On the Construction of point processes for classical and quantum gases

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Abstract

We propose a new construction of point processes P with a Laplace transform of the form $\exp\big(-L(1-\zeta_f)\big), f \in \mathcal{K}(X)$, where L is a signed modified Laplace functional (s.m.L.f.) in the sense of Mecke [3]. In case that L is of first order P is the unique solution of the equation $\mathcal{C}_P = \mathcal{C}_L \star P$. The case where L is a positive m.L.F. is well known. P then is infinitely divisible with Lévy measure L and describes for example Bosons or Pôlya sum processes. We are mainly interested in the case of signed m.L.f. L which lead to Fermions, Pôlya difference and other immanantal processes. We show that a large class of Papangelou processes belongs to these processes. Other constructions may by found in the foundational work of Soshnikov [10] and Shirai/Takahashi [9].

1 Notations and problems

Let $(X, \mathcal{B}, \mathcal{B}_0)$ be a Polish *phase space*, i.e. X is a Polish space, $\mathcal{B}$ its Borel $\sigma-$field and $\mathcal{B}_0$ the collection of its bounded Borel sets. On the next level we consider the sets $\mathcal{M}^{\cdot} \subset \mathcal{M}^{\cdot\cdot} \subset \mathcal{M}$ of Radon resp. (simple) Radon point measures μ on X. $\mathcal{M} = \mathcal{M}(X)$ is a nice phase space $(\mathcal{M}, \mathcal{B}(\mathcal{M}), \mathcal{B}_0(\mathcal{M}))$ with respect to the vague topology in $\mathcal{M}$, and $\mathcal{M}^{\cdot}, \mathcal{M}^{\cdot\cdot} \in \mathcal{B}(\mathcal{M})$. By $\mathcal{M}^{\cdot\cdot}_f$ we denote the set of finite point measures. On the third level we consider laws P on $\mathcal{M}^{\cdot}, \mathcal{M}^{\cdot\cdot}$ and $\mathcal{M}$, which are called (simple) point processes in resp. random measures on X. Denoting $\zeta_f : \mu \longmapsto \mu(f), f \in \mathcal{K}(X)$, we are interested in the random fields $(\mathcal{M}, \mathcal{B}(\mathcal{M}), P, (\zeta_f)_{f\in\mathcal{K}(X)})$. (Here $\mathcal{K}(X)$ denotes the collection of continuous functions with compact support.)

Starting point for the main concept is a kernel $\pi(\mu, \mathrm{d}\,x), \mu \in \mathcal{M}^{\cdot\cdot}$, from $\mathcal{M}^{\cdot\cdot}$ to X. (I.e. $\pi(\mu, .) \in \mathcal{M}$ for any $\mu \in \mathcal{M}^{\cdot\cdot}$; and $\pi(., f)$ is measurable for any $f \in \mathcal{K}(X)$.) Then a point process P in X is called a *Papangelou process with* *conditional intensity* π if P is a solution of the following integration-by-parts formula

$$(\Sigma_\pi) \quad \mathcal{C}_P(h) = \int\int h(x, \mu + \delta_x)\, \pi(\mu, \mathrm{d}\, x) P(\mathrm{d}\, \mu), h \in F_+.$$

(Here $\mathcal{C}_P$ is the Campbell measure of P and F_+ the collection of non-negative measurable functions on the underlying space.)

π is a conditional intensity for P in the sense that $P(\pi(., f)) = P(\zeta_f), f \in \mathcal{K}(X)$, where the right hand side is given by the intensity of P. Equation (Σ_π) may have no solution, exactly one or a continuum of solutions.

