

# Solutions of the Schlesinger equation in a neighborhood of a singular point\*

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## Abstract

A local behavior of solutions of the Schlesinger equation is studied. We obtain expansions for this solutions, which converge in some neighborhood of a singular point. As a corollary the similar result for the sixth Painlevé equation was obtained. In our analysis, we use the isomonodromic approach to solve this problem.

## 1 Introduction

We study a local behavior of solutions of the Schlesinger equation. We present solutions of this equation in the form of power series or logarithmic-power series. This series are converge in some neighborhood of a singular point. As a corollary we obtain a similar result for description of the behavior of solutions of the sixth Painlevé equation in some sectorial neighborhood. We use the isomonodromic approach to solve this problem.

Let us consider the following system of analytical partial differential equations

$$dB_i = - \sum_{j=1, j \neq i}^n \frac{[B_i, B_j]}{a_i - a_j} d(a_i - a_j), \quad i = 1, \dots, n, \quad (1)$$

where  $B_i$  ( $i = 1, \dots, n$ ) — are analytical  $p \times p$ -matrix functions of the variable  $a = (a_1, \dots, a_n)$ ,  $[B_i, B_j]$  denotes the commutator of matrices  $B_i$  and  $B_j$ . The matrix-functions  $B_i(a)$  are defined and meromorphic (see B. Malgrange [1], R. Gontsov and I. Vyugin [4]) on the space

$$\{a \mid a = (a_1, \dots, a_n) \in \overline{\mathbb{C}}^n \setminus \bigcup_{i,j} A_{ij}\}, \quad A_{ij} = \{a \mid a_i = a_j\}.$$

This system is called *Schlesinger equation* (read more in A.A. Bolibruch [7]). Divisor of the Schlesinger equation is the following set

$$\Omega = \bigcup_{i,j} A_{ij}. \quad (2)$$

We are going to describe a local form of solutions of Schlesinger equation (1) in a neighborhood of the point  $a^0 = (a_1^0, \dots, a_n^0)$ , which belongs to the following singular set

$$a^0 \in \Omega' = \Omega \setminus \left( \bigcup_{i,j,k} A_{ijk} \right), \quad A_{ijk} = \{a \mid a_i = a_j = a_k\}. \quad (3)$$

We obtain the local expansions of the solutions of the system (1) in the form of power and logarithmic-power series of  $(a_s - a_r)$  (if  $a^0 \in A_{sr}$ ), which converges in some neighborhood of the point  $z = a^0$  (the first version of these results see [6]). These series have terms of complex degrees.

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**Theorem 1.** Any solution of two dimensional Schlesinger equation (1) can be represented in the neighborhood of a point  $a^0 = (a_1^0, \dots, a_n^0) \in \Omega'$ , where  $a_r^0 = a_s^0$ ,  $r \neq s$ , in one of two following forms:

- $b_{kl}^i(a) = F_1(a) + (a_s - a_r)^\varphi F_2(a) + (a_s - a_r)^{-\varphi} F_3(a)$ ,  $\varphi \in \mathbb{C}$  in the general case;
- $b_{kl}^i(a) = F_1(a) + F_2(a) \ln(a_s - a_r) + F_3(a) \ln^2(a_s - a_r)$  in the degenerate case,

where  $F_1^{kli}(a), F_2^{kli}(a), F_3^{kli}(a)$  are meromorphic (holomorphic in the generic case) functions,  $i = 1, \dots, n$ , and  $k, l \in \{1, 2\}$ .

The notions of “general case” and “non-general case” are explained below. Notice that the measure of the systems of non-general case is equal to zero.

Now consider the case  $n = 4$ ,  $p = 2$ , which is equivalent to case of the sixth Painlevé equation (4). Without loss of generality, let us fix three variable  $a_1 = 0$ ,  $a_2 = 1$ ,  $a_3 = \infty$  and denote  $a_4$  by  $t$ . We obtain the system of ordinary differential equations with variable  $t$  and unknown matrix-functions

$$B_i(t) = \begin{pmatrix} b_{11}^i(t) & b_{12}^i(t) \\ b_{21}^i(t) & b_{22}^i(t) \end{pmatrix}, \quad i = 0, t, 1, \infty.$$

With restrictions above the following corollary holds.

