A Markov process of tectonic plate motions

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Abstract

The model we propose can describe a movement of one plate on another being also relatively compatible with the elastic spring-slider model used in the study of some friction laws. It also deals with the stick-slip mechanism which has been associated to the mechanism of earthquakes. The model is rather universal to describe tectonic plate motions as also the friction between different material plates. The proposed model is stochastic model. The stochasticity of the model is described by the birth and the death of contact points of the plates. All diversities of plate interactions are presented by two parameters of the model, related to birth and death of the contacts.

1 Introduction

Since the late 50s, when the movement of tectonic plates is not already considered as absurd, the stick-slip mode of motion of plates relatively to each other is considered as the principal cause of earthquakes (see for example review [4]). A lot of models was appeared for that type of the motion. We propose another model in which the key role plays the Markov process of contact points of both plates.

A similar Markov process was studied in [1].

2 Model description. Assumptions. Properties

We consider a plate sliding on a solid substrate. The plate can be, for example, a rectangle. The plate is subjected to the action of a constant force F in the direction parallel to one of the plates sides.

A1 The contact of the plate and the substrate is realized by a set ω of points on the surface area of the plate. The points of ω is called *contact points*. They are the places through which the plate and the substrate interact. For example, they may be caused by asperities of the plates surfaces.

A2 The set ω is a random set and it changes over time. At any time moment ω is a point process realization.

A3 Under any force G the plate is moving with the velocity v proportional to G: $v = \gamma G$.

This assumption means that we do not consider Newton mechanics in the model. Such relation between forces and velocities is often called Archimedes mechanics. It can take place, for example, if the plate is immersed in a viscous medium.

A4 The contact points undergo deformations. It means that the contact points are moving in the direction of the velocity v. The displacement is proportional to the velocity: in a small time interval Δt the following standard relation

$$\Delta x(\zeta) = v\Delta t,$$

where $\Delta x(\zeta)$ is the shift value,holds for any $\zeta \in \omega$.

A5 The displacements of the contact points create a resistance force

$$R_\omega = \min\Big\{\varkappa \sum_{\zeta\in\omega} x(\zeta), F\Big\} \tag{2.1}$$

Two positive constants $\bar{c}_b$ and $\bar{c}_u$ control the stochastic dynamics of ω which is birth and death of the contact points.

A6 New contact points appear with the t depending intensity

$$c_b(t) = \bar{c}_b v(t), \tag{2.2}$$

where $\bar{c}_b > 0$ and $v(t)$ is the velocity of the plate at the moment t. The intensity c_b is such that new contact points appear at the moment t if the plate has a positive velocity $v(t)$ at t (see **A3**).

A7 The death or vanishing of every contact point has the intensity

$$c_u = \bar{c}_u, \tag{2.3}$$

where $\bar{c}_u > 0$.

We consider here the death intensity which does not depend on the displacement of the contact point. Such choice is not physically justified. May be the death intensity of a contact point $\omega \in \Omega$ equal to

$$c_u(\zeta, t) = \overline{c}_u x(\zeta, t) \tag{2.4}$$

could be more reasonable. However, we consider c_u equal to a constant as a first model version to simplify the investigations. The case (2.4) will be studied elsewhere.

It follows from the above assumptions that the real force acting on the plate at a moment t depends on the displacements of all the contact points and is

$$G(t) = [F - R_\omega]_+ = [F - \varkappa \sum_{\zeta \in \omega} x(\zeta, t)]_+, \tag{2.5}$$

where $x(\zeta, t)$ is the displacement of the contact point $\zeta \in \omega$ at te moment t and $[\cdot]_+$ means the positive part of a value in the brackets.

The random process $G(t)$ is piece-wise deterministic (see [2, 3]). There exist a locally finite set of time points $D \subset \mathbb{R}_+$ called *events* such that $G(t)$ is a deterministic function between any nearest events. The events are split into two subsets:

a subset of *death* events where a path of $G(t)$ has a positive random jumps

a subset of *birth* events where a path derivative of $G(t)$ has a negative jump.

