Complexity of the minimum-time damping of a pendulum

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1 Introduction

The problem of minimum-time damping of a pendulum is a classical problem of control theory. In the linear case, described by the equation $\ddot{x} + x = u$, $|u| \leq 1$, its solution is stated in [1]. The optimal control is of bang-bang type, i.e. it takes values $u = \pm 1$, and the switching curve which separates the domain of the phase plane, where $u = -1$ from the domain $u = +1$ consists of unit semicircles centered at points of the form $(2k+1, 0)$, where k is an integer. The real physical pendulum controlled by a torque in the joint is governed by the equation $\ddot{x} + \sin x = \varepsilon u$, $|u| \leq 1$, where x is the vertical angle, and ε is the maximal amplitude of the control torque. The parameter ε is arbitrary: it might be large, small, of order 1. We are interested most in the case of a small ε. The maximum principle says that the optimal control has the form $u = \operatorname{sign} \psi$, where the "adjoint" variable satisfies the equation $\ddot{\psi} + (\cos x)\psi = 0$. Thus, the control is still of the bang-bang type, but the time instants of switchings are roots of a rather nontrivial function, a solution of the general Sturm–Liouville/Schrödinger equation. The complexity of a control is characterized mainly by the switching number. In the linear case this number for a trajectory connecting the initial point $(x, \dot{x})$ with $(0,0)$ is $\frac{T}{\pi} + O(1)$, where T is the duration of the motion. In its turn, $T = \pi\sqrt{\frac{E}{2}} + O(1)$, where $E = \frac{1}{2}\dot{x}^2 + \frac{1}{2}x^2$ is the energy. Thus, each trajectory possesses a finite number of switches, but if the initial energy is large this number is $\sqrt{\frac{E}{2}} + O(1)$ and is also large.

In the nonlinear case the switching number behaves quite differently. The best result, known to the author, is due to Reshmin [2]. It says that if the parameter ε is large enough, all optimal trajectories possess no more than a single switch.

2 Results

We show that for any ε the switching number for all optimal trajectories possesses a common bound.

Theorem 2.1 *Suppose $N_\varepsilon(x, \dot{x})$ is the number of zeroes of the adjoint variable ψ along an optimal trajectory connecting $(x, \dot{x})$ with (0,0). Then the quantity $N_\varepsilon = \sup N_\varepsilon(x, \dot{x})$, where* sup *is taken over the entire phase space is finite.*

*Supported by RFBR grant 11-08-00435.

Another result is an upper and lower bound for N_ε which is sharp with respect to the order of magnitude.

Theorem 2.2 *There exist positive constants* c_1, c_2, *such that*

$$\frac{c_1}{\varepsilon} \le N_\varepsilon \le \frac{c_2}{\varepsilon}$$

as the parameter ε *is small enough.*

Our main result is a promotion of inequalities of theorem 2.2 to an asymptotic equality:

Theorem 2.3 *As* $\varepsilon \to 0$ *there is the asymptotic equivalence*

$$N_\varepsilon \sim \frac{D}{\varepsilon}, \text{ where } D = \frac{1}{2}\,\mathrm{Si}(\pi) = \int_0^\pi \frac{\sin x}{2x}\,dx = 0.925968526\ldots \tag{2.1}$$

The theorem can be regarded, like the "Feigenbaum universality", as an asymptotic formula $\varepsilon_n \sim D/n$ for bifurcation values of the parameter ε. Here, the bifurcation is the increment by 1 of the maximal number of the control switches. Unlike the period-doubling bifurcation, studied by Feigenbaum, the relevant constant D can be expressed via standard mathematical constructions, and its computation with any accuracy is not a problem.

The paper is based on a lemma saying that in the large speed area the optimal trajectory possesses no more than a single switch. We use heavily the Sturm theory of zero loci for solutions of a Sturm–Liouville equation. It allows us to relate N_ε to the optimal time of motion from points with energy of order 1 to the point (0,0). The lower bound in Theorem 2.2 is based on energy considerations, which allows us to estimate this time. The upper bound is more complicated and follows from a computation of the elapsed time in a motion under a quasioptimal control. The asymptotic equivalence (2.1) stems from the idea of the Poincaré map control, coupled with a special nonlinear Sturm-like theorem.