Example 1 *Let $\varrho \in \mathcal{M}, z > 0$. (z may depend on x.)*

1 *(The Poisson process) Take $\pi(\mu, .) = \varrho$. (Σ_π) then is Mecke's equation having P_ϱ the Poisson process with (conditional) intensity ϱ as unique solution.*

2 *Take $\pi(\mu, .) = 1_{\{0\}}(\mu) \cdot \varrho$.*

3 *(The Polya sum process [6]) Take $\pi_+(\mu, .) = z \cdot (\varrho + \mu)$. The unique solution then is the Polya sum process for π_+.*

4 *(The Polya difference process [6]) Here the underlying space X is countable and ϱ is the counting measure on X. Take $\pi_-(\mu, .) = z \cdot (\varrho - \mu)$ if μ is simple, i.e. $\mu \in \mathcal{M}^{\cdot}$, and $\equiv 0$ otherwise. The unique solution now is the Polya difference process. It is a simple point process.*

5 *(Gibbs processes [4]) Take $\gamma(\mu, \mathrm{d}\, x) = \exp\big(-E(x, \mu)\big) \cdot \varrho(\mathrm{d}\, x)$ where $E(x, \mu)$ is a suitable energy of x given μ. The solutions of (Σ_γ) are the Gibbs states in the DLR-sense.*

The Problem to be discussed in this paper is the construction of such Papangelou processes, but also certain immanantal processes, and to indicate their characteristic properties.

2 Finite Papangelou processes

Given π, a natural number $m \geq 0$ and a configuration $\mu \in \mathcal{M}^{\cdot\cdot}$. Consider the kernel on X^m defined by

$$\pi^{(m)}(\mu; \mathrm{d}\, x_1 \ldots \mathrm{d}\, x_m) = \pi(\mu; \mathrm{d}\, x_1) \pi(\mu + \delta_{x_1}, \mathrm{d}\, x_2) \ldots \pi(\mu + \delta_{x_1} + \cdots + \delta_{x_{m-1}}, \mathrm{d}\, x_m),$$

with $\pi^{(0)} = \delta_\emptyset$.

Assume now the following *finiteness condition*:

$$(1) \qquad 0 < \Xi(\mu) = \sum_{m \geq 1} \frac{1}{m!} \pi^{(m)}(\mu, X^m) < \infty.$$

In this case the following finite point process is well defined:

$$P_\pi^{(\mu)}(\varphi) = \frac{1}{\Xi(\mu)} \sum_{m \geq 1} \frac{1}{m!} \int_{X^m} \varphi(\delta_{x_1} + \cdots + \delta_{x_m}) \pi^{(m)}(\mu, \mathrm{d}\, x_1 \ldots \mathrm{d}\, x_m), \varphi \in F_+.$$

Under the additional *cocycle condition*

$$\pi^{(m)}, m \geq 0, \text{ are all symmetric measures}$$

we then have the

Lemma 1 *([6]) $P_\pi^{(\mu)}$ is a solution Q of the equation*

$$\mathcal{C}_Q(h) = \int\int h(x, \eta + \delta_x)\,\pi(\eta + \mu, \mathrm{d}\,x) Q(\mathrm{d}\,\eta), h \in F_+.$$

(For a proof we refer to [6].)

In particular we see that $P_\pi^{(0)}$ is a (finite) Papangelou process for π. We remark that this theorem remains true in a more general context and allows the construction of *locally finite* Papangelou processes. For this one has to assume the cocycle condition and local integrability or local finiteness of the kernel (to be defined below). (cf. [6]) We now sketch a new construction also based on these assumptions on a given kernel π.

3 A new construction of point processes

Let $\pi(\mu, \mathrm{d}\,x), \mu \in \mathcal{M}^{\cdot\cdot}$, be a given kernel, locally integrable and satisfying the cocycle condition. $G \in \mathcal{B}_0$ denotes bounded Borel sets in X. And $\pi_G = 1_G \cdot \pi$ the restriction of π to G. Let Q_G be the finite Papangelou process $P_{\pi_G}^{(0)}$, i.e.

$$Q_G(\varphi) = \frac{1}{\Xi_G} \cdot \sum_{m \geq 0} \frac{1}{m!} \cdot \int_{G^m} \varphi(\delta_{x_1} + \cdots + \delta_{x_m})\,\varrho_m(\mathrm{d}\,x_1 \ldots \mathrm{d}\,x_m), \varphi \in \mathcal{M}_f^{\cdot\cdot},$$

where we wrote $\varrho_m(\mathrm{d}\,x_1 \ldots \mathrm{d}\,x_m) = \pi^{(m)}(0; \mathrm{d}\,x_1 \ldots \mathrm{d}\,x_m)$ for short. Here we use local integrability of π, i.e. finiteness of the normalizing constants Ξ_G. Note that the measures ϱ_k are symmetric.