**Corollary 1.** Any solution of the Schlesinger equation under the above constraints can be represented in the neighborhood of  $t = 0$  in one of two forms:

- $b_{kl}^i(t) = F_1^{kli}(t) + t^\varphi F_2^{kli}(t) + t^{-\varphi} F_3^{kli}(t)$ ,  $\varphi \in \mathbb{C}$  in the general case;
- $b_{kl}^i(t) = F_1^{kli}(t) + F_2^{kli}(t) \ln t + F_3^{kli}(t) \ln^2 t$  in the degenerate case,

where  $F_1^{kli}(t), F_2^{kli}(t), F_3^{kli}(t)$  are meromorphic in  $t = 0$  functions,  $i = 0, t, 1, \infty$ , and  $k, l \in \{1, 2\}$ .

Note that the well-known sixth Painlevé equation

$$\begin{aligned} \frac{d^2 w}{dt^2} = & \frac{1}{2} \left( \frac{1}{w} + \frac{1}{w-1} + \frac{1}{w-t} \right) \left( \frac{dw}{dt} \right)^2 - \left( \frac{1}{t} + \frac{1}{t-1} + \frac{1}{w-t} \right) \frac{dw}{dt} + \\ & + \frac{w(w-1)(w-t)}{t^2(t-1)^2} \left( \alpha + \beta \frac{t}{w^2} + \gamma \frac{t-1}{(w-1)^2} + \delta \frac{t(t-1)}{(w-t)^2} \right), \end{aligned} \quad (4)$$

$\alpha, \beta, \gamma, \delta \in \mathbb{C}$  is equivalent to the system (1), where

$$w(t) = \frac{tb_{12}^0}{(t+1)b_{12}^0 + tb_{12}^1 + b_{12}^t}. \quad (5)$$

Corollary 1 and (5) give the power expansions for solutions of the sixth Painlevé equation. A different asymptotics for sixth Painlevé equation was obtained in D. Guzzetti [2], A. Bruno and I. Goryuchkina [5], M. Mazzocco [3] and others.

For the sixth Painlevé equation, we have an analogue of Corollary 1.

**Corollary 2.** Any solution  $w(t)$  of sixth Painlevé equation (4) in the intersection of the given sector for  $t$  sufficiently close to singular point  $t = 0, 1, \infty$  can be represented as a converged power series or as a converged logarithmic-power series:

- if  $G_1 G_\infty$  is diagonalizable, then  $w(t) = S(t, t^\lambda, t^{-\lambda})$ , where  $\lambda = \lambda(\alpha, \beta, \gamma, \delta, t_0, w(t_0), w'(t_0))$  can be found approximately;
- if  $G_1 G_\infty$  is a Jordan block, then  $w(t) = S(t, \ln t)$ .

## 2 Schlesinger equation and isomonodromic deformations

In this section we give a description of the Schlesinger equation (1) as an isomonodromy condition for a family of Fuchsian systems. Let us consider a Fuchsian system

$$\frac{dy}{dz} = \left( \sum_{i=1}^n \frac{B_i^0}{z - a_i^0} \right) y, \quad B_i^0 \in \text{Mat}_{p \times p}(\mathbb{C}), \quad y(z) \in \mathbb{C}^p. \quad (6)$$

The family of such systems

$$\frac{dy}{dz} = \left( \sum_{i=1}^n \frac{B_i(a)}{z - a_i} \right) y \quad (7)$$

is called *isomonodromic* if the following conditions hold:

- $B_i(a)$  are continuous matrix-functions of  $a = (a_1, \dots, a_n)$ ;
- The Fuchsian system (7) with any fixed  $a$  has fixed monodromy representation

$$\chi : \pi_1(\overline{\mathbb{C}} \setminus \{a_1, \dots, a_n\}, z_0) \longrightarrow GL(p, \mathbb{C}).$$

*Schlesinger isomonodromic family* is a family defined by the equation (1). An isomonodromic fundamental matrix  $Y(z, a)$  of the Schlesinger isomonodromic family (7) satisfies the following condition

$$Y(\infty, a) \equiv Y(\infty, a_0).$$

The initial data of such family are the coefficients  $B_i(a^0) = B_i^0$ ,  $i = 1, \dots, n$  of system (6). It is known that the solutions of Schlesinger equation are meromorphic functions on the space  $a \in \overline{\mathbb{C}}^n \setminus \Omega$ .

