Positive definite functions and spherical codes

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Abstract

I. J. Schoenberg proved that a function is positive definite in the unit sphere if and only if this function is a nonnegative linear combination of Gegenbauer polynomials. This fact play a crucial role in Delsarte-Kabatiansky-Levenshtein's (DKL) method for finding bounds for the density of sphere packings on spheres and Euclidean spaces.

One of the most excited applications of DKL's method is a solution of the kissing number problem in dimensions 8 and 24. However, 8 and 24 are the only dimensions in which this method gives a precise result. For other dimensions (for instance, three and four) the upper bounds exceed the lower. We have found an extension of the DKL method that allows to solve the kissing number problem (as well as the one-sided kissing number problem) in dimensions three and four.

In this talk we also will discuss the maximal cardinalities of spherical s-distance sets. For $s = 2$ using the so-called polynomial method and DKL's method these cardinalities can be determined for all dimensions $n < 40$. For the case $s = 3, 4$ we are going to discuss several new tight bounds.

Recently, were found extensions of Schoenberg's theorem for multivariate positive-definite functions. Using these extensions and semidefinite programming can be improved some upper bounds for spherical codes.

Let M be a metric space with a distance function τ. A real continuous function $f(t)$ is said to be positive definite (p.d.) in M if for arbitrary points $p_1, \ldots, p_r$ in M, real variables $x_1, \ldots, x_r$, and arbitrary r we have

$$\sum_{i,j=1}^{r} f(t_{ij})\, x_i x_j \geq 0, \quad t_{ij} = \tau(p_i, p_j),$$

or equivalently, the matrix $\big(f(t_{ij})\big) \succeq 0$, where the sign $\succeq 0$ stands for: "is positive semidefinite".

Let $\mathbb{S}^{n-1}$ denote the unit sphere in $\mathbb{R}^n$, and let φ_{ij} denote the angular distance between points p_i, p_j. In 1942 Schoenberg proved that:
$f(\cos\varphi)$ is p.d. in $\mathbb{S}^{n-1}$ if and only if $f(t) = \sum_{k=0}^{\infty} f_k G_k^{(n)}(t)$ with all $f_k \geq 0$.
Here $G_k^{(n)}(t)$ are the Gegenbauer polynomials.

Schoenberg's theorem has been generalized by Bochner to more general spaces. Namely, the following fact holds: *f is p.d. in a 2-point-homogenous space M if and only if $f(t)$ is a nonnegative linear combination of the zonal spherical functions $\Phi_k(t)$.*

Note that the Bochner - Schoenberg theorem is widely used in coding theory and discrete geometry for finding bounds for error-correcting codes, constant weight codes, spherical codes, sphere packings and other packing problems in 2-point-homogeneous spaces.

The talk will be organized as follows:

First we recall definitions of Gegenbauer polynomials and considers DKL's method for spherical codes. Then we discuss applications of DKL's method for the kissing number problem. One of the most excited applications of DKL's method is a solution of the kissing number problem in dimensions 8 and 24. However, 8 and 24 are the only dimensions in which this method gives a precise result. For other dimensions (for instance, three and four) the upper bounds exceed the lower. We have found an extension of the DKL method that allows to solve the kissing number problem (as well as the one-sided kissing number problem) in dimensions three and four.

Next we discuss maximal cardinalities of spherical two-distance sets. Using the so-called polynomial method and DKL's method these cardinalities can be determined for all dimensions $n < 40$. We extend this method for s-distance sets. (These are join works, one with A. Barg and another one with H. Nozaki.)

Next we consider Sylvester's theorem and semidefinite programming (SDP) bounds for codes. DKL's method and its extensions allow to consider the upper bound problem for codes in 2-point-homogeneous spaces as a linear programming problem with perhaps infinitely many variables, which are the distance distribution. We show that using as variables power sums of distances this problem can be considered as a finite semidefinite programming problem. This method allows to improve some linear programming upper bounds.

The last part of the talk discusses an application of the extended Schoenberg's theorem for multivariate Gegenbauer polynomials. This extension derives new positive semidefinite constraints for the distance distribution which can be applied for spherical codes.

References

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