Reversible and irreversible random growth models

Vadim Shcherbakov*
Moscow State University

1 Introduction

A typical random growth model describes a Markov process whose state at time t, $X(t)$, is a subset either of lattice $\mathbf{Z}^d$ or continuous space $\mathbf{R}^d$. The models such as Eden model ([2]), Richardson model ([10], [1]), contact process ([5], [15]) are motivated by modeling biological growth or spread of epidemics. Another example of a random growth model is provided by cooperative sequential adsorption model (CSA) which is widely used in physics and chemistry for modelling various adsorption processes ([3]). In this note we announce new results (Section 2) and review some recent results (Sections 3 and 4) for several random growth models.

2 Equilibrium distributions of reversible growth models

Consider a finite lattice box $\Lambda \in \mathbf{Z}^d$ containing the origin and with a side length of L, where $L \geq 1$ is an integer. Denote $\Omega_{N,\Lambda} = \{0, 1, \dots, N\}^{\Lambda}$ and $\Omega_N = \{0, 1, \dots, N\}^{\mathbf{Z}^d}$, where $N \geq 1$ is a positive integer; $\Omega_{\Lambda} = \{0, 1, \dots, \}^{\Lambda}$ and $\Omega = \{0, 1, \dots, \}^{\mathbf{Z}^d}$. Let $\|y - x\|$ be the usual Euclidean distance between $x, y \in \mathbf{Z}^d$ and write $x \sim y$ for $x, y \in \mathbf{Z}^d$, if $\|x - y\| = 1$. Given configuration $\xi \in \Omega_{N,\Lambda}$ denote

$$n(x, \xi) = \xi_x + \sum_{y \sim x} \xi_y. \tag{1}$$

A quantity $\xi_x(t)$ is called spin and interpreted as a number of particles located at site $x \in \Lambda$ at time t. Call two particles neighbors if the distance between their locations is either 0 (i.e., both of them are located at the same site) or 1. Given configuration ξ denote

$$s(\xi) = \frac{1}{2} \sum_{x \in \Lambda} \xi_x(\xi_x - 1) + \sum_{x \sim y} \xi_x \xi_y \tag{2}$$

the total number of neighboring particles in the configuration, and

$$h(\xi) = \sum_{x \in \Lambda} \xi_x, \tag{3}$$

the total number of particles in the configuration ξ.

Consider a continuous-time Markov chain $\xi(t) = \{\xi_x(t),\, x \in \Lambda\} \in \Omega_{N,\Lambda}$ evolving as follows. Given a state $\xi(t) = \xi$,

*Laboratory of Large Random Systems, Faculty of Mechanics and Mathematics, Moscow State University, Moscow, Russia. Email address: v.shcherbakov@mech.math.msu.su

1. at rate 1 a spin $\xi_x > 0$ decreases by 1;

2. at rate $c(n(x,\xi))$ a spin $\xi_x < N$ increases by 1, where $c(k) > 0$, $k = 0, \ldots, (2d+1)N-1$, is a fixed set of positive coefficients.

The following statement is proved in [16] by applying the Kolmogorov's reversibility criteria.

Theorem 1 *The Markov chain $\xi(t)$ is time-reversible if and only if*

$$c(k) = a\gamma^k,\ k = 0, 1, \ldots, (2d+1)N - 1, \tag{4}$$

where γ and a are arbitrary positive numbers.

If $c(k) = a\gamma^k$, $k = 0, 1, \ldots, (2d+1)N - 1$, for some $\gamma > 0$ and $a > 0$, then probability measure

$$\mu_{\gamma,N,\Lambda}(\xi) = Z^{-1}_{\gamma,N,\Lambda} a^{h(\xi)}\gamma^{s(\xi)}, \quad \xi \in \Omega_{N,\Lambda}, \tag{5}$$

where $s(\xi)$ and $h(\xi)$ are defined by equations (2) and (3) respectively, and where

$$Z_{\gamma,N,\Lambda} = \sum_{\xi \in \Omega_{N,\Lambda}} a^{h(\xi)}\gamma^{s(\xi)},$$

is the stationary distribution of the Markov chain.