Let us consider the Painlevé VI case ( $n = 4, p = 2, a_1 = 0, a_2 = 1, a_3 = \infty, a_4 = t$ ). Usually the following family

$$\frac{dy}{dz} = \left( \frac{B_0(t)}{z} + \frac{B_t(t)}{z - t} + \frac{B_1(t)}{z - 1} \right) y \quad (8)$$

is considered, where

$$\text{tr}B_0 = \text{tr}B_t = \text{tr}B_1 = \text{tr}B_\infty = 0, \quad B_\infty = -(B_0 + B_t + B_1),$$

and the matrices  $B_0, B_t, B_1, B_\infty = \text{diag}(\delta, -\delta)$  are diagonalizable.

The formula (5) gives a solution of sixth Painlevé equation (4) with the following constants

$$\alpha = \frac{(2\lambda_\infty - 1)^2}{2}, \quad \beta = -2\lambda_0^2, \quad \gamma = 2\lambda_1^2, \quad \delta = \frac{1}{2} - 2\lambda_t^2,$$

where  $\lambda_0, \lambda_t, \lambda_1, \lambda_\infty$  are eigenvalues of matrices  $B_0, B_t, B_1, B_\infty$ .

## 3 Sketch of the proof

At first, we study the simplest isomonodromic family. Consider the following family of Fuchsian systems

$$\frac{dy}{dz} = \left( \frac{B_0}{z} + \frac{B_t}{z - t} \right) y, \quad B_0, B_t \equiv \text{const}. \quad (9)$$

It is easy to see that it is an isomonodromic family. The systems of this family are mutually equivalent by a linear mapping of  $z$ . This family is non-Schlesinger if  $B_\infty = -B_0 - B_t \neq 0$ , but this family can be

transformed to a Schlesinger family by the gauge transformation  $\tilde{y} = t^{B_\infty} y$ . The transformed family has the following form

$$\frac{dy}{dz} = \left( \frac{B'_0(t)}{z} + \frac{B'_t(t)}{z-t} \right) y, \quad B'_i = t^{B_\infty} B_i t^{-B_\infty}, \quad i = 0, t. \quad (10)$$

We call this family *canonical normalized family*.

Let us write the coefficients  $B'_0, B'_t$  explicitly. There are two cases: in the first case when the matrix  $B_\infty$  is diagonalizable, and in the second case  $B_\infty$  is a Jordan block:

- First case,  $B_\infty = \begin{pmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{pmatrix}$  and  $B'_i = \begin{pmatrix} b_{11}^i & b_{12}^i t^{\lambda_1 - \lambda_2} \\ b_{21}^i t^{\lambda_2 - \lambda_1} & b_{22}^i \end{pmatrix}$ .
- Second case,  $B_\infty = \begin{pmatrix} \lambda & 1 \\ 0 & \lambda \end{pmatrix}$  and  $B'_i = \begin{pmatrix} b_{11}^i + b_{21}^i \ln t & b_{12}^i + (b_{22}^i - b_{11}^i) \ln t - b_{21}^i \ln^2 t \\ b_{21}^i & b_{22}^i - b_{21}^i \ln t \end{pmatrix}$ .

Now we study the limit of the family (10) as  $t \rightarrow 0$ . We would like to find a limit Fuchsian system. For the existence of this limit, we impose the following condition on the real part of  $\lambda_1 - \lambda_2$

$$|\operatorname{Re}(\lambda_1 - \lambda_2)| < 1. \quad (11)$$

The condition (11) implies the following equation (see [7])

$$\frac{B'_0(t)}{z} + \frac{B'_t(t)}{z-t} = t^{B_\infty} \left( \frac{B_0}{z} + \frac{B_t}{z-t} \right) t^{-B_\infty} = t^{B_\infty} \left( \frac{B_0 + B_t}{z} + O(t) \right) t^{-B_\infty} = \frac{B_\infty}{z} + o(1).$$

The limit system as  $t \rightarrow 0$  of the family (10) under the condition (11) is

$$\frac{dy}{dz} = \frac{B_\infty}{z} y. \quad (12)$$

The proof is similar to that given in A.A. Bolibruch [7].

