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An explicit Lipschitz constant for the joint spectral radius ☆

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ABSTRACT

In 2002, Wirth has proved that the joint spectral radius of irreducible compact sets of matrices is locally Lipschitz continuous as a function of the matrix set. In the paper, an explicit formula for the related Lipschitz constant is obtained.

1. Introduction

Information about the rate of growth of matrix products with factors taken from some matrix set is of great importance in various problems of control theory [1–3] wavelet theory [4–6] and other fields of mathematics. One of the most prominent values characterizing the exponential rate of growth of matrix products is the so-called joint or generalized spectral radius.

Let $\mathbb{K} = \mathbb{R}, \mathbb{C}$ be the field of real or complex numbers, and $\mathcal{A} \subset \mathbb{K}^{d\times d}$ be a set of $d \times d$ matrices. As usual, for $n \geqslant 1$ denote by $\mathcal{A}^n$ the set of all n-products of matrices from $\mathcal{A}$; $\mathcal{A}^0 = I$.

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Given a norm $\|\cdot\|$ in $\mathbb{K}^d$, the limit

$$\rho(\mathcal{A}) = \limsup_{n\to\infty} \|\mathcal{A}^n\|^{1/n}, \tag{1}$$

with $\|\mathcal{A}^n\| = \sup_{A\in\mathcal{A}^n} \|A\|$ is called *the joint spectral radius* of the matrix set $\mathcal{A}$ [7]. The limit in (1) is finite for bounded matrix sets $\mathcal{A} \subset \mathbb{K}^{d\times d}$ and does not depend on the norm $\|\cdot\|$. As shown in [7], for any $n \geqslant 1$ the estimates $\rho(\mathcal{A}) \leqslant \|\mathcal{A}^n\|^{1/n}$ hold, and therefore the joint spectral radius can be defined also by the following formula:

$$\rho(\mathcal{A}) = \inf_{n\geqslant 1} \|\mathcal{A}^n\|^{1/n}. \tag{2}$$

If $\mathcal{A}$ is a singleton set then (1) turns into the known Gelfand formula for the spectral radius of a linear operator. By this reason sometimes (1) is called the generalized Gelfand formula [8].

Besides (1) and (2), there are quite a number of different equivalent definitions of $\rho(\mathcal{A})$ in which the norm in (1) is replaced by the spectral radius [4,5,9] or the trace of a matrix [10], or by a uniform non-negative polynomial of even degree [11]. Sometimes $\rho(\mathcal{A})$ is defined in terms of existence of specific norms [2,12] (the Barabanov and Protasov norms). Unfortunately, the common feature of all the mentioned definitions is their nonconstructivity. In all these definitions the value of $\rho(\mathcal{A})$ is defined either as a certain limit or as a result of some "existence theorems", which essentially complicates the analysis of properties of the joint spectral radius.

In the paper, we are concerned with properties of the joint spectral radius $\rho(\mathcal{A})$ as a function of $\mathcal{A}$ for compact (i.e. closed and bounded) matrix sets $\mathcal{A}$. In this case it is convenient to denote the set of all nonempty bounded subsets of $\mathbb{K}^{d\times d}$ by $\mathcal{B}(\mathbb{K}^{d\times d})$, and the set of all nonempty compact subsets of $\mathbb{K}^{d\times d}$ by $\mathcal{K}(\mathbb{K}^{d\times d})$. Both of these sets become metric spaces if to endow them with the usual Hausdorff metric

$$H(\mathcal{A},\mathcal{B}) := \max\left\{\sup_{A\in\mathcal{A}}\inf_{B\in\mathcal{B}} \|A-B\|, \sup_{B\in\mathcal{B}}\inf_{A\in\mathcal{A}} \|A-B\|\right\}.$$

In doing so the space $\mathcal{K}(\mathbb{K}^{d\times d})$ is proved to be complete while the set $\mathcal{I}(\mathbb{K}^{d\times d})$ of all irreducible compact matrix families is open and dense in $\mathcal{K}(\mathbb{K}^{d\times d})$.