Without loss of generality we assume $a = 1$ in the rest of this section. It is clear that the interaction between particles is repulsive, when $0 < \gamma < 1$, and is attractive, when $\gamma > 1$. Also, if $\gamma = 1$, then the corresponding measure $\mu_{1,N,\Lambda}$ is the uniform distribution on $\Omega_{N,\Lambda}$. We are interested in the asymptotic behavior of the equilibrium measure $\mu_{\gamma,N,\Lambda}$ in the case of the attractive interaction.

Theorem 2 [16]. *For any $\gamma \geq 1$ there exists the limit measure*

$$\mu_{\gamma,N} = \lim_{\Lambda \uparrow \boldsymbol{Z}^d} \mu_{\gamma,N,\Lambda},$$

where convergence is understood in a sense of the weak convergence of the finite-dimensional distributions.

Percolation properties of the limit measure.

Definition 1 *Given a configuration $\xi \in \Omega_N$*

1. *we call a site $x \in \boldsymbol{Z}^d$ occupied, if $\xi_x > 0$, and call it empty otherwise;*

2. *a set of occupied sites $U = \{x, y, \ldots\}$ is called an occupied cluster, if for any $x', x'' \in U$, there exists a finite subset $\{y_1, \ldots, y_n\} \subseteq U$, such that $x' = y_1$, $y_n = x''$ and $y_i \sim y_{i+1}$, $i = 1, \ldots, n-1$.*

Theorem 3 [16]. *1) If $d \geq 2$ and $N \geq 3$, then*

$$\mu_{\gamma,N}\,(\textit{there exists a unique occupied cluster}) = 1$$

for any $\gamma \geq 1$.

2) If $d \geq 2$ and $N = 1$ or 2, then there exists a critical value $\gamma_c \geq 1$ such that

$$\mu_{\gamma,N}\,(\textit{there exists a unique occupied cluster}) = \begin{cases} 0, & \textit{if } \gamma < \gamma_c, \\ 1, & \textit{if } \gamma > \gamma_c. \end{cases}$$

The proofs of both Theorem 2 and Theorem 3 are based on the fact that the family of probability measures $\{\mu_{\gamma,N,\Lambda}\}$ posses certain stochastic monotonicity properties. For instance, if $1 \leq \gamma_1 \leq \gamma_2$, then probability measure $\mu_{\gamma_1,N,\Lambda}$ is stochastically dominated by $\mu_{\gamma_2,N,\Lambda}$.

3 Irreversible model for particle deposition

Let $\{1, 2, \ldots, M\}$ be a lattice segment with periodic boundary conditions, i.e. a one dimensional lattice torus with M points. Assume that $M \geq 3$. The growth process is a discrete-time Markov chain $\xi(t) = (\xi_1(t), \ldots, \xi_M(t))$, $t \in \mathbf{Z}_+ = \{0, 1, 2, \ldots\}$ with values in $\mathbf{Z}_+^M$, specified by the following transition probabilities:

$$\mathsf{P}\{\xi_i(t+1) = \xi_i(t) + 1,\ \xi_j(t+1) = \xi_j(t)\ \forall j \neq i | \xi(t)\} = \frac{\beta^{u_i(t)}}{\sum_{j=1}^{M} \beta^{u_j(t)}},$$

$$u_i(t) = \sum_{j \in U_i} \xi_j(t), \quad i = 1, 2, \ldots, M,$$

where $\beta > 0$ and U_i is a certain neighbourhood of site i.

Definition 2 *The quantity $u_i(t)$ is called a* potential *of site i at time t.*

We consider the following three possibilities for neighbourhood U_i: $U_i = \{i\}$ (no interaction); $U_i = \{i, i+1\}$ (asymmetric interaction); $U_i = \{i-1, i, i+1\}$ (symmetric interaction).

The question of interest is *stability* of the growth process. Loosely speaking, stability means that the "profile" $\xi_i(t)$, $i = 1, \ldots, M$, is "approximately flat", i.e. there are no extraordinary peaks. To describe this property in a formal way a process of differences $\zeta(t) = (\zeta_1(t), \ldots, \zeta_{M-1}(t)) \in \mathbf{Z}^{M-1}$, $t \in \mathbf{Z}_+$, is introduced ([13]), where

$$\zeta_i(t) = \xi_i(t) - \xi_M(t),\ i = 1, \ldots, M-1.$$

It is easy to see that $(\zeta(t),\, t \in \mathbf{Z}_+)$ is also a Markov chain.

