PERSISTENT MASSIVE ATTRACTORS OF SMOOTH MAPS

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Consider a smooth map F of a compact manifold X (possibly with boundary) into itself. Discrete-time dynamical system is the semigroup generated by F. In the theory of dynamical systems, we are interested in the limit behavior of the orbits. In a wide class of dynamical systems, one can find a proper subset $\Lambda \subset X$ and an open set $B \supset \Lambda$ such that the trajectories of almost every point in B accumulate (in some sense) onto Λ. Then Λ is said to be an *attractor* and B to be its basin. Various formalizations of the word "accumulate" give rise to different (usually non-equivalent) definitions of attractors. The proper identification of an attractor is very important in applications because it is often possible to reduce the dimensions or size of the system by restricting it to its attractor.

We do not consider the trivial case, $\Lambda = X$. In this situation, the notion of attractor fails to provide any additional information on the system. In non-trivial case Λ can be as small as a single attracting fixed point of F. Now we address the following natural question: if $\Lambda \neq X$, how *big* can be Λ?

There are many ways to tell what is "big". The incomplete list of approaches includes

- having full Hausdorff dimension
- having positive Lebesgue measure
- having non-empty interior

Given full freedom, one can construct dynamical systems with very weird behavior. Our motivation comes from physics, thus we are interested only in (counter)examples which admit some sort of non-degeneracy.

This topic has been very hot in the recent years. Abdenur, Bonatti, Díaz [1] conjecture that C^1-generic diffeomorphisms whose non-wandering set has a non-empty interior are (globally) transitive. They give the proofs for three cases: hyperbolic diffeomorphisms, partially

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hyperbolic diffeomorphisms with two hyperbolic bundles, and tame diffeomorphisms. They mention that in the first case, the conjecture is folklore; in the second one, they adapt the proof of Brin [4]. There are more of interesting examples given by Fisher [5].

The big drawback of these examples is that none of them actually provide an *attractor*. Classical result by Bowen [3] states that every *hyperbolic* attractor of a diffeomorphism has zero Lebesgue measure. But in the non-hyperbolic settings the question is still open. Ilyashenko [8] showed there exists an open set in the space of boundary preserving step skew products (cocycles) over a full shift with a fiber $[0, 1]$, such that any map in this set has a big attractor: it has positive but not full Lebesgue measure.

In the realm of smooth many-to-one maps, the situation is even more interesting. The model examples of such maps are skew products over expanding circle maps. Tsujii, Avila, Gouëzel [12], [2] and Rams [10] show that in certain families of such skew products the attractor may stably carry an invariant SRB measure which is absolutely continuous w.r.t. Lebesgue measure. Thus the attractor is big.

Our example shows big attractors robustly appear in much wider class.

Definition 1. We say that F has *a massive attractor* if

(1) it has an ergodic hyperbolic invariant SRB measure μ;
(2) $\operatorname{supp}\mu \neq X$;
(3) $\operatorname{supp}\mu$ is a maximal attractor of its neighborhood;
(4) $\operatorname{supp}\mu$ is a closure of a bounded non-empty connected open set in X.

Now let M be any smooth manifold, and fix any $m \geq 3$. The dimension of M and m are the main parameters of our construction. Let $S^1 = \mathbb{R}/\mathbb{Z}$ be the standard circle, $X := S^1 \times M$. Denote by $C^1(X)$ the space of C^1-smooth m-to-1 coverings of X by itself, with C^1-topology.

Theorem 2. *There exists a nonempty open set $U \subset C^1(X)$ such that each $F \in U$ has a massive attractor $\mu(F)$.*

The attractor is not only stable. Due to its hyperbolicity and classical result by Ruelle [11], $\mu(F)$ depends on F in a differentiable way.

Our proof is based on the theory of partially hyperbolic perturbations started by Hirsch, Pugh, Shub [7], and continued by Gorodetski [6] and Ilyashenko, Negut [9]. We construct a special skew product over an expanding circle map, and show that every smooth map which is C^1-close to it has a massive attractor.

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