The Laplace transform of Q_G is

$$\mathcal{L}_{Q_G}(f) = \frac{1}{\Xi_G} \cdot \sum_{m \geq 0} \frac{1}{m!} \cdot \int_{G^m} \prod_{j=1}^{m} \mathrm{e}^{-f(x_j)}\,\varrho_m(\mathrm{d}\,x_1 \ldots \mathrm{d}\,x_m), f \in \mathcal{K}(G).$$

The aim now is to construct some point process Q as the weak limit of the sequence $(Q_G)_{G \in \mathcal{B}_0}$. Define the (generalized) *cummulant measures* Θ_m of $(\varrho_m)_m$ by

$$\Theta_m(f_1 \otimes \cdots \otimes f_m) = \frac{1}{(m-1)!} \sum_{\mathcal{J}} c(\mathcal{J}) \cdot \prod_{J \in \mathcal{J}} \varrho_{|J|}(\otimes_{j \in J} f_j). \tag{2}$$

($f_1, \ldots, f_m \in \mathcal{K}(X)$.) Here the sum is taken over all partitions of $\{1, \ldots, m\}$ into pairwise disjoint non-empty subsets; and $c(\mathcal{J}) = (-1)^{|\mathcal{J}|-1}(|\mathcal{J}|-1)!$. Then define the corresponding *cluster measure* L on $\mathcal{M}_f^{\cdot\cdot}$ by

$$L(\varphi) = \sum_{m \geq 1} \frac{1}{m} \int_{X^m} \varphi(\delta_{x_1} + \cdots + \delta_{x_m})\ \Theta_m(\mathrm{d}\,x_1 \ldots \mathrm{d}\,x_m), \varphi \in F_+.$$

We assume here that L is a *signed modified Laplace functional* (s.m.L.f.) in the sense of Mecke ([3]), i.e. $|L|$ is a Radon measure in the sense that it is finite on all $\mathcal{M}^{\cdot\cdot}_G = \{\zeta_G \geq 1\}, G \in \mathcal{B}_0$, and moreover

$$|L|(1 - \mathrm{e}^{-\zeta_f}) < -\infty, f \in \mathcal{K}(X). \tag{3}$$

Note that this is a condition on the kernel π. A sufficient condition for this to happen is that L is of first order, i.e. the intensity measure $\nu^1_{|L|}$ of $|L|$ is Radon. Then for any $G \in \mathcal{B}_0$ $|L_G|$ is a finite measure with

$$|L_G|(1 - \mathrm{e}^{-\zeta_f}) < \infty, f \in \mathcal{K}(X).$$

Here L_G denotes the restriction of L to $\mathcal{M}^{\cdot\cdot}_f(G)$.

A combinatorial argument then shows that

$$\mathcal{L}_{Q_G}(f) = \exp\big(-L_G(1 - \mathrm{e}^{-\zeta_f})\big),\ f \in \mathcal{K}(X). \tag{4}$$

This step resembles the well known problem in statistical mechanics to represent the log-partition function by means of the Ursell functions (or cummulants in probability theory).

We now go to the thermodynamic limit $G \nearrow X$ in (4) and can see by means of Lebesgue's dominated convergence theorem that the limiting function $\mathcal{L}(f) = \exp\big(-L(1-\mathrm{e}^{-\zeta_f})\big) f \in \mathcal{K}(X)$, satisfies all conditions of Levy's continuity theorem for random measures (cf. [3]). Thus we see that there exists some point process P in X such that $P_G \Rightarrow_G P$. Here $\Rightarrow$ denotes weak convergence. Its Laplace transform is given by

$$\mathcal{L}_P(f) = \exp\big(-L(1 - \mathrm{e}^{-\zeta_f})\big), f \in \mathcal{K}(X). \tag{5}$$

Note here that the class of point processes is closed in the weak topology.