Definition 3 *Say that the growth process is* stable *if the process of differences is an ergodic (positive recurrent) Markov chain. Otherwise the growth process is called* unstable.

Theorem 4 [13].

(I) Suppose $U_i = \{i\}$, $i = 1, \ldots, M$.

(1) If $0 < \beta < 1$, then Markov chain $(\zeta(t),\, t \in \boldsymbol{Z}_+)$ is ergodic.

(2) If $\beta > 1$, then Markov chain $(\zeta(t),\, t \in \boldsymbol{Z}_+)$ is transient.

(II) Suppose $U_i = \{i, i+1\}$, $i = 1, \ldots, M$.

(1) If $M = 3$ and $0 < \beta < 1$, then Markov chain $(\zeta(t),\, t \in \boldsymbol{Z}_+)$ is ergodic. Consequently, $\xi_1(t) = \xi_2(t) = \xi_3(t)$ for infinitely many t's almost surely.

(2) If $\beta > 1$, then Markov chain $(\zeta(t),\, t \in \boldsymbol{Z}_+)$ is transient.

(III) Suppose $U_i = \{i-1, i, i+1\}$, $i = 1, \ldots, M$. Then Markov chain $(\zeta(t),\, t \in \boldsymbol{Z}_+)$ is transient for any $M \geq 3$ and for any $\beta \in (0, 1) \cup (1, \infty)$. Moreover, if $\beta > 1$, then with probability 1 there is a $k \in \{1, \ldots, M\}$ such that

$$\lim_{t \to \infty} \xi_i(t) = \infty,\ \textit{if and only if } i \in \{k-1, k\},\ \textit{and} \lim_{t \to \infty} \frac{\xi_k(t)}{\xi_{k-1}(t)} = \beta^c,$$

where $c = \lim_{t \to \infty}[\xi_{k+1}(t) - \xi_{k-2}(t)] \in \boldsymbol{Z}$.

The proof of this theorem in [13] is a combination of constructive methods for studying the asymptotic behaviour of Markov chains ([4]) with the methods typical for studying the reinforced random processes ([17]).

The model with $\beta = 0$. Consider the following deposition rule. Given a configuration of sites potentials $u_i(t)$ at time t next particle is deposited at a site with the minimal potential. If there are more than one minimum, then a site with the minimal potential is chosen at random. This model can be regarded as the limit case of the previous model as $\beta \to 0$.

Theorem 5 [14].

(I) *Suppose $U_i = \{i, i+1\}$. Then, with probability 1, there is a $t_0 = t_0(\omega)$ (depending also on the initial configuration) such that for all $t \geq t_0$*

$$|\xi_i(t) - \xi_{i+2}(t)| \leq 2$$

for $i = 1, \ldots, M$. Moreover,

$$\xi_i(t) = \frac{t}{M} + \eta_i(t) + \begin{cases} 0, & M+1 \text{ is odd,} \\ (-1)^i\, Z(t), & M+1 \text{ is even,} \end{cases}$$

where $|\eta_i(t)| \leq 2M$ and for some $\sigma > 0$

$$\lim_{n\to\infty} \frac{Z(\lfloor sn \rfloor)}{\sigma\sqrt{n}} = B(s),$$

where $\lfloor x \rfloor$ denotes the integer part of x and $B(s)$ is a standard Brownian motion.