By the Riemann–Hilbert theory gives that for almost all monodromy representations (8) with generators  $G'_0, G'_1, G'_\infty$  there exists a Fuchsian system (9) with this monodromy data and given asymptotics (see [8]). In all other cases, we can construct such system with one regular singular point (see [9]). This cases were called in Theorem 1 “general case” and “non-general case”.

We will use the family (10) for the proof of the theorem 1.

Let us consider a family of holomorphic vector bundles with logarithmic connection having the following description

$$(F_t, \nabla_t) = (D_0, D_\infty, g_{0\infty}^t(z), \omega_0^t, \omega_\infty),$$

where  $D_0, D_\infty$  — circles with centers 0 and  $\infty$ , which has a nonempty intersection  $K = D_0 \cap D_\infty$ ,  $g_{0\infty}^t(z)$  is a holomorphic cocycle  $g_{0\infty}^t(z) : K \rightarrow \operatorname{GL}(p, \mathbb{C})$  and  $\omega_0^t, \omega_\infty$  is a differential 1-forms of logarithmic connection  $\nabla_t$ .

Define the pairs  $(F_t, \nabla_t)$  by the following description:

- $\omega_\infty$  is a 1-form of coefficients of the initial system (8), when  $t = t_0$ , which has the monodromy representation (8) and generators  $G_0, G_t, G_1, G_\infty$ ;
- $\omega_0^t$  is a 1-form of coefficients of system (10), with monodromy (8) with generators  $G'_0, G'_t, G'_\infty$  ( $G'_0 = G_0, G'_t = G_t, G'_\infty = G_1 G_\infty$ );
- cocycle  $g_{0\infty}^t(z)$  is a ratio  $g_{0\infty}^t(z) = Y_0^t(z) Y_\infty^{-1}(z)$ , where  $Y_0^t(z)$  and  $Y_\infty(z)$  are fundamental matrices of the systems

$$dy = \omega_0^t y, \quad dy = \omega_\infty y,$$

normalized in  $z = \infty$ .

**Proposition 1.** *Assume that the family*

$$(F_t, \nabla_t) = (D_0, D_\infty, g_{0\infty}^t(z), \omega_0^t, \omega_\infty^t),$$

*holomorphically depends on the valuable  $t$ . If the limit*

$$(F_0, \nabla_0) = (D_0, D_\infty, g_{0\infty}^0(z), \omega_0^0, \omega_\infty^0), \quad t \rightarrow 0$$

*exists and it is a trivial bundle with trivialization  $V^0(z)$ ,  $W^0(z)$ , then bundles  $F_t$  are trivial bundles, for sufficiently small  $t$ , and their trivializations  $V^t(z)$ ,  $W^t(z)$  have the limits*

$$\lim_{t \rightarrow 0} V^t(z) = V^0(z), \quad \lim_{t \rightarrow 0} W^t(z) = W^0(z),$$

*which are uniform for  $z \in D_0 \cap D_\infty$*

The family (8) can be represented in the neighborhood  $D_0$  as

$$\frac{dy}{dz} = \left( V(t, z) \omega_0^t V^{-1}(t, z) + \frac{dV}{dz} V^{-1} \right) y, \quad V(t, z) = V_0(t) + V_1(t)z + \dots, \quad (13)$$

and in the neighborhood  $D_\infty$  as

$$\frac{dy}{dz} = \left( W(t, z) \omega_\infty W^{-1}(t, z) + \frac{dW}{dz} W^{-1} \right) y, \quad W(t, z) = W_0(t) + W_1(t)z + \dots \quad (14)$$

We obtain that in the general case the matrix-coefficients of the system (8) are

$$B_i = V_0(t) B_i'(t) V_0^{-1}(t), \quad i = 0, t,$$

and

$$B_i = W_0(t) B_i^0(t) W_0^{-1}(t), \quad i = 1, \infty.$$

From this formulas and from the form (10) we obtain Theorem 1 and Corollary 1 in the general case.

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