On the Vlasov-Maxwell equations, the Lagrange identity, Godunov's double-divergence form and critical mass value.

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We describe the derivation of Vlasov-Maxwell equation from classical Lagrangian, and a similar derivation of the Vlasov-Poisson-Poisson charged gravitating particles. We derive electromagnetic hydrodynamic equations and present them to the Godunov's double divergence form. For them we get generalized Lagrange identity and compare it. We analyzes the steady-state solutions of the Vlasov-Poisson-Poisson equation: their types changes at a certain critical mass $m^2 = e^2 / G$ having a clear physical meaning with different behavior of particles - recession or collapse trajectories.

I. Introduction.

Vlasov-Maxwell equation is fundamental for the description of plasma. Written by A.A. Vlasov in 1938 [1] it is increasingly used instead of the MHD equations (this transition is noticeable abroad, and is particularly strong in the U.S.) in the calculation of complex plasma problems due to the growth power of computers. But this is not the main reason - the main thing, apparently, is that it has "super-fundamental nature". Vlasov-type equations contain a solution of N-body problem for any N. This property makes it super-fundamental and noted by many authors. It was known to A.A. Vlasov, and N.N. Bogolyubov, in the preface to the book [2] notes: "The Vlasov equation is the foundation of plasma physics. It seems very significant that the Vlasov equation has microscopic solutions corresponding to exact solutions of classical mechanics." This property is used in the derivation from Bogolyubov's chain of equations [3] and in approximations using these microscopic solutions (in the form of a sum of Dirac delta functions - [4-9]). In the derivation of the Vlasov-Maxwell equations we can't use this approach. It is has the microscopic solutions, but no one used them, since the corresponding potential – Lienard-Wiechert potential [10]. It is desirable to derive this equation from an exact Lagrangian, and to circumvent the microscopic solutions, and understand the nature of the approximations, as well as the influence of relativistic expressions. The equations written in different ways in different guidelines, so the exact derivation is highly desirable. We derive it from the Lagrangian of classical electrodynamics [10] and follow [11].

The number of versions for magnetohydrodynamics (MHD) equations, even more than the equations of Vlasov type. By using an exact substitution we derive some versions of the equations of the electromagnetic hydrodynamics (EMHD - equations of MHD in the presence of an electric field [13]) from Vlasov-Maxwell equations and automatically obtain for them the Lagrange identity. The Lagrange identity is convenient here as a test to compare different forms of equations. In [12] V.V. Kozlov proved the Lagrange identity for equations of Vlasov type, and we study its form for different types of Vlasov's equation and the MHD.

We discuss exact solutions of the Vlasov- Poisson-Poisson system of equations. The last term we use for combination of electrostatic and gravitational forces. We derive it from the Lagrangian of classical electrodynamics plus Lagrangian of the gravitational interaction. Reduction of the stationary Vlasov equation leads us to a system of nonlinear elliptic equations. It turns out that if the value of a critical mass is $m^2 = e^2 / G$ type of these equations changes (G – is the gravitational constant and e – electron charge.

The first section is devoted to derivation of equations of the Vlasov-Maxwell and Vlasov-Poisson-Poisson, in second we study the Lagrange identity and the third is devoted to equations of the type EMHD.

II. Derivation of equations of the Vlasov-Maxwell and Vlasov-Maxwell-Poisson.

Under Vlasov equation simply imply the following equation for an arbitrary K(x, y) pair interaction potential of particles ([2-9]):

$$\frac{\partial F}{\partial t} + \left(v, \frac{\partial F}{\partial x} \right) - \left(\nabla_x \int K(x,y) F(y,v,t) dv dy, \frac{\partial F}{\partial v} \right) = 0$$

Let us consider the substitution $F(t,v,x) = \sum_{i=1}^{N} \rho_i \, \delta(v - V_i(t)) \, \delta(x - X_i(t))$.

