Analysis of thresholds for braided block codes in BEC

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The ensemble of *braided block codes* (BBC) [1] using Hamming codes as component codes is considered. A method of calculation of thresholds for these codes is suggested. Results of calculation of thresholds for BBC with the (7,4), (15,11), and (31,26) Hamming codes as component codes in binary erasure channel (BEC) are presented.

1 Introduction

Low-density parity-check (LDPC) codes, invented by Gallager [2], who described and analyzed a block variant of the codes, arouse great interest of researchers. These codes are widely used in practice since they can be decoded by using relatively simple iterative decoding algorithms. Several modifications of these codes are known, in particular, *generalized LDPC codes* [3]. In contrast to Gallager's LDPC codes, using *single parity-check codes* as component codes, the generalized LDPC codes use as component codes arbitrary block codes.

The convolutional version of Gallager's codes, called *convolutional LDPC codes*, was described in [4]. These codes have better decoding reliability performance then the related LDPC block codes. Particularly, if the density of the parity-check matrices of convolutional LDPC codes increases, then their *threshold* (see below) tends to the Shannon limit [5], [6].

Braided block codes [1] and *braided convolutional codes* [7] can be considered as special constructions of *generalized LDPC convolutional codes*.

In this paper, we will analyze the thresholds of braided block codes with Hamming component codes. Thresholds are most important for the asymptotical performance of LDPC codes, characterizing the iterative decoding reliability when the blocklength (memory) goes to infinity.

2 Braided Block Codes

A BBC is constructed by the interconnection of two block codes, a horizontal component code and a vertical component code. These component codes are usually chosen to have relatively small a posteriori probability decoding complexity as, e.g., the Hamming codes. In this paper we consider BBCs based on the (ν, κ) Hamming component codes. We describe *symmetric* BBC, using the same Hamming code as both horizontal and vertical component code.

It is convenient to define a BBC by means of a two-dimensional infinite array. The array representation of BBCs is based on *multiple convolutional permutors.* The code symbols are placed into the cells of the permutor and checked row-wise and column-wise by the horizontal and vertical components codes.

A symmetric multiple convolutional permutor of memory m and multiplicity ν can be described as an infinite binary matrix of width m having ν 1s in each row and each column (the other elements of the matrix are 0s). The nonzero elements of the matrix represent memory cells capable to store the input symbols. Therefore, each row and each column of the array stores ν symbols. Symbols stored in each row and in each column form codewords of the component Hamming code such that $2\kappa - \nu$ symbols of a codeword are information symbols and $2\nu - 2\kappa$ symbols are parity-check symbols. For example, if the component codes are the $(7, 4)$ Hamming codes, each row/column stores 1 information symbol and 6 parity symbols; if they have parameters $(15, 11)$, then each row/column stores 7 information and 8 parity symbols; if they are $(31, 21)$ codes, then each row/column stores 21 information and 10 parity symbols. Correspondingly, the rates R of BBCs are equal to 1/7, 7/15, and 21/31.

Consider a communication over a BEC with erasure probability δ. We assume that a binary BBC with (ν, κ)-Hamming component codes is used and that the girth of the Tanner graph of the code goes to infinity when the memory of the code goes to infinity[1]. Then we can assume that the iterative decoder operates on a cycle-free graph.

The variable nodes of the Tanner graph are associated with code symbols; half of the check nodes (first group) are associated with horizontal component codes and the other half of the check nodes (second group) are associated with vertical component codes.

The iterative message-passing decoding algorithm of BBC codes is based on a posteriori probability (APP) decoding of component codes and operates as follows. Assume that at the beginning of the decoding process, the symbols of the received sequence $\mathbf{r}$ are associated with the corresponding variable nodes. After each iteration the decoder reconstructs some erased symbols and then uses the modified received sequence $\mathbf{r}'$ for the next iteration.

[1]A proof of the existence of such BBC is analogous to the proof given for conventional regular LDPC codes (see, for example, [8]).

Let us explain the decoding using the Tanner graph of the code. In the first phase of the first iteration all variable nodes send messages to their connected check nodes of the first group. The message is the value of the corresponding received symbol , i.e., 0, 1, or the erasure symbol Δ. In the second phase of the first iteration, the decoder performs APP decoding of the horizontal component codes. Then, the check nodes of the first group send the corresponding messages to their connected variable nodes.

In the first phase of the second iteration, variable nodes corresponding to nonerased symbols retain these symbols independently of the messages received from their connected check nodes and continue to pass the same messages. If a variable node corresponding to an erased symbol receives from a connected check node of the first group a 0 or 1 symbol, it corrects the erasure, resulting in a modified received sequence $\mathbf{r}'$, and sends the corresponding message to a connected check node of the second group. In the second phase of the second iteration the decoder performs APP decoding of the vertical component codes. Then the check nodes of the second group send the corresponding messages to their connected variable nodes.