$G(t)$ is a random process since the set ω is random (see (2.5)). A typical path of $G(t)$ is a piece-wise continuous function between of two nearest deaths of contacts. If t is a time of a birth of a new contact ζ' then $G(t)$ is still continuous at t, but the derivative of $G(t)$ is discontinuous at t because the number of the contact points $|\omega|$ is increasing at t to $|\omega'| = |\omega| + 1$. The new contact points in $\omega' = \omega \cup \{\zeta'\}$ does not create the resistance force immediately. This resistance force is increasing from 0. If t is a time of the death moment then $G(t)$ has a positive jump at t because one of the terms in $\sum_{\zeta \in \omega} x(\zeta, t)$ disappears (see the figure 1).

Deterministic paths of $G(t)$ satisfy the equation

$$\frac{\mathrm{d}G(t)}{\mathrm{d}t} = -\varkappa\gamma|\omega|G(t)$$

and thus

$$G(t) = G(0)e^{-\varkappa\gamma|\omega|t}. \tag{2.6}$$

$\square$

If $G(0) = [F - \varkappa \sum_{\zeta \in \omega} x(\zeta, 0)]_+$ is positive then the real force $G(t)$ is kept positive at any moment $t > 0$. In this case the sum of displacements $\sum_i x_i < \frac{F}{\varkappa}$. However the number of contacts n is not bounded.

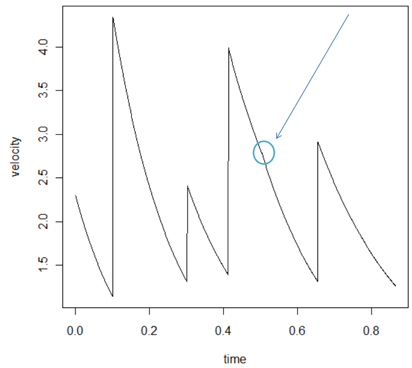

Figure 1: A trajectory of velocity. Positive jumps correspond to deaths, the arrow shows a birth

3 Markov process

Next we give the exact description of the model. It is a Markov process with a rather complicate set of its states.
The configuration set (the set of states) is

$$\mathcal{X} = \cup_{n=0}^{\infty} \left[\{n\} \times \left[0, \frac{F}{\varkappa}\right]^n \right] = \left\{ (n, x_1, ..., x_n) : \ n \in \mathbb{N}, \ (x_1, ..., x_n) \in \left[0, \frac{F}{\varkappa}\right]^n \right\} \tag{3.1}$$

Infinitesimal generator. We introduce a set of functions $\mathbf{F} = \{f = (f_n)\}$ on $\mathcal{X}$ such that every $f_n : \{n\} \times \left[0, \frac{F}{\varkappa}\right]^n \to \mathbb{R}$ is a continuous function. We shall omit the index n if it does not lead to misunderstanding and write $f(n, x_1, ..., x_n)$ instead $f_n(n, x_1, ..., x_n)$.

The infinitesimal operator of the Markov process L defined on $\mathbf{F}$ is

$$\begin{aligned} Lf(n, \mathbf{x}) = \ & v \sum_{i=1}^{n} \frac{\partial f}{\partial x_i} + c_b v [f(n+1, x_1, ..., x_n, 0) - f(n, x_1, ..., x_n)] \\ & + c_u \sum_{j=1}^{n} [f(n-1, x_1, ..., \widehat{x}_j, ..., x_n) - f(n, x_1, ..., x_n)], \end{aligned} \tag{3.2}$$

where $\widehat{x}_j$ means that the variable x_j is not presented in the list of variables, and we recall that $v(n, x_1, ..., x_n) = \gamma[F - \varkappa \sum_{i=1}^{n} x_i]_+$. The first term in (3.2) corresponds the deterministic plate motion between the events. The second term reflects the birth event and the third term reflects the death event.