3 Proofs

The control system takes the form

$$\begin{cases} \dot{x} &= y, \\ \dot{y} &= -\sin x + \varepsilon u, \ |u| \le 1 \end{cases} \tag{3.1}$$

We are interested in the minimum-time damping: fastest motion from a given point $(x, y) \in S^1 \times \mathbf{R}$ to the stable equilibrium (lower) point $(0, 0)$.

According to the Pontryagin maximum principle this problem is associated with adjoint variables (ϕ, ψ) and the Hamiltonian

$$H = y\phi + (-\sin x + \varepsilon u)\psi - 1$$

so that, the maximum of the Hamiltonian is attained at the optimal control u, the optimal motion is governed by the corresponding canonical system, and

$H \equiv 0$ along the optimal trajectory. In other words, besides the system (3.1) the following relations hold:

$$u = \operatorname{sign}\psi, \tag{3.2}$$
$$\dot{\phi} = (\cos x)\psi, \tag{3.3}$$
$$\dot{\psi} = -\phi \tag{3.4}$$
$$y\phi + (-\sin x + \varepsilon u)\psi - 1 \equiv 0. \tag{3.5}$$

The following bound for the number of switches is implied by (3.3), (3.4), and the Sturm theory [4]:

Lemma 3.1 *In the optimal arc of duration T no more than $\frac{T}{\pi} + 1$ switches of control is possible.*

Note that the duration of optimal motion can be arbitrary large, if the initial energy is large enough. In particular, the Lemma does not immediately imply the finiteness Theorem 2.1.

3.1 Basic Lemma

We begin with a lemma which implies (almost total) absence of switches at high energy states.

Lemma 3.2 (Basic Lemma) *Suppose t_1, t_2 are adjacent zeroes of the adjoint variable $\psi = \psi(t)$. Then, the velocities $y(t_1)$, $y(t_2)$ have opposite directions.*

Corollary 3.1 *Under conditions of the Lemma there is a time t between t_1, t_2 such that $y(t) = 0$.*

Notice that the statement of Corollary is a Sturm-like theorem. In section 3.5 we will prove a strengthening of Corollary 3.1, where the uniqueness of t is asserted.

The energy $E = \frac{1}{2}y^2 + (1 - \cos x)$ of the pendulum cannot be large at point, where $y = 0$; at that point $E = |E| \leq 2$. Therefore, even before the second switch the optimal motion takes place in the bounded energy area.

3.2 Bounds for the damping time

Let K be a compact in the phase space $S^1 \times \mathbf{R}$, and $T_\varepsilon = T_\varepsilon(K)$ the maximum of damping times over all initial conditions $(x, y) \in K$. Assume that K is not the singleton (0,0). The next estimate for the time T_ε makes a ground for the finiteness Theorems 2.1 and 2.2:

Theorem 3.1 *There exist positive constants $C_i = C_i(K)$, $i = 1, 2$ such that*

$$\frac{C_1}{\varepsilon} \leq T_\varepsilon \leq \frac{C_2}{\varepsilon}$$

as the (positive) ε is small enough.

Theorems 2.1 and 2.2 follows from the above bounds relatively easily.

3.3 Proof of Theorem 3.1: lower estimate

To prove the lower estimate it suffices to take a singleton for the compact K. Take this point p_0 as the initial one of an optimal trajectory $p(t) = (x, y)(t)$, and consider the energy E of the running point on the trajectory as a function of time. We have

$$\dot{E} = \varepsilon y u, \ |y| \leq \sqrt{2E}, \tag{3.6}$$

which implies that $\left|\frac{d}{dt}\sqrt{E}\right| = \left|\frac{1}{2}\frac{\dot{E}}{\sqrt{E}}\right| \leq \frac{1}{\sqrt{2}}\varepsilon$. Since the initial value of energy is $E(p_0)$, and the final one is zero, we get a lower estimate for the elapsed time $T_\varepsilon \geq \frac{\sqrt{2E(p_0)}}{\varepsilon}$. ▶

