Non-ergodicity with unique stationary state: how come?

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In this paper we will discuss the following question: are there stochastic dynamics with unique stationary state, which are not ergodic?

Without any further restrictions this question is easy and not very interesting, with the answer being YES. For the discrete time case it is given by the two-state Markov chain, $\sigma_\tau = \pm 1$, with degenerate transition probabilities,

$$p\left(\sigma_{\tau+1} = \mp 1 | \sigma_\tau = \pm 1\right) = 1. \tag{1}$$

For the continuous time case the corresponding (degenerate) process is the rotation over a circle with a constant speed v,

$$\sigma_\tau \in \mathbb{S}^1, \ \sigma_{\tau+t} = \sigma_\tau + vt \mod 2\pi. \tag{2}$$

One wonders whether one can construct non-degenerate random processes, which exhibit the above prototypical behavior. Of course, in order to be able to do this, one has to consider interacting infinite-volume systems – i.e. Probabilistic Cellular Automates (PCA) for the discrete time and Interacting Particle Systems (IPS) for the continuous time – since finite state non-degenerate Markov processes are always ergodic.

For the case of PCA-s such a construction, built upon (1), is presented in a recent paper by [CM]. One must add, however, that the example of [CM] still has some degeneracy. Namely, for any time T one can present two local events, A and B, such that the transition probability $p_T\left(A|B\right)$ in T steps vanishes. Thus, we feel that a truly non-degenerate example is still missing.

In what follows, we will discuss the nondegenerate realizations of the circle rotation prototype, The Rotating States. The first example with that flavour is described in [RSV]. There it was considered an infinite queuing network with several types of clients and with exponential service times. In the high load regime the system exhibits a coherent behavior. That means that if the initial state of the network is close to the 'coherent' one, characterized by a given value of the 'phase' observable (which takes values on the circle), then in the process of evolution this phase evolves with a constant speed, is never 'forgotten', and the initial synchronization is never broken. Yet the system has unique stationary state, with the phase in that state being uniformly distributed over the circle, which implies in particular that the coherent states never approach the stationary state. (In the language of the queuing networks we have thus an example of the Poisson hypothesis violation.)

The [RSV] example of the IPS is not yet a completely satisfactory example. First, it lives not on a lattice, but on a mean-field graph, which is, in some sense, an infinite complete graph. Next, this IPS has countably many states, and not just finitely many.

In the present paper we want to exhibit a non-degenerate finite state IPS on $\mathbb{Z}^3$ with nearest-neighbor interaction, which has unique stationary state, and which is non-ergodic. The reason we have to go to 3D lies in the fact that we will use the existence of magnetized phases in the XY-model at low temperatures, and this breaking of continuous symmetry starts only from dimension 3. The magnetized phases will play the role of the points of the circle, as in (2).

1 The N-Clock Models

The proposed IPS lives on $\mathbb{Z}^3$. At each site $t \in \mathbb{Z}^3$ there is a spin $\sigma_t \in \mathbb{Z}_N$, where $\mathbb{Z}_N \subset \mathbb{S}^1 \subset \mathbb{C}^1$ is the group of N-th roots of unity. Each spin σ_t has its clock, and when the clock rings, the spin jumps to one of the two 'nearest' values:

$$\sigma_t \to \sigma_t^{\pm} = \exp\left\{\pm\frac{2\pi i}{N}\right\}\sigma_t.$$

The particles σ_t are interacting, with the energy given by

$$H(\sigma) = -\sum_{s,t \text{ n.n.}} \cos(\sigma_s - \sigma_t). \tag{3}$$

Now we define the rates $c(\sigma, t, \pm)$ of the jumps $\sigma_t \to \sigma_t^{\pm}$ of the particle σ_t in the environment σ by

$$c(\sigma, t, \pm) = p_{\pm} \exp\left\{ \beta \sum_{s \text{ n.n. } t} \left[\cos\left(\sigma_s - \sigma_t^{\pm}\right) - \cos\left(\sigma_s - \sigma_t\right) \right] \right\}, \qquad (4)$$

where the numbers $p_+ \geq p_-$ are two extra non-negative parameters, with $p_+ + p_- = 1$. We will use the name 'drift' for the difference $d = p_+ - p_- \geq 0$. We will call the above model the 'N-Clock model' with a drift. We first describe the properties of the symmetric Clock model, when the drift $d = 0$. (Note that in this case the evolution defined is detailed balance.) These properties are given by the theorem of [FILS].