Here $\delta(x)$ – Dirac delta function, $X_i(t)$ and $V_i(t)$ – a function of time (co-ordinates and velocity of particles), $\rho_i > 0$ – numbers (the weight of the particles). A.A.Vlasov already knew that such a substitution takes place if $X_i(t)$ and $V_i(t)$ satisfy N-body equations of motion [2], [4-11]:

$$\begin{cases} \dot{X}_i = V_i, \\ \dot{V}_i = -\sum_{j=1}^{N} \nabla_1 K(X_i, X_j) \, \rho_j, \end{cases}$$

Where ∇_1 – is a gradient vector over the first argument. Such solutions are called microscopic (or substitution in the form of a finite number of particles, or substitution of a sum of Dirac delta functions). In the original A.A. Vlasov work [1], it was introduced what they now called the Vlasov-Maxwell systems. It was introduced properly on time, specifically to describe the plasma, and shows its effectiveness on the example of small oscillations of the plasma. Their simplest derivation from classical Lagrangian is highly desirable, it is provides us a firm basis: for the classification of equations with the same name; to assess their validity; the nature of the approximations made by various authors. It should be noted that the derivation from Bogolyubov's chains [3] or Brawn-Hepps-Maslov- Neunzert-Dobrushin method [4-9] in terms of microscopic-solutions are not so direct, and for the Vlasov-Maxwell is inappropriate.

The system of Vlasov-Maxwell equations describes the motion of particles in self-consistent electromagnetic field. It can be obtained from the Lagrangian of classical electrodynamics. We start with the usual action of the electromagnetic field ([10], [18], the action of Lorentz-Shwartzchield as it explained in W.Pauli book [18]) (repeated upper and lower indices are summed over):

$$S_L = S_{VM} = -\sum_\alpha m_\alpha c^2 \sum_q \int_0^T \sqrt{g_{\mu\nu}\dot{X}^\mu_\alpha(q,t)\dot{X}^\nu_\alpha(q,t)}dt + \tag{1}$$

$$+\sum_\alpha \frac{e_\alpha}{c}\sum_q \int_0^T A_\mu\left(X_\alpha(q,t),t\right)\dot{X}^\mu_\alpha(q,t)dt + \frac{1}{16\pi c}\int F_{\mu\nu}F^{\mu\nu}d^4x = S_{\text{p}} + S_{\text{p-f}} + S_{\text{f}}$$

S_{p} – is a particle action, S_{f} – is a field action, $S_{\text{p-f}}$ – is a particles-fields action.

Here α – is a sort of particles, which is characterized by mass m_α and charge e_α; q – numerate particles inside each sort; $X^\mu_\alpha(q,t)(\mu = 0,1,2,3; q = 1,2...,N_\alpha; \alpha = 1,2...,r)$ – are four-coordinates of q-particle in α-sort; $A_\mu(x)$ – are four-potential; $F_{\mu\nu} = \partial_\nu A_\mu - \partial_\mu A_\nu$ – electromagnetic fields; $g_{\mu\nu}$ – is a Minkowski metric: $g_{\mu\nu}$ = diag(1,-1,-1,-1). We have to seek a variation in a special way [10-18]: first we obtain the particle motion in a field $\delta(S_{\text{p}} + S_{\text{p-f}}) = 0$, than evolution of fields with a given particles motion $\delta(S_{\text{p-f}} + S_{\text{f}}) = 0$. However, for particles, we proceed to distribution functions that give us the desired system of equations.

1) The variation $S_{\text{p}} + S_{\text{p-f}}$ gives us an equation of charges motion in the field:

$$\frac{dp_{\alpha i}}{dt} = -\frac{e_\alpha}{c}\frac{\partial A_i}{\partial t} - \frac{\partial A_0}{\partial x^i}e_\alpha - \frac{e_\alpha}{c}F_{ij}\dot{x}^j_\alpha\ ,\ p_{\alpha i} = \frac{\partial L_p}{\partial x^i_\alpha} = \frac{m_\alpha \dot{x}_{\alpha i}}{\sqrt{1-\dot{x}^2_\alpha / c^2}}, F_{ij} = \frac{\partial A_i}{\partial x^j} - \frac{\partial A_j}{\partial x^i}$$