3.4 Proof of Theorem 3.1: upper estimate

General strategy. We divide the phase space into three parts: of high energy $\{E > 2\}$, of low energy $\{E < 2\}$, and the standstill zone $S_{2\varepsilon} = \{|\sin x| < 2\varepsilon, |y| < 2\varepsilon\}$. For small ε the standstill zone consists of two connected components, the neighborhoods of the upper and lower equilibrium points. To estimate the damping time we use a particular "quasioptimal" control which is given by the formula

$$u = -\operatorname{sign} y \tag{3.7}$$

outside the standstill zone. It reflects the idea of steepest local energy descent. Note that on an interval of a constant velocity sign the controlled motion is governed by the Hamiltonian of the form $\frac{1}{2}y^2 + (1 - \cos x) \pm \varepsilon x$. We will show that one can make it to the upper standstill zone from a high energy state in time of order $O(1/\varepsilon)$, make it to the lower standstill zone from a low energy state in time of the same order $O(1/\varepsilon)$, make it to the low energy from the upper standstill zone in time of order $O(\log 1/\varepsilon)$, and, finally, make it to the lower equilibrium point from the lower standstill zone in time of order $O(1)$.

To this end we use the logarithmic bound for the oscillation period of the uncontrolled pendulum. It has the following form. Let p be a point of the phase space, denote by $\tau(p)$ the time required for the next hit of the point p in the uncontrolled motion of the pendulum. Then, if the energy $E(p) = 2 + h$, then $\tau(p) = O(\log\left|\frac{1}{h}\right|)$ as $h \to 0$. Note that if $h = 0$ the pendulum might stay forever in the upper equilibrium state, so that $\tau = \infty$. Analytically, the estimate has the form

$$\int_0^{2\pi} |\cos s + 1 + h|^{-1/2} ds = O(\log |h|^{-1}) \text{ as } h \to 0. \tag{3.8}$$

Standstill zones. In order to understand the motion inside and in the vicinity of the standstill zone we use linearization of the control system in a neighborhood of an equilibrium point. The manner of passage of the standstill zones is different in the upper and lower parts. The situation in the lower part is simpler: The corresponding linearized system is globally controllable in spite of the control bound $|u| \leq 1$. Therefore, it is possible to reach the lower equilibrium point from any point of the lower standstill zone in time of order $O(1)$.

In order to get from a point p of the upper standstill zone to the low-energy zone one can do as follows: Because of the local controllability of the linearized system there is a positive constant c such that if the point p is at the distance less than $c\varepsilon$ from the upper equilibrium point, we can move it in time $O(1)$ to any point at the distance exactly $c\varepsilon$ from the upper equilibrium. If p is at the distance more than $c\varepsilon$ from the upper equilibrium it stays intact. Thereafter we switch the control off, and wait for the time t_1, when the x-coordinate of the point $p(t_1)$ become zero. Then we apply the control (3.7) up to the time t_2, when the velocity y of the point $p' = p(t_2)$ become zero. The energy decrease $E(p)-E(p') = (\pi+o(1))\varepsilon$. Therefore, if $E(p) = 2+O(\varepsilon^2)$, then, $E(p') < 2-\frac{1}{2}\pi\varepsilon$, provided that the parameter ε is sufficiently small. This means that the point p' is within the low-energy zone. In view of the estimate (3.8) for the period of oscillations the maneuver takes time of order $O(\log 1/\varepsilon)$.

To estimate the duration of motion within high and low energy zones we use the Poincaré section technique coupled with the logarithmic bound for the oscillation period of the uncontrolled pendulum.