In the first phase of the odd-numbered iterations the variable nodes send corresponding messages to the check nodes of the first group; in the first phase of the even-numbered iterations they send corresponding messages to the check nodes of the second group. In the second phase of the odd-numbered iterations decoder performs APP decoding of the horizontal component codes. In the second phase of the even-numbered iterations decoder performs APP decoding of the vertical component codes. Then check nodes send the corresponding messages to their connected variable nodes. Decoding continues until either all symbol nodes have 0 or 1 symbols, i.e., the modified received sequence $\mathbf{r}'$ does not contain any erased symbols, or the sequence $\mathbf{r}'$ becomes a stopping set.

3 Shannon Limit, Iterative Limit and Threshold

The Shannon limit δ_{sh} for the BEC is defined as the maximal erasure probability δ for which reliable communication over the channel with code rate R and blocklength $N \to \infty$ is possible. It is connected to the code rate R as

$$\delta_{\text{sh}} = 1 - R. \tag{1}$$

The iterative limit of the BBC δ_{id} for BEC is also a function of the code rate R and it characterizes communication when $N \to \infty$. But in contrast to the Shannon limit, we assume that the parameters of the code and iterative decoding method are fixed. The iterative limit is defined as the maximal erasure probability δ for which reliable communication over the BEC is possible, conditioned that the parameters of used BBC and decoding method are fixed. Calculating the iterative limit for LDPC-like codes is a complicated problem, therefore, in practice a lower bound on $\delta_{\rm id}$ is found. This bound is called the *threshold*,

$$\delta_{\rm id} \geq \delta_{\rm th}. \tag{2}$$

The standard method of threshold calculation is called *density/probability evolution*. In the case of analysis of the communication over the BEC it is reduced to calculating the code symbol erasure probability $f_i, i = 1, 2, \ldots,$ after ith decoding iteration.

4 Thresholds Calculation

Our goal is to calculate the thresholds $\delta_{\rm th}$ for a BBC for the BEC with erasure probability δ. By definition, if $\delta \leq \delta_{\rm th}$, then there exists iteratively decodable BBC of sufficiently large memory m such that the bit-error probability of the code goes to zero as $m \to \infty$.

We assume that the decoder operates on a cycle-free graph, i.e., the code has girth $g = O(\log m) \to \infty$, and the number of decoding iterations I does not exceed $\ell_0 = \lceil (g-4)/4 \rceil$. Then we get the following recurrent equations for the probability f_i that a code symbol is erased after the ith iteration:

for the BBC code with $(7,4)$ component Hamming codes

$$f_i = \delta[1-(1-f_{i-1})^6 - 6f_{i-1}(1-f_{i-1})^5 - 12f_{i-1}^2(1-f_{i-1})^4]; \tag{3}$$

for the BBC code with $(15,11)$ component Hamming codes

$$f_i = \delta[1-(1-f_{i-1})^{14} - 14f_{i-1}(1-f_{i-1})^{13} - 84f_{i-1}^2(1-f_{i-1})^{12} - 224f_{i-1}^3(1-f_{i-1})^{11}]; \tag{4}$$

for the BBC code with $(31,21)$ component Hamming codes

$$\begin{aligned} f_i &= \delta[1-(1-f_{i-1})^{30} - 30f_{i-1}(1-f_{i-1})^{29} - 420f_{i-1}^2(1-f_{i-1})^{28} - \\ &- 3360f_{i-1}^3(1-f_{i-1})^{27} - 13440f_{i-1}^4(1-f_{i-1})^{26}]. \end{aligned} \tag{5}$$

In all cases the initial condition is

$$f_0 = \delta. \tag{6}$$

Using the recurrent equations (3)–(5), we can find the maximum δ for which the function $f_i \to 0$ as $i \to \infty$. This value represents the iterative decoding threshold of the corresponding BBC.

We get the following results: for the rate $R = 1/7$ BBC with(7,4) Hamming component codes $\delta_{\rm th} = 0.7032506$; for the rate $R = 7/15$ BBC with (15,11) Hamming component codes $\delta_{\rm th} = 0.4116977$; for the rate $R = 21/31$ BBC with (31,21) Hamming component codes $\delta_{\rm th} = 0.2385489$. The Shannon limits for the BEC are: for the rate $R = 1/7$ codes $\delta_{\rm sh} = 0.8571428$; for the rate $R = 7/15$ codes $\delta_{\rm sh} = 0.5333333$; for the rate $R = 21/31$ codes $\delta_{\rm sh} = 0.3225806$.

Note that ratios of the thresholds $\delta_{\rm th}$ and the Shannon limits $\delta_{\rm sh}$ are close to those of LDPC codes with the same rates and parity-check densities.

5 Conclusion

In this paper, we calculated the thresholds of BBC with Hamming component codes. In our analysis we do not take into account the convolutional nature of the BBC codes. Because of this, our analysis is valid for the corresponding constructions of generalized LDPC block code [3], i.e., there exist generalized LDPC codes with Hamming component codes having the same thresholds as given in Section 4. We expect that application of more ingenious analytical methods, analogous to those used in [5], will give thresholds very close to the Shannon limits.

References

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