2) Here, we cease to follow [10], and proceed to the distribution function. The equation for the distribution function is obtained as the equation of translation along the trajectories of the resulting dynamic system of charges in the field. It is seen that is convenient to take the distribution function of the impulse (instead of velocity). It should express velocity through impulses:

$$p_i = \frac{mv_i}{\sqrt{1-v^2/c^2}} \Rightarrow p^2 = \frac{m^2v^2}{1-v^2/c^2}.$$

Let $1-(v^2/c^2) = \gamma^{-2}$, and $\gamma^{-2} = 1 + p^2/(m^2c^2)$ and $v_i = p_i/(\gamma m)$. Hence we can find the equation for the distribution function $f_\alpha(x,p,t)$:

$$\frac{\partial f_\alpha}{\partial t} + \left(v_\alpha, \frac{\partial f_\alpha}{\partial x}\right) + \left(-\frac{e_\alpha}{c}\frac{\partial A_i}{\partial t} - e_\alpha\frac{\partial A_0}{\partial x^i} - \frac{e_\alpha}{c}F_{ij}v^j_\alpha\right)\frac{\partial f_\alpha}{\partial p_i} = 0. \tag{2}$$

Where $v_{\alpha j} = \frac{p_j}{m_\alpha}\frac{1}{\sqrt{1+p^2/(m^2_\alpha c^2)}}$ and we use that $\text{div}_p(F^i_j v^j) = 0$.

3) Fields equations. Follow [10], but now we use the distribution function instead of density.

If $\delta(S_{\text{p-f}} + S_{\text{p}}) = 0$ we have

$$\partial_\mu F^{\mu\nu} = -\frac{4\pi}{c}\sum_\alpha e_\alpha \int v^\mu_\alpha f_\alpha(x,p)d^3p. \tag{3}$$

System of equations (2),(3) is Vlasov-Maxwell [1], [17] with some small adjustments: we have explicite expression of velocities over momentum.

Similarly, we can derive the system of equations of the Vlasov-Poisson with gravitation in the nonrelativistic case. The Lagrangian of electrostatics derived from the general Lagrangian (1), and gravitation part is derived by analogy with electrostatics. At the same time we also check the constants in the original Lagrangian (1). So, in the nonrelativistic case:

$$S=\sum_{\alpha,q}\frac{m_\alpha \dot{x}_\alpha^2(q,t)}{2}-\sum_\alpha e_\alpha\int\varphi(x,t)\,f_\alpha(x,p,t)dxdpdt-\sum_\alpha m_\alpha\int U(x,t)\,f_\alpha(x,p,t)dxdpdt+\frac{1}{8\pi}\int(\nabla\varphi)^2dxdt-\frac{1}{8\pi G}\int(\nabla U)^2dxdt$$

Varying this expression as before we obtain a system of Vlasov-Poisson-Poisson plasma with gravitation:

$$\frac{\partial f_\alpha}{\partial t}+\left(\frac{p}{m_\alpha},\frac{\partial f_\alpha}{\partial x}\right)-\left(m_\alpha\frac{\partial U}{\partial x}+e_\alpha\frac{\partial\varphi}{\partial x},\frac{\partial f_\alpha}{\partial p_i}\right)=0, \tag{4}$$

$$\Delta U=4\pi G\sum_\alpha m_\alpha\int f_\alpha(x,p,t)dp\,,\quad \Delta\varphi=-4\pi\sum_\alpha e_\alpha\int f_\alpha(x,p,t)dp\,.$$

