Tiling by rectangles, discrete analytic functions, electrical networks

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The talk is on the following interrelated subjects: tilings of polygons by rectangles, discrete analytic functions, electrical networks. Celebrated physical interpretations of tilings by rectangles (R.L. Brooks, C.A.B. Smith, A.H. Stone, W.T. Tutte [2]) and discrete analytic functions on rhombic lattices (R.J. Duffin [7]) use direct-current networks. Our new approach is application of alternating-current networks and energy estimates for direct-current networks.

Previous work. Discrete analytic functions on a square lattice were introduced by R. Ph. Isaacs [13] and J. Ferrand [9]; see Figure 1a. They were studied further by R. J. Duffin [7] (cf. [14]) who also generalized the notion to *rhombic lattices*, i.e., planar graphs with rhombic faces; see Figure 1b. His version of the finite element method [6] is equivalent to discrete complex analysis on special "*kite lattices*". C. Mercat [16] studied discrete analytic functions in a more general context of *orthogonal lattices*, i.e., planar graphs having convex quadrilateral faces with orthogonal diagonals; see Figure 1c. Other discretizations of complex analysis were introduced by I. A. Dynnikov–S. P. Novikov [8] and A. I. Bobenko–C. Mercat–Y. B. Suris [1].

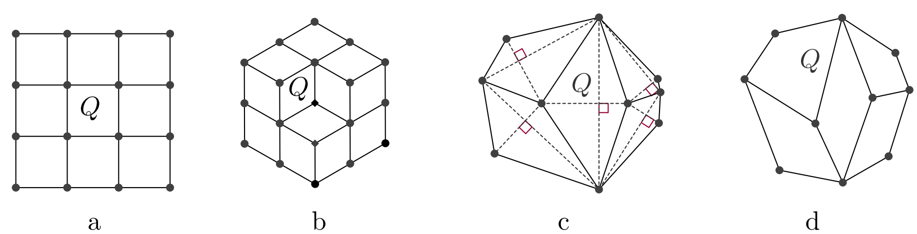

Figure 1: Examples of lattices Q: (a) a square lattice; (b) a rhombic lattice; (c) an orthogonal lattice; (d) a generic quadrilateral lattice.

Motivation. We develop complex analysis on arbitrary *quadrilateral lattices*, i.e., planar graphs whose bounded faces are convex quadrilaterals; see Figure 1d. This generalization is motivated by the following reasons:

- it provides a new approximation algorithm for numerical solution of the Dirichlet boundary value problem;
- it gives an approach to statistical physics models on more general lattices than the ones studied earlier;
- it has an interesting physical interpretation (alternating-current networks) degenerating (to direct-current networks) for the lattices studied earlier;
- it leads sometimes to easier proofs of known results, not relying on particular properties of the lattices.

Description of the results. We prove that the Dirichlet boundary value problem for the real part of a discrete analytic function on a quadrilateral lattice has a unique solution. Our main result is that in the case of orthogonal lattices this solution converges to a harmonic function in the scaling limit (under certain regularity assumptions); see Convergence Theorem below.

This was proved earlier for square lattices by R. Courant–K. Friedrichs–H. Lewy [5, §4], for special kite lattices implicitly by P.G. Ciarlet–P.-A. Raviart [4, Theorem 2], and for rhombic lattices by D. Chelkak–S. Smirnov [3, Proposition 3.3]. In concert we simplify the known proofs. Our result solves a problem of S. Smirnov [19, Question 1] on convergence of discrete holomorphic functions in a more general setup than rhombic lattices (although a less general result of P.G. Ciarlet–P.-A. Raviart is already sufficient for a solution).

Our result provides a new approximation algorithm for numerical solution of the Dirichlet boundary value problem; for other algorithms see [20, 12]. It also has probabilistic corollaries.

Statements. A complex-valued function f on the vertices of a quadrilateral lattice $Q \subset \mathbb{C}$ is *discrete analytic*, if

$$\frac{f(z_1) - f(z_3)}{z_1 - z_3} = \frac{f(z_2) - f(z_4)}{z_2 - z_4}$$

for each quadrilateral face $z_1 z_2 z_3 z_4$ with the vertices listed clockwise; see Figure 2. The real part of a discrete analytic function is called a *discrete harmonic function*.

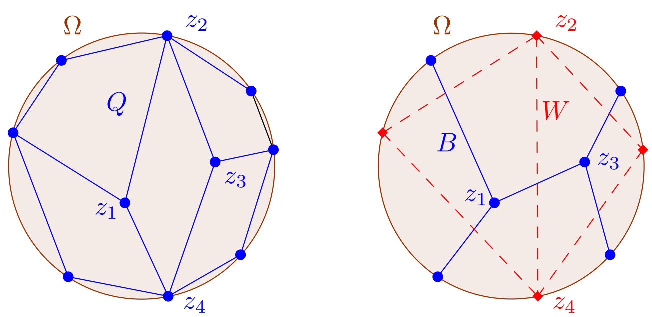

Figure 2: A quadrilateral lattice Q in a domain Ω and associated dual graphs B and W.

