sigma; |\lambda|^{\alpha-d}u, |\lambda|^{2\alpha-3d}v)$. The linear transformation $(u, v) \to (|\lambda|^{\alpha-d}u, |\lambda|^{2\alpha-3d}v)$ in the coupling–constant plane (u, v) we also denote by $S_\lambda(\alpha)$. It is easy to see that in the space of the Hamiltonians $H(\sigma; u, v)$ we have relation

$$R_\lambda(\alpha)P(\alpha) = P(\alpha)S_\lambda(\alpha). \tag{4}$$

On expanding of projection Hamiltonian$P(\alpha)H(\sigma; u, v)$ in Feynman diagrams some coefficient functions have a poles in $\varepsilon = \alpha - \frac{3}{2}d$ at zero [9, 11]. In that problem the operation of analytic renormalization $A.R.$ is most natural regularization procedure [11]. Let $H(\sigma; u, v)$ denotes bosonic(or fermionic) Euclidean (or p-adic) Hamiltonian.

Theorem 2 *There are non-trivial formal series $u(\varepsilon)$ and $v(\varepsilon)$ such that the Hamiltonian*

$$H_0(\sigma; \alpha) + A.R.P(H(\sigma; u(\varepsilon), v(\varepsilon)) \tag{5}$$

is invariant under RG-transformation.

Using this formalism, we have obtained [13]the following expansion for the critical exponent ν in the bosonic Euclidean and p-adic models: $\nu^{-1} = d/2 + (1/3)\varepsilon + (8/9)A(d)\varepsilon^2 + O(\varepsilon^3)$. In the Euclidean case $A(d) = -\frac{1}{2}\left(-\gamma - 2\psi(d/4) + \psi(d/2)\right)$, $\quad \psi(x) = \Gamma'(x)/\Gamma(x)$, where $\Gamma(x)$ is Euler gamma-function,γ is Euler constant. In the p-adic case $A(d) = A_p(d) = -\frac{1}{2}\left(\ln p - 2\psi_p(d/4) + \psi_p(d/2)\right)$, where $\psi_p(x) = f_p'(x)/f_p(x)$, $\quad f_p(x) = \left(1 - p^{-2x}\right)^{-1}$. Using that

$$-\gamma = \lim_{x\to 0}\left(\frac{\Gamma(x)}{\mathrm{Res}\ \Gamma(0)} - \frac{1}{x}\right), \qquad \ln p = \lim_{x\to 0}\left(\frac{f_p(x)}{\mathrm{Res}\ f_p(0)} - \frac{1}{x}\right),$$

we see an interesting similarity of Euclidean and p-adic ε-expansions up to a second order. Relation between $\Gamma(x)$ and p-adic analog of gamma-function $f_p(x)$ is well known in the theory of Riemann Zeta-function.

It turns out that in the fermionic hierarchical (p-adic) case all constructions can be performed explicitly (see[10, 2, 3, 12]).

One can snow that in the fermionic hierarchical case projection Hamiltonian has the same structure: $P(\alpha)H(\sigma;u,v) = H(\sigma;r(u,v),g(u,v))$, where coefficients $r(u,v)$ and $g(u,v)$ are series of (u,v). We also denote the transformation $(u,v) \to (r(u,v),g(u,v))$ by $P(\alpha)$. The action of RG-transformation $R_\lambda(\alpha)$ on the Hamiltonian $H(\sigma;r,g)$ reduces to the transformation of the coupling constants. Particularly, for $\lambda = p^{-1}$