III. Lagrange identity.

As shown, a complete system of Vlasov-Maxwell equations is obtained by varying the action of electro-magnetism with the transition to the distribution function:

$$\frac{\partial f_\alpha}{\partial t}+\left(v_\alpha,\frac{\partial f_\alpha}{\partial x}\right)+\left(-\frac{e_\alpha}{c}\frac{\partial A_i}{\partial t}-e_\alpha\frac{\partial A_0}{\partial x^i}-\frac{e_\alpha}{c}F_{ij}v_\alpha^j\right)\frac{\partial f_\alpha}{\partial p_i}=0\,. \tag{5}$$

$$\frac{\partial F^{\mu\nu}}{\partial x_\nu}=-\frac{4\pi}{c}\Sigma_\alpha e_\alpha\int v_\alpha^\mu f_\alpha(t,x,p)dp\,,\; F_{\mu\nu}=\frac{\partial A_\mu}{\partial x_\mu}-\frac{\partial A_\nu}{\partial x_\nu}\,,(\mu,\nu:1,\ldots,4),$$

$$E_i=-\frac{1}{c}\frac{\partial A_i}{\partial t}-\frac{\partial A_0}{\partial x^i},\quad [v_\alpha,H]=-F_{ij}v_\alpha^j,\; v_\alpha=\frac{p}{m_\alpha\gamma_\alpha},\;\gamma_\alpha=\sqrt{1+\frac{p^2}{m_\alpha^2c^2}}.$$

Lagrange's identity is defined as the second time derivative of the moment of inertia through the kinetic and potential energy. Following to [12] we show that the Lagrange identity can be extended to the case of Vlasov-Maxwell equations. Let us introduce the moment of inertia of the particles respect to origin of coordinates:

$$I(t)=\sum_\alpha\int f_\alpha(t,x,p)x^2d^3pd^3x,$$

And functional

$$T(t)=\frac{1}{2}\sum_\alpha\int f_\alpha(t,x,p)v_\alpha^2d^3pd^3x,$$

$$\Pi=\sum_\alpha\int\frac{e_\alpha}{\gamma_\alpha m_\alpha}\left(x,E+\frac{1}{c}[v_\alpha,H]\right)f_\alpha d^3pd^3x-\sum_\alpha\int\frac{e_\alpha}{\gamma_\alpha^3m_\alpha^3c^2}(p,x)(p,E)\,f_\alpha d^3pd^3x.$$

Lagrange identity is valid as:

$$\ddot{I}=4T-2\Pi. \tag{6}$$

Prove.

$$\ddot{I}=-2\sum_\alpha\int\left(\frac{\partial f_\alpha}{\partial x},v_\alpha\right)(v_\alpha,x)d^3pd^3x-2\sum_\alpha\int(v_\alpha,x)e_\alpha\left(E+\frac{1}{c}[v_{\alpha,}H]\right)\frac{\partial f_\alpha}{\partial p_i}d^3pd^3x.$$

The first integral in this expression with integrating by parts can be transformed to

$$2\sum_\alpha\int(v_\alpha,v_\alpha)\,f_\alpha d^3pd^3x=4T.$$

The second integral can be transformed, if we count

$$\frac{\partial v_i}{\partial p_j},\text{ где }v_i=\frac{p_i}{m_\alpha\gamma_\alpha},\;\gamma_\alpha=\sqrt{1+\frac{p^2}{m_\alpha^2c^2}},\;\frac{\partial v_i}{\partial p_j}=\frac{\delta_{ij}}{\gamma_\alpha m_\alpha}-p_j\frac{p_i}{\gamma_\alpha^3m_\alpha^3c^2}\;\Rightarrow\;F_{ij}\frac{\partial v_j}{\partial p_i}=0$$

is the convolution of the symmetric and skew tensors. Then we can get:

$$-2\sum_\alpha\int\frac{\partial v_{\alpha j}}{\partial p_i}x_je_\alpha\left(-\frac{1}{c}\frac{\partial A_i}{\partial t}-\frac{\partial A_0}{\partial x^i}-\frac{1}{c}F_{ij}v_\alpha^j\right)f_\alpha d^3pd^3x=$$

$$=-2\sum_\alpha\int\left(\frac{\delta_{ij}}{\gamma_\alpha m_\alpha}-\frac{p_ip_j}{\gamma_\alpha^3m_\alpha^3c^2}\right)x_je_\alpha\left(-\frac{1}{c}\frac{\partial A_i}{\partial t}-\frac{\partial A_0}{\partial x^i}-\frac{1}{c}F_{ij}v_\alpha^j\right)f_\alpha d^3pd^3x=$$