To summarize the above reasoning we obtained the following

Theorem 1 *(cf. [5]) Let π be locally integrable and satisfy the cocycle condition. We assume also that the associated L is a s.m.L.f.. Then there exists a unique point process P in X with a Laplace transform of the form* (5).

If L is a signed modified Laplace functional *of first order*, then equation (5) for a random measure P is equivalent to the condition that P is a solution of the following equation :

$$\mathcal{C}_P = \mathcal{C}_L \star P. \tag{6}$$

Here the operation $\star$ is a version of a convolution defined by

$$\mathcal{C}_L \star P(h) = \int h(x, \kappa + \mu)\ \mathcal{C}_L(\mathrm{d}\,x, \mathrm{d}\,\kappa)P(\mathrm{d}\,\mu), h \in F_+.$$

(For a proof see [3, 7].) This equation is well known in the case that L is a positive measure on $\mathcal{M}^{\cdot\cdot}_f$ satisfying condition (3). P is then infinitely divisible with socalled Lévy measure L; and P can the represented as the image of the Poisson process with intensity measure L under the transformation which dissolves the clusters into its particles. (cf. [3]) In case of a signed measure L the above construction seems to be new.

If we assume in addition to the conditions above that π is a Feller kernel in the sense that $\pi(.,f)$ is a continuous and bounded function for any $f \in \mathcal{K}(X)$ then the theorem, combined with the generalized Palm-Chinchin theorem, implies that P is even a Papangelou process with conditional intensity π.

If L is simple, i.e. concentrated on $\mathcal{M}^{\cdot}_f$, with diffuse first moment measure then P is simple too. ([5])

It is evident from the above reasoning that one can start with a family of signed measures $(\Theta_m)_m$ on X^m such that the corresponding L is a s.m.L.f, and define the measures $(\varrho_k)_k$ by means of the inversion of formula (2):

$$\varrho_k(f_1 \otimes \dots f_k) = \sum_{\sigma \in \mathcal{S}_k} \prod_{\omega \in \sigma} \Theta_{|\omega|}(\otimes_{j\in\omega} f_j). \tag{7}$$

If these measures are all positive then the assertions of the theorem remain valid.

4 Bosons, Fermions, Polya and immanantal processes

Poisson and other point processes

Given a Radon measure ϱ on X the kernel $\pi(\mu,.) \equiv \varrho$ leads to $\varrho_m = \varrho^{\otimes m}$ and the Poisson process $Q_G = P_{\varrho_G}$ in $\mathcal{M}^{\cdot\cdot}(G)$. Furthermore, $\Theta_1 = \varrho$ whereas all other Θ_m are the zero measure. Therefore L is the non-signed m.L.f. $L(\varphi) = \int_X \varphi(\delta_x)\,\varrho(\mathrm{d}\,x), \varphi \in F_+$. The above construction immediately leads to P_ϱ, the Poisson process with intensity measure ϱ.

Consider one of the simplest Gibbs resp. Boltzmann kernels $\pi(\mu,.) = 1_{\{0\}}(\mu)\cdot\varrho$, where we now assume that $0 < \varrho(X) < 1$. In this case $\varrho_1 = \varrho$ and all other ϱ_k are the zero measure. The corresponding point process is $Q_G = \frac{1}{\Xi_G}\cdot\big(\delta_0 + \int_G \delta_{\delta_x}\,\varrho(\mathrm{d}\,x)\big)$. For L we then find the signed m.L.f.

$$L(\varphi) = \sum_{m\geq 1} \frac{(-1)^{m-1}}{m} \int_{X^m} \varphi(\delta_{x_1} + \dots + \delta_{x_m})\,\varrho(\mathrm{d}\,x_1)\dots\varrho(\mathrm{d}\,x_m).$$

And the corresponding simple point process is Q_X.

Polya processes

Let ϱ be a Radon measure on X and $0 < z < 1$ a given parameter. Consider the kernel $\pi_{+1}(\mu,.) = z\cdot(\varrho+\mu)$. Simultaneously consider also $\pi_{-1}(\mu,.) = z\cdot(\varrho-\mu)\cdot 1_{[0,\varrho]}(\mu)$. Here $[0,\varrho]$ is the collection of all Radon point measures κ which are subconfigurations of ϱ. (We assume here that $[0,\varrho] \neq \emptyset$.) A special case was presented in example 4 above.