(II) *Suppose $U_i = \{i-1, i, i+1\}$. Then with probability 1 there exists the limit $\mathbf{x} = \lim_{t\to\infty} \xi(t)/t$, which takes a finite number of possible values with positive probabilities. The set of limiting configurations consists of those $\mathbf{x} = (x_1, \ldots, x_M) \in \boldsymbol{R}^M$ which simultaneously satisfy the following properties:*

- *there exists an $\alpha > 0$ such that $x_i \in \{0, \alpha/2, \alpha\}$ for all $i = 1, 2 \ldots, M$; also $\sum_{i=1}^M x_i = 1$;*
- *if $x_i = 0$, then $x_{i-1} > 0$ or $x_{i+1} > 0$, or both;*
- *if $x_i = \alpha/2$, then $(x_{j-3}, x_{j-2}, x_{j-1}, x_j, x_{j+1}, x_{j+2}) = (\alpha, 0, \alpha/2, \alpha/2, 0, \alpha)$, where $j \in \{i, i+1\}$;*
- *if $x_i = \alpha$, then $x_{i-1} = x_{i+1} = 0$;*
- *if $M = 3K$ is divisible by 3, then $\min\{x_j, x_{j+3}, x_{j+6}, \ldots, x_{j+3(K-1)}\} = 0$, for $j = 1, 2, 3$.*

Moreover, the adsorption eventually stops at all $i = 1, \ldots, M$ where $x_i = 0$, that is $\sup_{t\geq 0} \xi_i(t) = \infty$ if and only if $x_i > 0$. Additionally, if the initial configuration is empty, then for each $x_i = 0$ we must have that **both** *$x_{i-1} > 0$* **and** *$x_{i+1} > 0$.*

It should be noted that in contrast to [13] this statement is proved in [14] by the purely combinatorial methods.

4 On modelling spatial time series by CSA

Originally motivated by adsorption processes in physics and chemistry [3], CSA dynamics seem to be relevant to many applications. It was first noticed by physicists (e.g., see [3], p.1285) that this type of model can be used for modelling the spatial-temporal processes similar to the irreversible spread of disease or epidemics, and biological growth was mentioned in [3] as another potential application. The main peculiarity of CSA is that the likelihood for a point (particle) to appear at a given location in space depends on the number of previous particles of the series nearby; depending on the parameters, the new particle may be attracted to, or repelled by, previous particles.

Mathematically CSA is formulated as a random finite sequential allocation of particles in a bounded region of space (the observation window). A parametric family of CSA was introduced in [12] as follows. Consider a sequence of points $X_1, X_2, \ldots, X_\ell$ located randomly in a bounded convex region D of Euclidean space $\mathbf{R}^d$. Let parameters $R > 0$ (the interaction radius) and $c(i) \geq 0$ $(i = 0, 1, 2, \ldots)$ be fixed. Given the first k points $X_1, \ldots, X_k$, let the conditional probability density of X_{k+1} at $x \in D$ be proportional to $c(i)$, if x has i points among $X_1, \ldots, X_k$ within distance R of it. The special case of CSA with $c(0) = 1$ and $c(i) = 0$ for $i \geq 1$ is known as random sequential adsorption (RSA). RSA is the most popular sequential adsorption model in physics and serves as a benchmark for modelling various time irreversible processes.

It is argued in ([6], [7]) that CSA appear to be suitable for modeling sequential point patterns in disciplines such as ecology, biology and geophysics in situations, where a data set is presented by a sequential or ordered point pattern, i.e., a collection of *spatial events* which appear sequentially. Fitting the model to real-life data necessarily requires developing statistical inference for the model. Statistical inference of the model parameters is developed in [6] and [7] for CSA parametrized by a finite number of parameters. Namely, it is assumed in those papers that $c(k) > 0$, $k = 0, \ldots, N$, and $c(k) = 0$ for $k > N$, where $N \geq 0$ can also be unknown. The interaction radius R is assumed to be known (or already estimated). Statistical inference for the parameters $c(k)$, $k = 0, \ldots, N$ and N is based on maximum likelihood estimation.

Existence, uniqueness, consistency and asymptotic normality of MLE are proved in [6] and [7] under assumption that the amount of the observed information increases in the following natural sense. Namely, it is assumed that the observation window D expands to the whole space and the number of observed points grows linearly in the volume of D. This limit regime is known as the thermodynamic limit in statistical physics and as the increasing domain asymptotic framework in spatial statistics.

It should be noted that analysis of asymptotic properties of MLEs in [6], [7] is based on the observation that the MLE equations are determined by statistics of a special type, namely, sums of locally determined functionals over a configuration of points ([8]). This allows to combine classic Cramer's technique (originally developed for i.i.d. observations) with the modern limit theory for random sequential packing and deposition developed in [8].

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