$$=-2\sum_\alpha\int\frac{e_\alpha}{\gamma_\alpha m_\alpha}\left(x,E+\frac{1}{c}[v_{\alpha,}H]\right)f_\alpha d^3pd^3x+2\sum_\alpha\int\frac{p_ip_j}{\gamma_\alpha^3m_\alpha^3c^2}x_je_\alpha E_i\,f_\alpha d^3pd^3x=-2\Pi$$

because $p_iF_{ij}v_\alpha^i=0$. And the second term is transformed into: $+2\sum_\alpha\int\frac{e_\alpha}{\gamma_\alpha^3m_\alpha^3c^2}(p,x)(p,E)\,f_\alpha d^3pd^3x$

IV. On the Vlasov-Maxwell equations and the Lagrange identity.

There are many forms of equations of magnetohydrodynamics (MHD): various authors understand by this completely different system of equations [13-17]. In deriving the equations of MHD they add usually [13-17] the collision integral and obtain the equations of hydrodynamic type (with non-zero temperature) using the procedure of Maxwell-Chapman-Enskog. This is called the derivation of the MHD or EMHD, but the approximate equations are obtained. Derivation of the EMHD equations with zero temperature (by the substitution of a delta-function multiplied by the density – see below), is known (see, eg, [7], where it is connected with Lagrangian submanifolds). We adapted it for the classification and determine the nature of the approximations by comparing it with the derivation of the equations with non-zero temper-ature.

1. We first consider the case of zero temperature, to obtain the corresponding equations of the exact consequences of Vlasov-Maxwell equations by using the following substitution:

$$f_\alpha(t,x,p)=n_\alpha(x,t)\delta(p-P_\alpha(x,t)) \qquad (7)$$

This is the ultimate form of Maxwell distribution when temperature $T_\alpha \to 0$

$f_\alpha(t,x,p)=\dfrac{n_\alpha(x,t)}{\left(2k\pi T_\alpha m_\alpha\right)^{\frac{3}{2}}}e^{-\frac{(p-P_\alpha)^2}{2kT_\alpha m_\alpha}}\xrightarrow[T_\alpha\to 0]{} n_\alpha(x,t)\delta(p-P_\alpha(x,t))$. Where k – is Boltzmann's constant and T_α – is a temperature of α -component. We obtain the equations of the electromagnetic form of multifluid hydrodynamics (EMHD – so now became known as the MHD equations in the presence of an electric field). These equations have the form

$$\frac{\partial n_\alpha}{\partial t}+\mathbf{div}(v_\alpha(P_\alpha)n_\alpha)=0;\ \frac{\partial P_\alpha}{\partial t}+v_\alpha^i(P_\alpha)\frac{\partial P_\alpha}{\partial x_i}-\left(E+\frac{1}{c}[v_\alpha(P_\alpha),H]\right)=0; \qquad (8)$$

Here $v_\alpha^i=\mathrm{P}_\alpha\dfrac{\gamma_\alpha}{m_\alpha},\quad \gamma_\alpha=\sqrt{1+\dfrac{\mathrm{P}_\alpha^{\ 2}}{m_\alpha^2c^2}}$. These equations are supplemented by Maxwell's equations

$$\nabla\times E-\frac{\partial B}{\partial t}=0;\quad \nabla\cdot H=0;\ \nabla\cdot E=4\pi\sum_\alpha e_\alpha n_\alpha;\ \nabla\times H-\frac{1}{c}\frac{\partial E}{\partial t}=-\frac{4\pi}{c}\sum_\alpha e_\alpha n_\alpha v_\alpha(P_\alpha); \qquad (9)$$