Let $\widehat{m}(n, x_1, ..., x_n) = \sum_{i=1}^{n} x_i$ and $\widehat{n}(n, x_1, ..., x_n) = n$ then applying operator L we obtain

$$\begin{aligned} M(n, x_1, ..., x_n) &:= L\widehat{m}(n, x_1, ..., x_n) \\ &= v(n, x_1, ..., x_n)\widehat{n}(n, x_1, ..., x_n) - c_u \widehat{m}(n, x_1, ..., x_n), \\ N(n, x_1, ..., x_n) &:= L\widehat{n}(n, x_1, ..., x_n) \\ &= c_b v(n, x_1, ..., x_n) - c_u \widehat{n}(n, x_1, ..., x_n). \end{aligned} \tag{3.3}$$

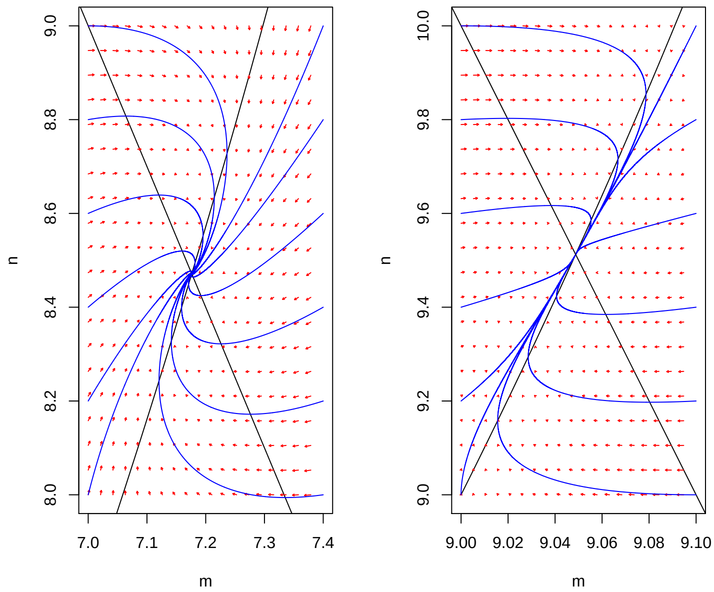

Figure 2: The vector field (3.3)The curves are integral curves of the field

The functions M and N expressed in the terms $\widehat{m}$ and $\widehat{n}$ are functions on $\left[0, \frac{F}{\varkappa}\right] \times \mathbb{N}$ since $\widehat{n}$ takes integers values. We extend both functions on $\left[0, \frac{F}{\varkappa}\right] \times \mathbb{R}_+$

Essential role in the following plays the combination of the parameters

$$a = \gamma \varkappa F \frac{c_b}{c_u^2}, \tag{3.4}$$

which we call an *order parameter*.

Theorem 3.1. *There exists only solution* (m_0, n_0) *in* $\mathbb{R}_+^2$ *of the equations* $M = N = 0$*:*

$$\begin{aligned} m_0 &= \frac{F}{\varkappa} - \frac{F}{2\gamma\varkappa a}\left[\sqrt{1+4\gamma a} - 1\right] \\ n_0 &= \frac{c_b}{c_u}\frac{F}{2a}\left[\sqrt{1+4\gamma a} - 1\right] \end{aligned} \tag{3.5}$$

The linearized at (m_0, n_0) *equations (3.3) are*

$$\begin{aligned} M &= -(c_u + \varkappa n_0)(\widehat{m} - m_0) + (F - \varkappa m_0)(\widehat{n} - n_0) \\ N &= -\varkappa c_b(\widehat{m} - m_0) - c_u(\widehat{n} - n_0) \end{aligned} \tag{3.6}$$

The determinant of the matrix

$$\mathcal{M} = \begin{pmatrix} -(c_u + \varkappa n_0) & F - \varkappa m_0 \\ -\varkappa c_b & -c_u \end{pmatrix}$$

is positive. Therefore the trace (the sum of the eigenvalues) of M is negative.

If $a < 20$ then the eigenvalues are complex, if $a > 20$ the both eigenvalues are negative, where a is defined by (3.4).

The vector field (3.3) is shown in the figure 2. The left picture is for $a = 9$ and the right one is for $a = 100$

Acknowledgements

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