The *boundary* ∂Q of a quadrilateral lattice Q is the boundary of its outer face. Hereafter assume for simplicity that ∂Q is a closed curve without self-intersections. Denote by Q^0 the set of vertices of the graph Q.

Let $u\colon \mathbb{C} \to \mathbb{R}$ be a smooth function. The *Dirichlet (boundary value) problem on* Q is to find a discrete harmonic function $f = f_{Q,u}\colon Q^0 \to \mathbb{R}$ such that $f(z) = u(z)$ for each vertex $z \in \partial Q$. The function $f_{Q,u}\colon Q^0 \to \mathbb{R}$ is called a *solution* of the Dirichlet problem.

Uniqueness Theorem. *The Dirichlet boundary value problem on any finite quadrilateral lattice has a unique solution.*

Let $\Omega \subset \mathbb{C}$ be a domain. The *Dirichlet (boundary value) problem on* Ω is to find a continuous function $f = f_{\Omega,u}\colon \operatorname{Cl}\Omega \to \mathbb{R}$ harmonic in Ω and such that $f(z) = u(z)$ for each point $z \in \partial\Omega$. The harmonic function $f_{\Omega,u}\colon \Omega \to \mathbb{R}$ is called a *solution* of the Dirichlet problem.

Let $Q_1, Q_2, \dots \subset \operatorname{Cl}\Omega$ be a sequence of quadrilateral lattices. The *boundary gap* of Q_n is the maximal distance from a point of ∂Q_n to the set $\partial\Omega$. The *maximal* (*minimal*) *size* of the lattice Q_n is the maximal (minimal) length of its edges. The *eccentricity* of a quadrilateral is the maximum of secants of the angles between the sides and the diagonals. The *eccentricity* of the lattice Q_n is the maximum of its face eccentricities and of pairwise ratios of the edge lengths. A sequence of quadrilateral lattices $Q_1, Q_2, \dots$ is *nondegenerate*, if their eccentricities are bounded. A sequence of functions $f^n\colon Q_n^0 \to \mathbb{C}$ *converges* to a function $f\colon \Omega \to \mathbb{C}$ *uniformly on compact sets*, if for each compact set $K \subset \Omega$ we have $\max_{z\in K\cap Q_n^0} |f^n(z) - f(z)| \to 0$ as $n \to \infty$.

Convergence Theorem. *Let $\Omega \subset \mathbb{C}$ be a domain bounded by a smooth closed curve $\partial\Omega$ without self-intersections and let $u\colon \mathbb{C} \to \mathbb{R}$ be a smooth function. Take a nondegenerate sequence of finite orthogonal lattices $Q_1, Q_2, \dots \subset \operatorname{Cl}\Omega$ of both maximal size and boundary gap approaching zero. Then the solution $f_{Q_n,u}\colon Q_n^0 \to \mathbb{R}$ of the Dirichlet boundary value problem on Q_n converges to the solution $f_{\Omega,u}\colon \Omega \to \mathbb{R}$ of the Dirichlet boundary value problem on Ω uniformly on compact sets.*

Physical interpretation. Joining the opposite vertices in each quadrilateral face of Q, we get two connected graphs B and W; see Figure 2. *Conductance* (or *admittance*) of an edge $z_1z_3 \subset B$ is the complex number

$$c(z_1z_3) := i\frac{z_2 - z_4}{z_1 - z_3},$$

where $z_1z_2z_3z_4$ is a quadrilateral face of Q with the vertices listed clockwise. A graph B with prescribed edge conductances (having positive real parts) can be considered as an *alternating-current network*; see [18, §2.4] for the details, [10] for a survey, and [17] for an elementary introduction. The proof of main results is based on energy estimates inspired by alternating-current networks theory.

Application to tiling by rectangles. In the talk we are also going to discuss applications of alternating-current networks to tiling problems [18]. As an example, we sketch a new short proof of the following theorem by C. Freiling–M. Laczkovich–D. Rinne–G. Szekeres.

Shape Tiling Theorem. [11, 15] *For $c > 0$ the following 3 conditions are equivalent*:

- *a square can be tiled by similar rectangles of side ratio c*;
- *the number c is algebraic and all its algebraic conjugates have positive real parts*;

- *for certain positive rational numbers* $d_1, \ldots, d_m$ *we have*

$$d_1 c + \cfrac{1}{d_2 c + \cdots + \cfrac{1}{d_m c}} = 1.$$

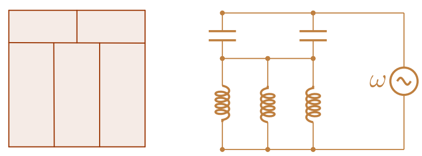

Figure 3: Correspondence between tilings by similar rectangles and LC alternating-current networks.

Idea of the proof. To prove the theorem, we give a one-to-one correspondence between tilings of a square by similar rectangles and LC alternating-current networks; see Figure 3. The Foster reactance theorem completely describes possible admittances $A(\omega)$ of such networks as functions in the frequency ω. This gives the required restriction on the side ratio c.

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