We should notice that these equations are exact consequences of Vlasov-Maxwell equations, so the Lagrange identity obtained for the system (8) – (9) by substituting (7) in the Lagrange identity (6):

$$\ddot{I}_2=4T_2-2\Pi_2, \qquad (10)$$

$$I_2(t)=\sum_\alpha\int n_\alpha(x,t)x^2d^3x,\quad T_2(t)=\frac{1}{2}\sum_\alpha\int n_\alpha(x,t)v_\alpha^2(P_\alpha(x,t))d^3x,$$

$$\Pi_2=\sum_\alpha\int\frac{e_\alpha}{\gamma_\alpha m_\alpha}n_\alpha(x,t)\left(x,E+\frac{1}{c}\left[v_\alpha(P_\alpha),H\right]\right)d^3x-\sum_\alpha\int\frac{e_\alpha}{\gamma_\alpha^3m_\alpha^3c^2}n_a(x,t)(P_\alpha,x)(P_\alpha,E)d^3x.$$

There is a generalization of Lagrange's identity when, instead of an arbitrary function of x^2 is taken arbitrary function $\varphi(x)$

$$I(t)=\sum_\alpha\int f_\alpha\varphi(x)d^3pd^3x,$$

For this functional from the Vlasov-Maxwell system (5) we have:

$$\ddot{I}=\sum_\alpha\int\frac{\partial\varphi}{\partial x_i\partial x_j}v_{\alpha i}v_{\alpha j}f_\alpha d^3pd^3x+\sum_\alpha\int\frac{e_\alpha}{\gamma_\alpha m_\alpha}\left(\frac{\partial\varphi}{\partial x},E+\frac{1}{c}\left[v_\alpha,H\right]\right)f_\alpha d^3pd^3x-$$

$$-\sum_\alpha\int\frac{e_\alpha}{\gamma_\alpha^3m_\alpha^3c^2}(p,\frac{\partial\varphi}{\partial x})(p,E)f_\alpha d^3pd^3x.$$

3. Two-fluid and regular MHD (or EMGD) with non-zero temperature have a lot of modifications [13–16]. Usually obtained from the equations of hydrodynamic type from system of kinetic equations, by introducing the following momentums and integrating the Vlasov-Maxwell system:

$$n_\alpha=\int f_\alpha\left(t,x,p\right)d^3p,\ \ P_{\alpha i}=\frac{1}{n_\alpha}\int p_if_\alpha\left(t,x,p\right)d^3p,\ D_\alpha=\frac{1}{n_\alpha}\int\left(p-P_\alpha\right)^2f_\alpha\left(t,x,p\right)d^3p, \qquad (11)$$

Here $n_\alpha(x,t)$ – density numbers of particles of α -sorts, $P_{\alpha i}(x,t)$ – mathematical expectation of impulse, $D_\alpha(x,t)$ – variance of the impulses of all particles of each kind, which is proportional to the energy of random motion.

$$\frac{\partial n_\alpha}{\partial t}+\frac{\partial}{\partial x}(n_\alpha v_\alpha)=0,\ \frac{\partial}{\partial t}(n_\alpha P_\alpha)+\frac{\partial}{\partial x_i}(n_\alpha P_{\alpha i}P_{\alpha j}+\sigma_{\alpha ij})-n_\alpha e_\alpha\left(\vec{\mathbf{E}}+\frac{1}{c}[\vec{v}_\alpha,\vec{\mathbf{H}}]\right)=0,$$

$$\sigma_{\alpha ij}=\int(p_i-P_{\alpha i})(p_j-P_{\alpha j})f_\alpha(t,x,p)d^3p \text{ – stress tensor. } \frac{\partial}{\partial t}(n_\alpha D_\alpha)+\frac{\partial}{\partial x_i}q_i=0, \quad (14)$$

$$q_i=\int\frac{p_i}{m}(p-P_\alpha)^2 f_\alpha(t,x,p)d^3p \text{ – heat flow vector.}$$

In this system, the first equation represents the continuity equation; the second is the equation of motion for momentum, and the last equation for the energy of random motion (or for temperature). This precise system of equations, but it is not closed. To close it, we must add the collision integral, or (from interaction with the environment) add a linear collision integral. This means that higher-order momentums are determined through the lower with the Maxwell distribution.

$$f_\alpha(t,x,p)=\frac{n_\alpha(x,t)}{(2k\pi T_\alpha m_\alpha)^{\frac{3}{2}}}e^{-\frac{(p-P_\alpha)^2}{2kT_\alpha m_\alpha}},$$

