On Ultrafilter Extensions of First-Order Models

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Ultrafilter extensions of arbitrary first-order models were introduced in [1]. If $(X, F, \ldots, P, \ldots)$ is a model with the universe X, operations $F, \ldots$, and relations $P, \ldots$, then it canonically extends to the model $(\boldsymbol{\beta} X, \widetilde{F}, \ldots, \widetilde{P}, \ldots)$ (of the same language) where $\boldsymbol{\beta} X$ is the set of ultrafilters over X, the operations $\widetilde{F}, \ldots$ extend the operations $F, \ldots$, and the relations $\widetilde{P}, \ldots$ extend the relations $P, \ldots$. Here X is considered as a subset of $\boldsymbol{\beta} X$ by identifying each element x in X with the principal ultrafilter $\widetilde{x}$ given by x, and $\boldsymbol{\beta} X$ is endowed with the standard compact Hausdorff topology generated by basic open sets $\{u \in \boldsymbol{\beta} X : S \in u\}$ for all $S \subseteq X$. Thus $\boldsymbol{\beta} X$ is the Stone–Čech compactification of X with the discrete topology, or else, the spectral space of the Boolean algebra of subsets of X.

The extension lifts homomorphisms between models: continuous extensions of homomorphisms of $\mathfrak{A}$ into $\mathfrak{B}$ are homomorphisms of $\boldsymbol{\beta}\mathfrak{A}$ into $\boldsymbol{\beta}\mathfrak{B}$. Moreover, if a model $\mathfrak{C}$ carries a compact Hausdorff topology which is compatible with its structure (like the topology and the structure of $\boldsymbol{\beta}\mathfrak{A}$), then continuous extensions of homomorphisms of $\mathfrak{A}$ into $\mathfrak{C}$ are homomorphisms of $\boldsymbol{\beta}\mathfrak{A}$ into $\mathfrak{C}$. These statements remain true for embeddings and some other relationships between models (although not for elementary embeddings, as Shelah has recently shown in [3]). This shows that, roughly speaking, the construction smoothly generalizes the Stone–Čech compactification of a discrete space to the situation when the space carries a first-order structure.

The principal precursor of this construction was ultrafiter extensions of semigroups, the technique invented in 60s and then used to obtain significant results in number theory, algebra, and topological dynamics; the book [2] is a comprehensive treatise of this field. In the talk, we plan to describe the construction, discuss main results from [1], and mention some applications.

References

[1] D. I. Saveliev. *Ultrafilter extensions of models.* Lecture Notes in AICS, Springer, 6521 (2011), 162–177. An expanded version in: S.-D. Friedman et al. (eds.). *The Infinity Project Proceedings.* CRM Documents 11, Barcelona, 2012, 599–616.

[2] N. Hindman, D. Strauss. *Algebra in the Stone–Čech compactification.* Second edition, revised and expanded, Walter de Gruyter, Berlin, 2012.

[3] S. Shelah. *On does $\boldsymbol{\beta}\mathfrak{M}$ preserve elementary embeddings.* Unpublished manuscript, May 2013.