Theorem 1 ***(symmetric Clock model)*** *There exists a value β_0 of the inverse temperature, such that for every $\beta > \beta_0$, every $N \geq 2$, the symmetric N-Clock model has N different stationary states, $\langle \cdot \rangle_{k,\beta}$, $k = 1, ..., N$. These states are translation-invariant, exhibit long-range order and are magnetized:*

$$\langle \sigma_t \rangle_{k,\beta} = m_N(\beta) \frac{2\pi i k}{N}, \text{ with } m_N(\beta) > 0 \text{ for } \beta > \beta_0.$$

(Here we interpret the spins σ_t as elements of $\mathbb{C}^1$.)

The proof of this theorem uses the reflection-positivity and infrared bounds; it goes via the Gaussian domination.

On a first glance this theorem does not seem to be so surprising, since the N-Clock model has N ground state configurations, which satisfy the Peierls condition, and so it surely has N different low-temperature magnetized Gibbs states. (The case $N = 2$ is the well known Ising model.) However, the standard proof of this statement, via the Peierls argument or the Pirogov-Sinai (PS) theory [PS, S] would establish this fact only for the range of inverse temperatures $\beta \geq \beta_N^{PS}$, where $\beta_N^{PS} \to \infty$ as $N \to \infty$. The reason for this weakening of the result is not only technical, but lies in the heart of the PS method: when applicable, it does not only prove the stated phase diagram of the model in question, but also shows various properties of the pure phases, among which is the exponential decay of the truncated correlation functions. However, we believe that such exponential decay holds only for the inverse temperatures $\beta > \beta_N^{BKT}$, with $\beta_N^{BKT} \to \infty$ as $N \to \infty$. Moreover, we think that the following is true:

Conjecture 2 *There exists a value $\bar{N}$, such that for each $N \geq \bar{N}$ the 3D N-clock model undergoes two phase transitions. Namely, for all $\beta > \beta_N^{BKT}$ it has N pure magnetized phases, with exponential decay of truncated correlations, with $\beta_N^{BKT} \to \infty$ as $N \to \infty$. For smaller intermediate values of $\beta \in \left(\beta_N^{cr}, \beta_N^{BKT}\right)$ it also has at least N* ***(and may be even infinitely many)*** *pure phases with non-zero magnetization; however, the correlation decay in these phases is only algebraic. Finally, for $\beta < \beta_N^{cr}$ the model has only one Gibbs state, again with exponential decay of correlations.*

Note that by the Theorem 1 we have $\beta_N^{cr} < \beta_0$ for all N large enough, and so the interval of β, where the intermediate phases do exist, is non-empty. Our conjecture does not hold for $N = 2$; indeed, it is known from the paper [ABF] that there is no intermediate phases for the Ising model. The conjectured behavior is somewhat similar to the one taking place for the 2D Clock models; it was proven in [FS] that the 2D Clock models indeed undergo the Berezinsky-Kosterlitz-Thouless phase transition (hence our notation β_N^{BKT}).

Let us now discuss the situation with non-zero drift d, (then we do not have the detailed balance). For this let us denote by δ_k the measure that gives the weight 1 to the configuration $\sigma_t \equiv e^{2\pi ki/N} \in \mathbb{C}^1,\ k = 1, ..., N$. Let us run our N-Clock model with a drift for the time duration T, starting in one of the states δ_k. Let us denote the resulting state by $\langle \cdot \rangle_{\beta,d}^{k,T}$.