It turns out that $\sigma_{\alpha ij}=\delta_{ij}kn_\alpha T_\alpha,\ D_\alpha=3kT_\alpha.$ More briefly those equations can be written in the Godunov's form, for this we introduce Godunov's function:

$$G^\alpha(\beta^\alpha_\mu)=\int f^0_\alpha(\beta^\alpha_\mu)d^3p,\quad \mu=(0,...,4), \quad (12)$$

$$f^0_\alpha(\beta^\alpha_\mu)=\exp[\beta^\alpha_0+\beta^\alpha_1 p_1+\beta^\alpha_2 p_2+\beta^\alpha_3 p_3+\beta^\alpha_4 p^2],$$

Compare $f^0_\alpha(\beta^\alpha_\mu)=\exp\left[\beta_0-\frac{\beta_1^2+\beta_2^2+\beta_3^2}{4\beta_4}\right]\exp\left[\beta_4\left(p+\frac{\beta}{2\beta_4}\right)^2\right],\ \beta=(\beta_1,\beta_2,\beta_3)$ with expression

$f_\alpha(t,x,P)=\frac{n_\alpha(x,t)}{(2k\pi T_\alpha m_\alpha)^{\frac{3}{2}}}\exp\left[-\frac{(p-P_\alpha)^2}{2kT_\alpha m_\alpha}\right]$ we obtain β_μ in terms of thermodynamic variables:

$$\beta^\alpha_0=\ln n_\alpha-\frac{3}{2}\ln(2\pi kT_\alpha m_\alpha)-\frac{P_\alpha^2}{2kT_\alpha},\ \ \beta^\alpha_1 p_1=\frac{P_{\alpha 1}}{kT_\alpha},\ \ \beta^\alpha_2 p_2=\frac{P_{\alpha 2}}{kT_\alpha},\ \ \beta^\alpha_3 p_2=\frac{P_{\alpha 3}}{kT_\alpha},\ \ \beta^\alpha_4 p^2=-\frac{1}{2kT_\alpha m_\alpha},$$

$$K^\alpha_\mu=\left(0,F^\alpha_1 n_\alpha,F^\alpha_2 n_\alpha,F^\alpha_3 n_\alpha,-2F^\alpha_i G^\alpha_{\beta_i}\right),\ F^\alpha=e_\alpha(E+[v_\alpha,H]),\quad i=(1,2,3),$$

and the equation with a non-zero temperatures in the zero approximation can be written in the form of Godunov:

$$\frac{\partial G^\alpha_{\beta_\mu}}{\partial t}+\frac{\partial G^\alpha_{\beta_\mu\beta_i}}{\partial x_i}+K^\alpha_\mu=0,\ \text{здесь}\ G^\alpha_{\beta_\mu}=\frac{\partial G^\alpha}{\partial\beta_\mu}.$$

A generalization of Lagrange's identity in this case has the following remarkable representation:

$$I(t)=\sum_\alpha\int f^0_\alpha(\beta^\alpha_\mu)\varphi(x)d^3p d^3x,\quad \ddot{I}=\sum_\alpha\int\frac{\partial\varphi}{\partial x_i\partial x_j}G^\alpha_{\beta_i}G^\alpha_{\beta_j}d^3x-\int\frac{\partial\varphi}{\partial x_i}G^\alpha F^\alpha_i d^3x.$$

4 Steady-state solutions and critical mass value

As shown, a complete system of Vlasov-Poisson-Poisson can be obtained from Lagrangian of the electrostatic plus gravitation (in nonrelativistic case) with the transition to the distribution function. Now we investigate the possible stationary solutions for (4). Assume that the distribution functions f_α are different functions of energy and are as follows:

