# On Interacting Brownian Loops

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*Dedicated to the eightieth anniversary of Professor R. A. Minlos and seventy fifth anniversary of Professor Ya. G. Sinai*

## 1 Introduction

The Feynman-Kac formula represents the statistical operator $e^{-\beta H}$ in terms of Wiener integrals and reduces the quantum mechanical model to a classical one where quantum particles become Brownian loops ( closed trajectories ) in one more dimension. Using this method in his pioneering work [3] Ginibre studied the reduced density matrices of quantum gases. In recent years there is an increased activity in studying different problems related to the model of Brownian loop gas. The major questions concern the occurrence of infinite loops which is motivated by its connection to the Bose-Einstein condensation. (See for example [1],[2] and references therein)

Below we consider a gas of interacting Brownian loops at low fugacity regime which corresponds to the phase without Bose-Einstein condensation.

We use cluster expansion method (see [9], [5],[6] and references therein) to study the decay of correlations in the loop gas. One has to introduce a new type of decay for functions of loops which compared with the classical counterparts are no more monotonically decreasing functions of the distance between the particles but have more sophisticated structure [7]. The decay of correlations are given in terms of bounds on two-point truncated correlation functions which are used to get the large volume asymptotic expansion of the log-partition function of a gas of interacting Brownian loops confined to

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a bounded domain. In the two-dimensional case we obtain the volume, the boundary and the constant term which is proportional to the Euler-Poincare characteristics.

## 2 Loop model

We work in the grand-canonical ensemble formalism where the parameters are the fugacity $z > 0$ and the inverse temperature $\beta > 0$. Let $\mathcal{X}_j = \{x \in \mathcal{C}([0, j\beta], \mathbb{R}^d) \mid x(0) = x(j\beta),\ j = 1, 2 \cdots\}$ be the space of Brownian loops of "length" $j$ in d-dimensional Euclidean space $\mathbb{R}^d$. The elements of $\mathcal{X}_j$ we call simple loops if $j = 1$ and composite loops if $j > 1$. We set $|X| = j$ if $X \in \mathcal{X}_j$ and we use the same notation $|\cdot|$ for the number of elements in a finite set. We will call $x \in \mathcal{C}([0, \beta], \mathbb{R}^d)$ an elementary constituent of a composite loop $X,\ x \in X$, if for all $t \in [0, \beta],\ x(t) = X(t + i\beta)$ for some $i$, $0 \le i < |X|$.

The underlying one particle space $\mathcal{X}$ is defined as a topological sum of the Polish spaces $\mathcal{X}_j$:

$$\mathcal{X} = \bigcup_{j=1}^{\infty} \mathcal{X}_j.$$

Let $\mathcal{X}^0 \subset \mathcal{X}$ be the set of loops which start and end at the origin $0 \in \mathbb{R}^d$. The subset of loops from $\mathcal{X}^0$ with a fixed length $j$ we denote by $\mathcal{X}_j^0$. In $\mathcal{X}_j^0$ we consider the measure $\frac{1}{j} P_j^0$ where $P_j^0$ is a non-normalized Brownian bridge measure with total mass $P_j^0\left(\mathcal{X}_j^0\right) = (\pi j \beta)^{-d/2}$. Using a natural bijection $\tau_j : \mathcal{X}_j^0 \times \mathbb{R}^d \to \mathcal{X}_j$ given by $\tau_j(X^0, u) = X^0 + u$ we define on $\mathcal{X}_j$ a $\sigma$-finite measure $\mu_j = (\lambda \times \frac{1}{j} P_j^0) \circ \tau_j^{-1}$. On $\mathcal{X}$ we will consider two measures $\mu = \sum_{j=1}^{\infty} \mu_j$ and $\bar{\mu}_z = \sum_{j=1}^{\infty} z^j \mu_j$. Hereafter we assume that $z \le 1$.

Let

$$\mathcal{M} = \mathcal{M}(\mathcal{X}) = \{\omega \subset \mathcal{X} \mid |\omega| < \infty\}$$

be the space of finite configurations of loops in $\mathbb{R}^d$. There is a natural $\sigma$-algebra in $\mathcal{M}(\mathcal{X})$ (see [7]) which we denote by $\mathcal{B}(\mathcal{X})$. Given any measure $\nu$ on $\mathcal{X}$ which is diffuse, i.e. $\nu(x) = 0,\ \forall\ x \in \mathcal{X}$, we define a measure $W_\nu$ on $\mathcal{M}(\mathcal{X})$ by

$$\int_{\mathcal{M}} W_\nu(d\omega) h(\omega) = \sum_{n=0}^{\infty} \frac{1}{n!} \int_{\mathcal{X}^n} h\left(X_1, \ldots, X_n\right) \nu(dX_1) \cdots \nu(dX_n) \tag{2.1}$$

where $h$ is any non-negative, measurable function on $\mathcal{M}$.