Low energies. Consider the controlled motion of the point p in the low energy zone $E(p) = 2-h$, $h > \varepsilon^2$ by using the Poincaré map associated to the Poincaré section $\Sigma_- = \{y = 0\}$. If a time interval under consideration is small compared to $1/\varepsilon$, the trajectory $p(t)$ is close to the trajectory of the uncontrolled motion with the same initial point. The Poincaré map $\mathcal{P}(u) : p \mapsto p'$, related to the Poincaré section $\Sigma_- = \{y = 0\}$, is close to the Poincaré map $\mathcal{P}(0)$ for the uncontrolled motion. In order to take into account the arising deviation of order $O(\varepsilon)$ it is convenient to invoke equation (3.6) for energy change. Suppose t_n are the hitting instances for the section Σ_-, $p_n = p(t_n) = (x_n, 0)$ is the sequence of points arising under iteration of the Poincaré map, $E_n = E(p_n)$ are the corresponding values of energy. Then

$$E_{n+1} - E_n = \varepsilon \int_{t_n}^{t_{n+1}} yudt = \varepsilon \int_{t_n}^{t_{n+1}} udx(t). \tag{3.9}$$

We fix the time instant t_n, the point $p_n = (x_n, 0)$, and study the influence of the control chosen upon the right-hand side of (3.9). In this equation $y = y(u,t)$ depends on control weakly: $y(u,t) = y(0,t) + O(\varepsilon\tau_n)$, where $\tau_n = t_{n+1} - t_n$ is the time interval between next hits of the section Σ_-. We know from (3.8) that in the low energy zone $\tau_n = O(\log 1/\varepsilon)$, and this bound is sharp in the vicinity of the standstill zone only; in the major part of trajectory τ_n is just bounded. If the time t_n is fixed the values of $t_{n+1}(u)$ and $\tau_n(u)$, like that of y, depend on u weakly. Put

$$\phi_n = \int\limits_{t_n}^{t_{n+1}(0)} |y(0,t)|dt = \int\limits_{t_n}^{t_{n+1}(0)} |dx(0,t)|.$$

This is a function of the initial position $\phi_n = \phi(x_n)$. An easy computation shows that $\phi(x) = 2|x|$. Thus, the right-hand side of (3.9) takes the form $\varepsilon\phi(x_n)U_n + o(\varepsilon)$, where U_n is arbitrary subject to $|U_n| \le 1$. In the upshot, if we pass to the variables $X_n = |x_n|$ we obtain a one-dimensional discrete control system

$$\cos X_n - \cos X_{n+1} = 2\varepsilon X_n U_n + o(\varepsilon),\ |U_n| \le 1, \tag{3.10}$$

or, equivalently,

$$\frac{\sin X_n}{2X_n}(X_{n+1} - X_n) = \varepsilon U_n + o(\varepsilon),\ |U_n| \le 1. \tag{3.11}$$

The obtained discrete system arises via the Euler approximation with step ε of the continuous control system

$$\frac{\sin X}{2X}\frac{dX}{dt} = U,\ |U| \le 1, \tag{3.12}$$

so that X_n approaches $X(n\varepsilon)$. The use of control (3.7) corresponds to $U \equiv -1$. The minimum-time damping problem corresponds to minimization of the functional $\sum_n \tau_n$. After normalization $\sum_n \tau_n \mapsto \varepsilon \sum_n \tau_n$ and passage to the limit $\varepsilon \to 0$, if the initial position belongs to the low energy zone, we get the problem of steering the system (3.12) to the point $X(T) = 0$ coupled with minimization of the functional

$$J_- = \int_0^T \tau_-(X(t))dt \to \min,\ \tau_-(X) = \int_0^X (\cos\phi - \cos X)^{-1/2} d\phi. \tag{3.13}$$

The finiteness of J_- corresponds to the upper bound $O(1/\varepsilon)$ for the duration of controlled motion in the low energy zone.

High energies. Quite similar but simpler arguments prove that one can get to the standstill zone from the high-energy zone in time of order $O(1/\varepsilon)$.

3.5 Proof of theorem 2.3

To prove our main result on asymptotics of N_ε we need two basic pieces: first, the next "Sturm-like" strengthening of Corollary 3.1:

Theorem 3.2 *Suppose ε is sufficiently small, t_1, t_2 are next zeroes of the adjoint variable $\psi = \psi(t)$, and the optimal motion in the interval $[t_1, t_2]$ of time does not hit the standstill zone. Then, there exists a single time instant t between t_1, t_2 such that $y(t) = 0$, so that the zeroes of y and the adjoint variable ψ are intermittent.*

Second, we use the reduction to the auxiliary one-dimensional control system (3.12).

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