$$R_{p^{-1}}(\alpha)H(\sigma;r,g) = H(\sigma;r',g'),$$

where

$$r' = p^{\alpha-d}\left(\frac{(r+1)^2-g}{(r+1)^2-g/p^d}(r+1)-1\right)$$

$$g' = p^{2\alpha-3d}\left(\frac{(r+1)^2-g}{(r+1)^2-g/p^d}\right)^2 g.$$

We denote the transformation $(r,g) \to (r',g')$ by $R(\alpha)$. From (4) it follows $R(\alpha)P(\alpha) = P(\alpha)S(\alpha)$, where $S(\alpha) = S_{p^{-1}}(\alpha)$. The mapping $S(\alpha)$ is given by the diagonal matrix whose eigenvalues are the eigenvalues of the differential of $R(\alpha)$ at the trivial(Gaussian) fixed point $r = 0, g = 0$. Hence we can treat the mapping $P(\alpha)$ as a normalizing transformation to the mapping $R(\alpha)$ at the zero point an can find functional integral $P(\alpha)$ as a solution of the classical two-dimensional functional equation. Particularly, ultrviolet poles of p-adic Feynman amplitudes are interpreted as resonance values of the corresponding normal form. The renormalization procedure can be defined as the mapping inverse to the normal mapping $P(\alpha)$. We can restore coupling constants of the continuum theory from the coupling constants of the discrete model using inverse map $P^{-1}(\alpha)$ for non-resonance values of α. We can prove rigorously [4]that discrete model is well defined for the whole plane of coupling constants and almost all values of α. But we can prove rigorously that continuum model is well defined only in some small neighborhood of trivial (zero) fixed point of renormalization group. In other words, continuum model is related to discrete model as normal form is related to map.

More natural to use RG-transformation in the space of non-normalized Grassmann-valued “densities” of single spin “distribution” $f(\psi^*;c_0,c_1,c_2) = c_0 + c_1(\bar\psi_1\psi_1 + \bar\psi_2\psi_2) + c_2\bar\psi_1\psi_1\bar\psi_2\psi_2$. If $c_0 \neq 0$, we can write the density f in the regular (exponential) form $f(\psi^*;c_0,c_1,c_2) = c_0\exp\{-L(\psi^*;r(c),g(c))\}$, where $c = (c_0,c_1,c_2)$, $r(c) = -c_1/c_0$, $g(c) = (c_1^2 - c_0c_2)/c_0^2$, $L(\psi^*;r,g)) = r(\bar\psi_1\psi_1 + \bar\psi_2\psi_2) + g\bar\psi_1\psi_1\bar\psi_2\psi_2$. If $c_0 = 0$, as, for example, in the case of Grassmann $\delta-$ function $\delta(\psi^*) = \bar\psi_1\psi_1\bar\psi_2\psi_2$, the exponential representation is impossible. RG-transformation in 2-dimensional projective c-space is given as $R(\alpha)c = c'$, where

$$c_0' = n^d(c_2-2c_1+c_0)^{n-2}(c_1-c_0)^2 + \frac{1}{n}(c_0c_2-c_1^2),$$

$$c_1' = \lambda_1 n^2(c_2-2c_1+c_0)^{n-2}\left[(c_1-c_0)(c_2-c_1) + \frac{1}{n}(c_0c_2-c_1^2)\right],$$

$$c_2' = \lambda_1^2 n^2(c_2-2c_1+c_0)^{n-2}\left[(c_2-c_1)^2 + \frac{1}{n}(c_0c_2-c_1^2)\right],$$

$\lambda_1 = p^{\alpha-d}, n = p^d$. If $c_2 - 2c_1 + c_0 \neq 0$, we can omit this factor. The mapping $R(\alpha)$ is correctly defined as the mapping from two-dimensional projective space to itself everywhere except the point $(1,1,1)$ because $R(\alpha)(1,1,1) = (0,0,0)$. In the (r,g) plane this is the point $(-1,0)$, and we call this point the singular point of the RG mapping. Using projective space representation we have described global picture of the renormalization group flow. RG-transformation in c-space has trivial (Gaussian) fixed point (1,0,0), δ-function fixed point (0,0,1), two non-trivial branches of fixed points and cycles of any order. The value $\alpha = d$ is special value. Particularly, when $\alpha \to d$ all non-trivial fixed points and cycles tend to the singular point.