In 2002, Wirth has proved [13, Corollary 4.2] that the joint spectral radius of irreducible compact matrix sets satisfies the local Lipschitz condition.

Wirth's Theorem. *For any compact set $\mathcal{P} \subset \mathcal{I}(\mathbb{K}^{d\times d})$ there is a constant C (depending on $\mathcal{P}$ and the norm $\|\cdot\|$ in $\mathbb{K}^{d\times d}$) such that*

$$|\rho(\mathcal{A}) - \rho(\mathcal{B})| \leqslant C \cdot H(\mathcal{A},\mathcal{B}), \quad \forall \mathcal{A},\mathcal{B} \in \mathcal{P}.$$

The aim of the present paper is to obtain an explicit expression for the constant C in the above inequality.

As demonstrated the following example the joint spectral radius is not locally Lipschitz continuous if to discard supposition about irreducibility of a matrix set.

Example 1. Consider the matrix set $\mathcal{A}_\varepsilon$ composed of a single matrix

$$A_\varepsilon = \begin{pmatrix} 1 & 1 \\ \varepsilon & 1 \end{pmatrix},$$

depending on the real parameter $\varepsilon > 0$.

Clearly, the singleton matrix set $\mathcal{A}_0$ is not irreducible. Besides, $\rho(A_\varepsilon) = 1 + \sqrt{\varepsilon}$, and therefore

$$|\rho(\mathcal{A}_\varepsilon) - \rho(\mathcal{A}_0)| = |\rho(A_\varepsilon) - \rho(A_0)| = \sqrt{\varepsilon},$$

whereas $H(\mathcal{A}_\varepsilon, \mathcal{A}_0) = \|A_\varepsilon - A_0\| = \varepsilon c$, where c is some constant.

2. Main result

Given a matrix set $\mathcal{A} \subset \mathbb{K}^{d\times d}$, for each $n \geqslant 1$ denote by $\mathcal{A}_n$ the set of all k-products of matrices from $\mathcal{A} \bigcup \{I\}$ with $k \leqslant n$, that is $\mathcal{A}_n = \cup_{k=0}^{n} \mathcal{A}^k$. Denote also by $\mathcal{A}_n(x)$ the set of all the vectors Ax with $A \in \mathcal{A}_n$. Let $\|\cdot\|$ be a norm in $\mathbb{K}^d$ then $\mathbb{S}(t)$ stands for the ball of radius t in this norm.

Let us call the *p-measure of irreducibility* of the matrix set $\mathcal{A}$ (with respect to the norm $\|\cdot\|$) the quantity $\chi_p(\mathcal{A})$ determined as

$$\chi_p(\mathcal{A}) = \inf_{\substack{x\in\mathbb{R}^d\\ \|x\|=1}} \sup\{t : \mathbb{S}(t) \subseteq \mathrm{conv}\{\mathcal{A}_p(x) \cup \mathcal{A}_p(-x)\}\}.$$

Under the name 'the measure of quasi-controllability' the measure of irreducibility $\chi_p(\mathcal{A})$ was introduced and investigated in [14–16] where the overshooting effects for the transient regimes of linear remote control systems were studied. The reason why the quantity $\chi_p(\mathcal{A})$ got the name 'the measure of irreducibility' is in the following lemma.

Lemma 1. *Let $p \geqslant d-1$. The matrix set $\mathcal{A}$ is irreducible if and only if $\chi_p(\mathcal{A}) > 0$.*

The proof of Lemma 1 can be found in [15,16]. In these works it is proved also that, for compact irreducible matrix sets, the quantity $\chi_p(\mathcal{A})$ continuously depends on $\mathcal{A}$ in the Hausdorff metric.