Conjecture 3 *For every $N \geq \bar{N}$, $\beta > \beta_N^{cr}$ there exists a critical value $d_{cr}(\beta, N)$ of the drift, $0 < d_{cr}(\beta, N) < 1$, such that the following holds:*

1. *if $d \leq d_{cr}(\beta, N)$, then the state $\langle \cdot \rangle_{\beta,d}^{k,T}$ goes to the limiting state $\langle \cdot \rangle_{\beta,d}^{k}$, as $T \to \infty$. The limiting state is magnetized: $\langle \sigma_0 \rangle_{\beta,d}^{k} \neq 0$;*

2. *if $d > d_{cr}(\beta, N)$, then the state $\langle \cdot \rangle_{\beta,d}^{k,T}$ is a 'rotating' state as $T \to \infty$ (in particular, it has no limit as $T \to \infty$). Namely, there exist the value $m = m(\beta, N, d) > 0$, the periodic function $\Phi(T) = \Phi(T; \beta, N, d)$, i.e.*

$$\Phi(T + \omega) = \Phi(T),$$

with period ω being the mean angular velocity, $\omega = \omega(\beta, N, d)$, and the phase shift $\varphi_k = \varphi_k(\beta, N, d)$, such that

$$\left| \langle \sigma_0 \rangle_{\beta,d}^{k,T} - m(\beta, N, d)\, e^{i(\Phi(T) + \varphi_k)} \right| \to 0 \text{ as } T \to \infty.$$

(Here we again are treating the spin σ_0 as belonging to $\mathbb{C}^1$.)

The next conjecture deals with the behavior of the critical drift.

Conjecture 4 *As the temperature increases, the critical drift decays. Let* $\beta_N^{drift} = \sup\{\beta : d_{cr}(\beta, N) > 0\}$. *Then*

$$\beta_{drift}(N) = \beta_N^{BKT}.$$

The rationale behind this conjecture is that at the temperatures above $\left(\beta_N^{BKT}\right)^{-1}$ the N-Clock model enters into the spin-wave Kosterlitz-Thouless phase and so qualitatively should behave like the XY-model, which is in the rotating phase for any non-zero value of drift, see below. This similarity of the intermediate phases with the XY-model is the basis of all our speculations. For the XY-model the analogs of the above statements are easy.

In simpler words, we think that for every β large enough there exists $N = N(\beta)$, such that for any $N \geq N(\beta)$ any $d > 0$ the N-Clock model with a drift d has a continuum of different rotating states.

One might wonder whether there is a difference between the structure of the stationary states in the PS regime and BKT regime, when $d > d_{cr}(\beta, N)$, i.e. when we are in the regime of rotating states. We expect the answer to be positive:

Conjecture 5

1. ***Rotating BKT.*** *In the regime* $\beta \in \left(\beta_N^{cr}, \beta_N^{BKT}\right)$, $d > 0$ *there is a unique stationary in time distribution,* $\langle\cdot\rangle_{\beta,d}^{st}$. *It is translation invariant and has zero magnetization.*

2. ***Rotating PS.*** *In the regime* $\beta > \beta_N^{BKT}$, $d > d_{cr}(\beta, N)$, *in addition to the time-stationary translation invariant state there are also time-stationary non-translation invariant states (the 'Dobrushin' states).*

The Dobrushin time-stationary non-translation invariant states have rigid interface. In a typical configuration drawn from such a state the spins on different sides of the interface are pointing in (approximately) opposite directions, though the direction itself can be arbitrary.

2 The 3D XY-rotators model

The dynamical XY-model with a drift – called dXY model below – can be obtained from the model (3), (4) by taking the limit $N \to \infty$, while rescaling the time by $t \to \frac{t}{N^2}$. Alternatively, it can be thought as a 3D model of Brownian motions φ_s on circles, $s \in \mathbb{Z}^3$, $\varphi_s \in \mathbb{S}^1 \subset \mathbb{C}^1$. The brownian motions φ_s have a constant drift, d, and they are interacting via the n.n. attraction (3) .

This driven XY-model can be studied due to the following simple statement.