The energy $U(\omega)$ of a configuration $\omega \in \mathcal{M}$ is given by

$$U(\omega) = \sum_{X \in \omega} U_1(X) + \frac{1}{2} \sum_{X,Y \in \omega; X \neq Y} U_2(X,Y) \tag{2.2}$$

where the self interaction

$$U_1(X) = \frac{1}{2} \sum_{x,y \in X, x \neq y} \int_0^\beta dt \Phi(x(t) - y(t), \text{ if} |X| > 1,\ U_1(X) = 0 \text{ if } |X| = 1$$

and the interaction between loops $X$ and $Y$,

$$U_2(X,Y) = \sum_{x \in X, y \in Y} \int_0^\beta dt \Phi(x(t) - y(t)). \tag{2.3}$$

We assume that
(a) $\Phi : \mathbb{R}^d \to \mathbb{R}^1$ is an even function,continuous outside of the origin,
(b) $\Phi$ is stable with stability constant $B \geq 0$,
(c) $\Phi$ has power decay at infinity:

$$||\Phi_l||_1 = \int_{\mathbb{R}^d} du |\Phi_l(u)| < \infty,\ \Phi_l(u) = \Phi(u)(1 + |u|)^l,\ l \geq 0.$$

For a given bounded domain $\Lambda \subset \mathbb{R}^d$ let

$$\mathcal{X}(\Lambda) = \{X \in \mathcal{X} \mid X(t) \in \Lambda, \forall t \in [0, \beta|X|]\}$$

be the set of loops in $\Lambda$ and $\mathcal{M}(\mathcal{X}(\Lambda))$ be the set of finite configurations of loops in $\Lambda$. Let $W_{\bar{\mu}_{z,\Lambda}}$ be the restriction of the measure $W_{\bar{\mu}_z}$ to the subspace $\mathcal{M}(\mathcal{X}(\Lambda))$.

The triple $(\mathcal{M}(\mathcal{X}(\Lambda)), \mathcal{B}(\mathcal{X}(\Lambda)), W_{\bar{\mu}_{z,\Lambda}})$ we call a loop gas in $\Lambda$ with Bose-Einstein (BE) statistics, interaction $\Phi$ and parameters $z$ and $\beta$, in the case of Maxwell-Boltzmann (MB) statistics we have $(\mathcal{M}(\mathcal{X}_1(\Lambda)), \mathcal{B}(\mathcal{X}_1(\Lambda)), W_{z\bar{\mu}_{1,\Lambda}})$.

# 3 Decay of correlations

We recall that truncated correlation functions $\rho^T(\omega)$ at low fugacity $z$ are given by (see [9], [3])

$$\rho^T(\omega) = \prod_{X \in \omega} z^{|X|} e^{-U_1(X)} \int_{\mathcal{M}} W_{\bar{\mu}_z}(d\bar{\omega}) \varphi(\omega, \bar{\omega}) \prod_{\bar{X} \in \bar{\omega}} e^{-U_1(\bar{X})} \tag{3.1}$$

where $(\omega, \bar{\omega})$ stands for $(\omega \cup \bar{\omega})$ and $\varphi$ is the Ursell function given by

$$\varphi(\omega) = \sum_{G \in \mathcal{C}_{|\omega|}} \prod_{\{X,Y\} \in G} (e^{-U_2(X,Y)} - 1), \text{if } |\omega| \geq 2; \;\; \varphi(\omega) = 1, \text{if } |\omega| = 1. \quad (3.2)$$

Here $\mathcal{C}_n$ is the set of connected graphs with n vertices and the product is over all edges of $G$.

The theorems 1, 2, and 3 below hold true for the MB statistics. For details see [8].

**Theorem 1.** If $\Phi$ satisfies the conditions (a) - (c) then for all $z$ from the interval

$$0 < z < [2^l e^{4\beta B+1} \beta (\pi\beta)^{-\frac{d}{2}} ||\Phi_l||_1]^{-1}$$

there exists a constant $C = C(\Phi, \beta, z, l)$ such that

$$\int_{\mathcal{X}_1^0} P_1^0(dX) \int_{\mathbb{R}^d} dv \int_{\mathcal{X}_1^0} P_1^0(dY) 1_{\mathcal{X}^c(B_R(0))}(Y+v) \rho^T(X, Y+v) \leq \frac{C}{(1+R)^l}$$

Here $A^c$ denotes the complement of a set $A$ and $B_R(0)$ is a ball of radius $R$ centered at the origin.