Theorem 1. *For any pair of matrix sets $\mathcal{A} \in \mathcal{I}(\mathbb{K}^{d\times d})$, $\mathcal{B} \in \mathcal{B}(\mathbb{K}^{d\times d})$ for each $p \geqslant d-1$ it is valid the inequality*

$$|\rho(\mathcal{A}) - \rho(\mathcal{B})| \leqslant \nu_p(\mathcal{A}) H(\mathcal{A},\mathcal{B}), \tag{3}$$

where

$$\nu_p(\mathcal{A}) = \frac{\max\{1, \|\mathcal{A}\|^p\}}{\chi_p(\mathcal{A})}.$$

Taking into account that the quantity $\nu_p(\mathcal{A})$ continuously depends on $\mathcal{A}$ in the Hausdorff metric, and hence it is bounded on any compact set $\mathcal{P} \subset \mathcal{I}(\mathbb{K}^{d\times d})$, Theorem 1 implies Wirth's Theorem. However, unlike to Wirth's Theorem, in Theorem 1 neither compactness nor irreducibility of the matrix set $\mathcal{B}$ is assumed.

As will be shown below under the proof of Theorem 1, in fact even more accurate estimate than (3) holds:

$$|\rho(\mathcal{A}) - \rho(\mathcal{B})| \leqslant \frac{\max\{1, (\rho(\mathcal{A}))^p\}}{\chi_p(\mathcal{A})} H(\mathcal{A},\mathcal{B}).$$

However, this last estimate is not quite satisfactory because practical evaluation of the quantity $\rho(\mathcal{A})$ is a problem. At the same time the quantity $\nu_p(\mathcal{A})$ in (3) *can be evaluated in a finite number of algebraic operations involving only information about $\mathcal{A}$.*

Remark also that whereas the value of the joint spectral radius is independent of a norm in $\mathbb{K}^{d\times d}$, the quantities $\nu_p(\mathcal{A})$, $\chi_p(\mathcal{A})$ and $H(\mathcal{A},\mathcal{B})$ in (3) do depend on the choice of the norm $\|\cdot\|$ in $\mathbb{K}^{d\times d}$.

At last, point out that in the case when both of the matrix sets $\mathcal{A}$ and $\mathcal{B}$ are irreducible and compact, that is $\mathcal{A},\mathcal{B} \in \mathcal{I}(\mathbb{K}^{d\times d})$, inequality (3) can be formally strengthened:

$$|\rho(\mathcal{A}) - \rho(\mathcal{B})| \leqslant \min\left\{\nu_p(\mathcal{A}), \nu_p(\mathcal{B})\right\} H(\mathcal{A},\mathcal{B}).$$

3. Auxiliary statements

To prove Theorem 1 we will need some auxiliary notions and facts related to the irreducible matrix sets. The principal technical tool in proving Theorem 1 will be the notion of the Barabanov norm mentioned above, existence of which follows from the next theorem [2].

Barabanov's Theorem. *The quantity* ρ *is the joint* (*generalized*) *spectral radius of the matrix set* $\mathcal{A} \in \mathcal{I}(\mathbb{K}^{d\times d})$ *if and only if there is a norm* $\|\cdot\|_b$ *in* $\mathbb{K}^d$ *such that*

$$\rho\|x\|_b \equiv \max_{A\in\mathcal{A}} \|Ax\|_b. \tag{4}$$

In what follows a norm satisfying (4) is called *a Barabanov norm* corresponding to the matrix set $\mathcal{A}$.

In the next elementary lemma, a simple way to get as upper as lower estimates for the joint spectral radius is suggested.

Lemma 2. *Let* $\mathcal{A}$ *be a nonempty matrix set from* $\mathbb{K}^{d\times d}$. *If, for some* α,

$$\sup_{A\in\mathcal{A}} \|Ax\| \leqslant \alpha\|x\|, \quad \forall x \in \mathbb{K}^d, \tag{5}$$

then $\rho(\mathcal{A}) \leqslant \alpha$. *If, for some* β,

$$\sup_{A\in\mathcal{A}} \|Ax\| \geqslant \beta\|x\|, \quad \forall x \in \mathbb{K}^d, \tag{6}$$

then $\rho(\mathcal{A}) \geqslant \beta$.

Proof. Clearly, the constants α and β may be thought of as non-negative. To prove the first claim note that (5) implies the inequality $\|\mathcal{A}\| = \sup_{A\in\mathcal{A}} \|A\| \leqslant \alpha$. Then $\|\mathcal{A}^n\| = \sup_{A_i\in\mathcal{A}} \|A_n \cdots A_2A_1\| \leqslant \alpha^n$, and $\rho(\mathcal{A}) \leqslant \alpha$ by the definition (1).