The branch, bifurcating from the trivial (gaussian) fixed point at $\alpha = 3d/2$ is called "plus"-branch and in (r,g)-coordinates is given by the formula

$$r^+(\alpha) = \frac{p^{d/2} - p^{\alpha-d}}{1 - p^{d/2}},$$

$$g^+(\alpha) = p^d \frac{1 - p^{\alpha-3d/2}}{1 - p^{\alpha-d/2}} \left(\frac{1 - p^{\alpha-d}}{1 - p^{d/2}} \right)^2 .$$

For $d/2 < \alpha < 3d/2$ "plus"-branch belongs to the lower half-plane $(r,g) : g < 0$. If $\alpha \to d/2$, then "plus"-fixed point tends to infinity (to the δ-function). Note, that δ-function fixed point corresponds to the "zero"automodel field. When $\alpha < d/2$ "plus"-fixed point belongs to the upper half-plane again. In physical papers usually α is fixed and is equal to $(d+2)$. In that case the Gaussian part of the Hamiltonian is given by the Laplace operator. Physicists consider $(4-d)$ -expansion and try to extrapolate the results of the expansion to the point $d = 3$ (they have a few lower order members of the series with zero convergence radius). If we do the same in the (r,g)-space of the coupling constants of the fermionic hierarchical model, we will see that $d = 4$ is bifurcation value of the parameter d and we can construct $(4-d)$-expansion from the Gaussian fixed point at the dimension $d = 4$. From the explicit formulas for the "plus"-fixed points it follows that $(\alpha - 3/2d)$-expansion and $(4-d)$-expansion describe the same non-Gaussian fixed point at the dimension $d = 3$. We have some arguments [14] that the same is true in the bosonic hierarchical model.

Taking into consideration algebraic similarity of the Euclidean and hierarchical models one can hope that some of the above-mentioned dynamical phenomena occur in the other types of models. It is interesting to know, how to describe bosonic models for $d/2 < \alpha < 3d/2$. Or, how to verify physically interesting conjecture about the equivalence of $(\alpha - 3/2d)$- and $(4-d)$-expansions in the Euclidean case?

Список литературы

[1] *Lerner E.Yu., Missarov M.D* Scalar models p-adic quantum field thery and Dyson's hierarchical model, Theor. Math. Phys. 1989, V.78, 248–257 .

[2] *Missarov M.D.* RG-invariant curves in the femionic hierarchical model, Theor. Math. Phys. 1998, V.114, 323–336.

[3] *Missarov M.D.* Critical phenomena in the femionic hierarchical model, Theor. Math. Phys. 1998, V.117, 471–488.

[4] *Missarov M.D.* Continuous limit in the femionic hierarchical model, Theor. Math. Phys . 1999, V.118, 40–50.

[5] *Ya.G. Sinai* Theory of Phase Transitions: Rigorous Results (Pergamon Press,Oxford, 1982).

[6] *Bleher P.M., Missarov M.D.* The equations of Wilson's RG and analytic renormalization I. General results, Commun. Math. Phys. 1980. V.74, 235–254.

[7] *Bleher P.M., Missarov M.D.* The equations of Wilson's RG and analytic renormalization II. Solution of Wilson's equations, Commun. Math. Phys. 1980. V.74, 255–272.

[8] *Bleher P.M., Sinai Ja.G.* Investigation of the critical point in models of the type of Dyson's hierarchical models, Commun. Math. Phys. 1973. V.33, 23.

[9] *Lerner E.Yu., Missarov M.D.* P-adic Feynman and String Amplitudes, Commun. Math. Phys. 1989. V.121, 35–48.

[10] *Lerner E.Yu., Missarov M.D.* Fixed points of renormalization group in the hierarchical fermionic model, J. Stat. Phys. 1994. V.76, 805–817.

[11] *Missarov M.D.* Renormalization group and renormalization theory in p-adic and adelic scalar models. Dynamical systems and statistical mechanics, ed. Ya.G. Sinai (Adv. Sov. Math. V.3, Amer. Matn. Soc., 1991), 143–161.

[12] *Missarov M.D.* Renormalization group solution of fermionic Dyson model, Asymptotic Combinatorics with Application to Mathematical Physics, V.A.Malyshev and A.M.Vershik (eds.)(Kluwer Academic Publishers, Printed in Netherlands, 2002, 151-166).

[13] *Missarov M.D., Stepanov R.G.* Critical exponents in p -adic φ^4 -model // p -Adic mathematical physics: Proc. 2nd Intern. Conf., Belgrade, 2005. Melville (NY): Amer. Inst. Phys., 2006. P. 129Ц139. (AIP Conf. Proc.; V. 826).

[14] *Missarov M.D., Stepanov R.G.* ϵ-expansions in the Dyson's hierarchical model, Theor. Math. Phys. 2004. V. 139, N 2,268-275.