In the case of BE statistics a similar bound holds true with natural modifications, for slightly different $z$-interval and for $\Phi \geq 0$.

For $n$-point truncated correlation functions, $n \geq 1$, we have

**Theorem 2.** If $\Phi$ satisfies the (a) - (c) with $l = 0$ then for

$$z < [e^{4\beta B+1} \beta (\pi\beta)^{-\frac{d}{2}} ||\Phi||_1]^{-1} \quad (3.3)$$

the following bound holds true

$$|\rho^T(\omega)| \leq \frac{|\omega|!}{e(e^{2\beta B}+1)} \left[ \frac{z e^{2\beta B+1}(e^{2\beta B}+1)}{1 - z e^{4\beta B+1} \beta (2\pi\beta)^{-\frac{d}{2}} ||\Phi||_1} \right]^{|\omega|}, \;\; \omega \in \mathcal{M}. \quad (3.4)$$

# 4 Asymptotic Expansion of the Log-Partition function

We define grand canonical partition function $\Xi(\Lambda, z)$ for the loop gas in a bounded domain $\Lambda$ as usual by

$$\Xi(\Lambda, z) = \int_{\mathcal{M}(\mathcal{X}(\Lambda))} dW_{\bar{\mu}_z}(\omega) e^{-U(\omega)} \quad (4.1)$$

The bound for the two-point truncated correlation function from Theorem 1 allows to obtain the asymptotic expansion of the $\ln \Xi(\Lambda_R, z)$ as $R \to \infty$ where $\Lambda_R = \{Ru \,|\, u \in \Lambda\}$.

Let $\Lambda \subset \mathbb{R}^d$ be an open convex bounded subset with finitely many convex closed holes such that the connected parts of the boundary $\partial\Lambda$ of $\Lambda$ are $d-1$ dimensional closed $C^3$ manifolds.

We assume that $\Phi$ satisfies the conditions (a) - (c) and in addition is rotation invariant, differentiable, uniformly bounded together with its derivatives so that

$$|\Phi(u)| \leq M, \;\; |\nabla\Phi(u)| \leq M' \;, ||\nabla\Phi||_1 = \int_{\mathbb{R}^d} du, |\nabla\Phi(u)| < \infty. \tag{4.2}$$

**Theorem 3.** For all $z$ from the interval

$$0 < z < [2^l \beta e^{\beta B+1} \bar{\gamma} \max(M, ||\Phi_l||_1, ||\nabla\Phi||_1)]^{-1}$$

where $\bar{\gamma} = \max(\gamma_1, \gamma_2)$ with $\gamma_k = \int_{\mathcal{X}_1^0} P_1^0(dX)(\sup|X|)^k, \;\; k = 1, 2$, the following asymptotic expansion is valid

$$\ln\, Z(\Lambda_R, z) = R^d \beta p(\phi, z)|\Lambda| + R^{d-1}\, b(\phi, z)|\partial\,\Lambda| + R^{d-2} c_1(\Lambda) c_2(\phi, z) + o(R^{d-2}).$$

Here $|\Lambda|$ is the volume, $|\partial\,\Lambda|$ the surface measure of the boundary of $\Lambda$ and

$$c_1(\Lambda) = (d-1)\int_{\partial\,\Lambda} \sigma(dr) k_m(r)$$

where $k_m(r)$ is the mean curvature of $\partial\Lambda$ at the point $r \in \partial\Lambda$.

The coefficients $p(\phi, z)$, $b(\phi, z)$ and $c_2(\phi, z)$ can be explicitly expressed as functional integrals and are analytic functions of the activity $z$ in a neighborhood of the origin; $p(\phi, z)$ can be interpreted as the pressure and $b(\phi, z)$ as the surface tension.

We note that in two dimensional case $c_1(\Lambda)$ is the Euler-Poincare characteristic of the domain $\Lambda$.

One can recover the large volume asymptotic expansion of Brownian integrals [4] (asymptotic expansion of $\ln_{id}\, Z(\Lambda_R, z$ for the ideal loop gas) as a special case of Theorem 3 by setting $\Phi \equiv 0$.

At the same time, we are not able to get more terms of the expansion. This is a familiar case also for the ideal gas and it is not clear whether the reason is technical or not (cf. [4], section VII).

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