Theorem 6 *There exists the angular speed $\varpi = \varpi(\beta, d)$, such that the evolution of the random variables $\psi_s(t) = \varphi_s(t) - \varpi t$ is that of the XY-model with zero drift.*

Therefore the stationary translation-invariant states of the dXY model with non-zero drift are translation-invariant Gibbs states of the XY-model, which are $\mathbb{S}^1$-invariant.

At low temperatures β^{-1} the 3D XY-model has continuum translation-invariant Gibbs states. They can be obtained as thermodynamic limits $\langle\cdot\rangle_{n,\beta}$ of the finite-volume Gibbs states with coherent boundary conditions $\bar{\varphi}_n \equiv n \in \mathbb{S}^1 \subset \mathbb{C}^1$. These translation invariant states have non-zero spontaneous magnetization,

$$\langle\varphi_0\rangle_{n,\beta} = m(\beta)\, n, \text{ with } m(\beta) > 0,$$

see [FSS]. One can construct thus a Gibbs field $\langle\cdot\rangle_{\mathbf{0},\beta}$, which is $\mathbb{S}^1$-invariant, by putting

$$\langle\cdot\rangle_{\mathbf{0},\beta} = \int_{\mathbb{S}^1} \langle\cdot\rangle_{n,\beta}\, dn.$$

We believe that the state $\langle\cdot\rangle_{\mathbf{0},\beta}$ is the only one translation-invariant Gibbs state of the XY-model, which is $\mathbb{S}^1$-invariant.

Theorem 7 *The stationary states of the XY-model correspond in an evident way to the rotating states of the dXY model. The rotating states do not converge to a stationary state of the dXY model.*

Proof. The proof is evident, since a stationary state of the dXY model has zero magnetization. ■

References

[ABF] M. Aizenman, D. J. Barsky and R. Fernández: The phase transition in a general class of Ising-type models is sharp. Journal of Statistical Physics, Volume 47, Numbers 3-4, 343-374, 1987.

[CM] Philippe Chassaing and Jean Mairesse: A non-ergodic probabilistic cellular automaton with a unique invariant measure, http://arxiv.org/abs/1009.0143.

[FSS] J. Fröhlich, B. Simon and T. Spencer: Infrared bounds, phase transitions and continuous symmetry breaking. Communications in Mathematical Physics, Volume 50, Number 1, 79-95, 1976.

[FILS] Jürg Fröhlich, Robert Israel, Elliot H. Lieb and Barry Simon: Phase transitions and reflection positivity. I. General theory and long range lattice models. Communications in Mathematical Physics, Volume 62, Number 1, 1-34, 1978.

[FS] Jürg Fröhlich and Thomas Spencer: The Kosterlitz-Thouless transition in two-dimensional Abelian spin systems and the Coulomb gas. Communications in Mathematical Physics, Volume 81, Number 4, 527-602, 1981.

[PS] S.A.Pirogov and Ya.G.Sinai, Phase Diagrams of Classical Lattice Systems, Theor. and Math. Phys. 25, 358-369, 1185-1192 (1975).

[RSV] Alexander Rybko, Senya Shlosman and Alexander Vladimirov: Spontaneous Resonances and the Coherent States of the Queuing Networks, Journal of Statistical Physics, Volume 134, Number 1, 67-104, 2009.

[S] Ya.G.Sinai, Theory of Phase Transitions, Budapest: Academia Kiado and London: Pergamon Press (1982).

[FP] Jurg Frohlich and Charles-Edouard Pfister: Spin Waves, Vortices, and the Structure of Equilibrium States in the Classical XY Model, Commun. Math. Phys. 89, 303-327 (1983).

[P] Pfister, C.-E.: Translation invariant equilibrium states of ferromagnetic abelian lattice systems. Commun. Math. Phys. 86, 375-390 (1982).

[SV] Senya Shlosman and Yvon Vignaud: Dobrushin Interfaces via Reflection Positivity, CMP Volume 276, Number 3, pp. 827-861, 2007.