Similarly, to prove the second claim note that (6) implies, for each $n = 1, 2, \ldots$, the inequality

$$\sup_{A_i\in\mathcal{A}} \|A_n \cdots A_2A_1x\| = \sup_{A_1\in\mathcal{A}} \sup_{A_2\in\mathcal{A}} \ldots \sup_{A_n\in\mathcal{A}} \|A_n \cdots A_2A_1x\| \geqslant \beta^n\|x\|, \quad \forall x \in \mathbb{K}^d.$$

Hence $\sup_{A_i\in\mathcal{A}} \|A_n \cdots A_2A_1\| \geqslant \beta^n$. Then $\|\mathcal{A}^n\| = \sup_{A_i\in\mathcal{A}} \|A_n \cdots A_2A_1\| \geqslant \beta^n$, and $\rho(\mathcal{A}) \geqslant \beta$ by the definition (1). The lemma is proved.

Following to [17], for convenience of comparison of norms in $\mathbb{K}^d$ let us introduce an appropriate notion. Given two norms $\|\cdot\|'$ and $\|\cdot\|''$ in $\mathbb{K}^d$, define the quantities

$$e^-(\|\cdot\|', \|\cdot\|'') = \min_{x\neq 0} \frac{\|x\|'}{\|x\|''}, \quad e^+(\|\cdot\|', \|\cdot\|'') = \max_{x\neq 0} \frac{\|x\|'}{\|x\|''}. \tag{7}$$

Since all norms in $\mathbb{K}^d$ are equivalent then the quantities $e^-(\cdot)$ and $e^+(\cdot)$ are well defined, and

$$0 < e^-(\|\cdot\|', \|\cdot\|'') \leqslant e^+(\|\cdot\|', \|\cdot\|'') < \infty.$$

Therefore the quantity

$$\mathrm{ecc}(\|\cdot\|', \|\cdot\|'') = \frac{e^+(\|\cdot\|', \|\cdot\|'')}{e^-(\|\cdot\|', \|\cdot\|'')} \geqslant 1, \tag{8}$$

called *the eccentricity* of the norm $\|\cdot\|'$ with respect to the norm $\|\cdot\|''$, is also well defined.

4. Proof of Theorem 1

We will prove Theorem 1 in two steps. First, slightly modifying the idea of the proof from [13, Corollary 4.2], we will show in Section 4.1 that under the conditions of Theorem 1 the eccentricity of any Barabanov norm $\|\cdot\|_{\mathcal{A}}$ for the matrix set $\mathcal{A}$ with respect to the norm $\|\cdot\|$ may serve as the Lipschitz constant for the joint spectral radius, that is

$$|\rho(\mathcal{A}) - \rho(\mathcal{B})| \leqslant \mathrm{ecc}(\|\cdot\|_{\mathcal{A}}, \|\cdot\|)H(\mathcal{A}, \mathcal{B}). \tag{9}$$

Then, using the techniques of the measures of irreducibility (see, e.g., [14,16,18]), we will prove in Section 4.2 the estimate

$$\operatorname{ecc}(\|\cdot\|_{\mathcal{A}},\|\cdot\|)\leqslant\nu_p(\mathcal{A}):=\frac{\max\{1,\|\mathcal{A}\|^p\}}{\chi_p(\mathcal{A})}. \tag{10}$$

4.1. Proof of estimate (9)

Let $\|\cdot\|_{\mathcal{A}}$ be some Barabanov norm for the matrix set $\mathcal{A}$. By definition of the Hausdorff metric, for any matrix $B\in\mathcal{B}$ there is a matrix $A_B\in\mathcal{A}$ such that $\|B-A_B\|\leqslant H(\mathcal{A},\mathcal{B})$. Then, by definition of the eccentricity of the norm $\|\cdot\|_{\mathcal{A}}$ with respect to the norm $\|\cdot\|$,

$$\|B-A_B\|_{\mathcal{A}}\leqslant C\cdot\|B-A_B\|\leqslant C\cdot H(\mathcal{A},\mathcal{B}), \tag{11}$$

where $C=\operatorname{ecc}(\|\cdot\|_{\mathcal{A}},\|\cdot\|)$.