The kernels $\pi_\varepsilon, \varepsilon \in \{+1,-1\}$, are locally integrable and satisfy the cocycle condition. The measures ϱ_k^ε resp. Θ_m^ε are given by

$$\begin{aligned}\varrho_k^\varepsilon(\mathrm{d}\,x_1\ldots\mathrm{d}\,x_k) = \quad & z^k\cdot 1^\varepsilon_{[0,\varrho]}\big(\delta_{x_1}+\cdots+\delta_{x_{k-1}}\big)\cdot\varrho(\mathrm{d}\,x_1)\big(\varrho+\varepsilon\delta_{x_1}\big)(\mathrm{d}\,x_2)\\ & \cdots\big(\varrho+\varepsilon(\delta_{x_1}+\cdots+\delta_{x_{k-1}})\big)(\mathrm{d}\,x_k),\\ \Theta_m^\varepsilon(\mathrm{d}\,x_1\ldots\mathrm{d}\,x_m) = \quad & z^m\cdot\varepsilon^{m-1}\cdot\delta_{x_m}(\mathrm{d}\,x_1)\cdots\delta_{x_m}(\mathrm{d}\,x_{m-1})\varrho(\mathrm{d}\,x_m).\end{aligned}$$

Here 1^ε denotes the usual indicator function if $\varepsilon = -1$; otherwise it is the constant 1. The corresponding point process P_ε is called Polya sum process if $\varepsilon = +1$ and Polya difference process otherwise. They had been analyzed in [8, 6]. P_ε is a Papangelou process for π_ε. P_{-1} is simple.

A class of Gibbs processes for classical systems

Consider a Boltzmann kernel of the form $\pi(\mu,\mathrm{d}\,x) = \exp\big(-E(x|\mu)\big)\varrho(\mathrm{d}\,x)$, where $E(x|\mu)$ is the energy of the particle in x given the environment μ defined by some pair potential Φ. In case that this potential is bounded from below, continuous, symmetric and with finite range then π is a Feller kernel. If it has additional properties implying that the corresponding L is a signed m.L.f. then by theorem 1 there exists a unique Gibbs process P with Boltzmann kernel π. It would be interesting to work out which class of pair potentials has these kernel properties.

The ideal Bose and Fermi gas

Another important class of examples is obtained as follows: The underlying phase space now is $\mathbb{R}^d$ with the Lebesgue measure $\mathrm{d}\,x$. Again $0 < z < 1$ is a parameter and

$$g(x) = \frac{1}{(2\pi\beta)^{d/2}}\exp\big(-\frac{\|x\|^2}{2\beta}\big), x\in\mathbb{R}^d,$$

the centered Gaußian density with covariance matrix βI, I denoting the identity and $\beta > 0$ a given parameter. Consider the kernel $K(x,y) = g(x-y)$ and denote by $\wp$ the coarsest resp. finest partition of $\{1,\ldots,k\}$. Define the measures

$$\varrho_k^\wp(\mathrm{d}\,x_1\ldots\mathrm{d}\,x_k) = z^k\sum_{\sigma\in\mathcal{S}_k}\chi^\wp(\sigma)\prod_{j=1}^k K(x_j,x_{\sigma(j)})\ \mathrm{d}\,x_1\ldots\mathrm{d}\,x_k,$$

which are in $1-1$–correspondence with the cummulant measures

$$\Theta_m^\wp(\mathrm{d}\,x_1\ldots\mathrm{d}\,x_m) = z^m\chi^\wp(e)\cdot K(x_1,x_2)\cdots K(x_m,x_1)\ \mathrm{d}\,x_1\ldots\mathrm{d}\,x_m.$$

Here e denotes the identity permutation in $\mathcal{S}_m$, and $\chi^\wp$ is the character of the corresponding irreducible representation of $\mathcal{S}_k$ resp. $\mathcal{S}_m$. In both cases the ϱ_k are positive, symmetric Radon measures on X^k. Thus we can construct locally the corresponding point processes $Q_G, G\in\mathcal{B}_0$.