$f_\alpha=g_\alpha(\frac{p^2}{2m_\alpha}+m_\alpha U+e_\alpha\varphi).$ Here g_α are arbitrary nonnegative function of proper energies. In this case the first of equations of the original system (4) are satisfied and we obtain a system of nonlinear elliptic equations for potentials $U(x)$ and $\phi(x)$:

$$\Delta U=V(U,\varphi),\ \text{ here }\ V(U,\varphi)=4\pi G\sum_{\alpha=1}^{N}m_\alpha\int g_\alpha(\frac{p^2}{2m_\alpha}+m_\alpha U+e_\alpha\varphi)d^3p,$$

$$\Delta\varphi = \Psi(U,\varphi)\,, \quad \text{and} \quad \Psi(U,\varphi) = -4\pi\sum_{\alpha=1}^{N} e_\alpha \int g_\alpha\left(\frac{p^2}{2m_\alpha} + m_\alpha U + e_\alpha\varphi\right)d^3p\,.$$

We investigate this system of equations. Let's start with the simplest case of one type of particles, when $N = 1$ and

$$\Delta U = 4\pi Gm\int g\left(\frac{p^2}{2m} + mU + e\varphi\right)d^3p\,, \qquad \Delta\varphi = -4\pi e\int g\left(\frac{p^2}{2m} + mU + e\varphi\right)d^3p$$

Therefore the system can be rewritten as

$$\Delta\left(mU + e\varphi\right) = (Gm^2 - e^2)\int g\left(\frac{p^2}{2m} + mU + e\varphi\right)d^3p\,, \quad \Delta\left(eU + Gm\varphi\right) = 0\,.$$

It turns out that the conditions for the solvability of the first equation, depends on the sign of expression $Gm^2 - e^2$. If this value is positive, the boundary problem is correct, otherwise there are global solutions (see [11, 20, 21]). Thus, the value of the mass $m^2 = e^2 / G$ – is critical. When $m^2 = e^2 / G$, the gravitational force stronger than the electrostatic forces. If e – electron charge, then this mass is $m \approx 10^{-12}$ grams.

If we consider Maxwell distribution function $f_\alpha = A\exp\left[-\beta\left(\frac{p^2}{2m} + mU + e\varphi\right)\right]$ than:

$$\Delta\left(mU + e\varphi\right) = (Gm^2 - e^2)B\exp\left[-\beta\left(mU + e\varphi\right)\right], \qquad \beta > 0,\ \ B > 0\,, \qquad \Delta\left(eU + Gm\varphi\right) = 0$$

This equation in two dimensions has a large group of symmetries (conformal group), on the basis of which it was solved by Liouville (see, for example, [11]). For our analysis of the trajectories is sufficient to consider one-dimensional case. Studying this system we find that when $Gm^2 - e^2 > 0$ the potential $mU + e\varphi$ is convex downward and particle trajectories are bounded, otherwise it is concave and the trajectories of the particles scatter. When $Gm^2 - e^2 = 0$ potentials $mU,\ e\varphi$ – are linear functions. V.A. Dorodnitsyn proposed, that this property could be used in technical improvement of car amd other sprays by choosing optimal size of particles (V.A.Dorodnitsyn spray).

Conclusion

We considered the derivation of the Vlasov-Maxwell from classical Lagrangian of electrodynamics and the Lagrange identity. This derivation is a convenient alternative to the methods of the BBGKY hierarchy [3] and the microscopic solutions methods [4–9], because it is simplier, can be used for important case of Vlasov Maxwell where other methods does not work, and give us classification of different types of equations of Vlasov type. We propose a derivation of MHD- and EMHD-type equations, for which variety only increases, and it allows us to monitor for the nature of the approximations made. We also examined the derivation of the Vlasov-Poisson-Poisson plasma with gravitation. Study of stationary solutions of these equations in the cases, where the distribution function is an arbitrary function of the energy integral show us that in this case the problem reduces to the elliptic system of nonlinear equations with different behavior.

Several new problems arises. One can ask on the existence and uniqueness theorems for equations obtained in analogy with the results [4-9]. In particular we can ask on the use of Kantorovich – Rubinstein metrics for proof of Dobrushin type uniqueness [8]. They could ask on the existence and uniqueness theorem for stationary solutions as in [20-21] for more general cases of this paper. Vlasov-type equations are more and more useful in different applications and theier role will be only increase.

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