Denote the corresponding cluster measure by $L^\wp$. Its Campbell measure $\mathcal{C}_{L^\wp}$ is the image of $\mathcal{K}^x(\mathrm{d}\,y)\varrho(\mathrm{d}\,x)$ under the mapping $(y,x)\longmapsto(x,\mu_y+\delta_x)$, where μ_y is the configuration built on the tupel y. Here $\mathcal{K}^x = \sum_{m\geq 1}K_m^x$ is a measure on $\mathbf{X} = \cup_{n\geq 0}X^n$ where

$$K_m^x(\mathrm{d}\,x_2\ldots\mathrm{d}\,x_m) = z^m\chi^\wp(e)\cdot K(x,x_2)\cdots K(x_m,x)\ \mathrm{d}\,x_2\ldots\mathrm{d}\,x_m.$$

This implies that the intensity measure of $|L^\wp|$ is $\mathcal{K}^x(\mathbf{X})\,\mathrm{d}\,\mathbf{x}$. This is a Radon measure because $\int_X K(x,y)K(y,z)\,\mathrm{d}\,y = K(x,z), x,z \in X$. In this case $L^\wp$ is a s.m.L.f of first order and theorem 1 implies the existence of a point process $P_\wp$ in X. We call it the *immanantal process for* $\wp$ *and* K. This process is a solution of equation (6) for $L_\wp$. Both processes are simple because their cluster measures have this property.

Fichtner (cf.[2]) considered the case where $\wp$ is the coarsest partition, i.e. $\chi^\wp \equiv 1$. The corresponding point process P_+ is infinitely divisible with Lévy measure $L^{\chi^\wp}$ and a solution of equation (6). It is called in [2] the *ideal Bose gas*. In this case $\varrho_k^\wp\big(\mathrm{d}\,x_1 \ldots \mathrm{d}\,x_k\big) = z^k \operatorname{per}\big(K(x_i,x_j)\big)_{1\le i,j\le k}\ \mathrm{d}\,x_1 \ldots \mathrm{d}\,x_k$, the *permanent* of the matrix in question.

When $\wp$ is the finest partition then $\chi^\wp$ is the *signum* of a permutation, and $\varrho_k\big(\mathrm{d}\,x_1 \ldots \mathrm{d}\,x_k\big) = z^k \det\big(K(x_i,x_j)\big)_{1\le i,j\le k}\,\mathrm{d}\,x_1 \ldots \mathrm{d}\,x_k$. This simple point process is called the *ideal Fermion gas*.

This class can be broadend considerably to *immanantal point processes* in the sense of [1] in considering also all other characters $\chi^\wp$ and a wider class of kernels K over abstract $\sigma-$finite measure spaces.

References

[1] Diaconis, P., Evans, S., Immanants and finite point processes. J. Combinatorial theory A, 91, 305-321 (2000).

[2] Fichtner, K.-H.: On the position distribution if the ideal Bose gas. Math. nachr. 151, 59-67 (1991).

[3] Mecke, J.: Random measures. In German. Unpublished Habilitation (1967). English version in preparation (2011).

[4] Nguyen, X.X., Zessin H.: Integral and differential characterizations of the Gibbs process. Math. Nachr. 88, 1055-115 (1979).

[5] Nehring, B.: On the structure of the moment measures of random measures with signed modified Laplace functionals. Thesis. In preparation.

[6] Nehring, B., Zessin, H.: The Papangelou process. A concept for Gibbs, Fermi and Bose processes. To appear in Izvestiya Nan Armenii: Matematika.

[7] Nehring, B., Zessin, H.: A representation of the moment measures of the general ideal Bose gas. To appear in Math. Nachr. (2011).

[8] Rafler, M.: Gaussian Loop- and Polya Processes. A Point Process Approach. Universitätsverlag Potsdam (2009).

[9] Shirai, T., Takahashi, Y.: Random point fields associated with certain Fredholm determinants I: Fermion, Poisson and boson processes. J. Functional Anal. 205, 414-463 (2003).

[10] Soshnikov, A.: Determinantal random point fields. Russ. Math. Surv. 55, 923-975 (2000).