Consider the obvious inequality

$$\|Bx\|_{\mathcal{A}}\leqslant\|A_Bx\|_{\mathcal{A}}+\|(B-A_B)x\|_{\mathcal{A}},\quad\forall x\in\mathbb{K}^d.$$

Here $\|A_Bx\|_{\mathcal{A}}\leqslant\rho(\mathcal{A})\|x\|_{\mathcal{A}}$ because $\|\cdot\|_{\mathcal{A}}$ is a Barabanov norm for the matrix set $\mathcal{A}$, and $\|(B-A_B)x\|_{\mathcal{A}}\leqslant C\cdot H(\mathcal{A},\mathcal{B})\|x\|_{\mathcal{A}}$ by inequality (11). Therefore

$$\|Bx\|_{\mathcal{A}}\leqslant(\rho(\mathcal{A})+C\cdot H(\mathcal{A},\mathcal{B}))\,\|x\|_{\mathcal{A}},\quad\forall x\in\mathbb{K}^d,$$

and, due to arbitrariness of $B\in\mathcal{B}$,

$$\sup_{B\in\mathcal{B}}\|Bx\|_{\mathcal{A}}\leqslant(\rho(\mathcal{A})+C\cdot H(\mathcal{A},\mathcal{B}))\,\|x\|_{\mathcal{A}},\quad\forall x\in\mathbb{K}^d.$$

From here by Lemma 2

$$\rho(\mathcal{B})\leqslant\rho(\mathcal{A})+C\cdot H(\mathcal{A},\mathcal{B}). \tag{12}$$

Now, let us prove that

$$\rho(\mathcal{B})\geqslant\rho(\mathcal{A})-C\cdot H(\mathcal{A},\mathcal{B}). \tag{13}$$

By definition of the Hausdorff metric, for any matrix $A\in\mathcal{A}$ there is a matrix $B_A\in\mathcal{B}$ such that $\|B_A-A\|\leqslant H(\mathcal{A},\mathcal{B})$. Then, as before,

$$\|B_A-A\|_{\mathcal{A}}\leqslant C\cdot\|B_A-A\|\leqslant C\cdot H(\mathcal{A},\mathcal{B}). \tag{14}$$

By evaluating with the help of (14) the second summand in the next obvious inequality

$$\|B_Ax\|_{\mathcal{A}}\geqslant\|Ax\|_{\mathcal{A}}-\|(B_A-A)x\|_{\mathcal{A}},\quad\forall x\in\mathbb{K}^d,$$

we obtain:

$$\|B_Ax\|_{\mathcal{A}}\geqslant\|Ax\|_{\mathcal{A}}-C\cdot H(\mathcal{A},\mathcal{B})\|x\|_{\mathcal{A}},\quad\forall x\in\mathbb{K}^d.$$

Maximizing now the both sides of this last inequality over all $A\in\mathcal{A}$ (which is possible due to arbitrariness of $A\in\mathcal{A}$), we get:

$$\sup_{A\in\mathcal{A}}\|B_Ax\|_{\mathcal{A}}\geqslant\sup_{A\in\mathcal{A}}\|Ax\|_{\mathcal{A}}-C\cdot H(\mathcal{A},\mathcal{B})\|x\|_{\mathcal{A}},\quad\forall x\in\mathbb{K}^d.$$

Here the left-hand part of the inequality does not exceed $\sup_{B\in\mathcal{B}}\|Bx\|_{\mathcal{A}}$, while by Barabanov's Theorem $\sup_{A\in\mathcal{A}}\|Ax\|_{\mathcal{A}}\equiv\rho(\mathcal{A})\|x\|_{\mathcal{A}}$. Hence,

$$\sup_{B\in\mathcal{B}}\|Bx\|_{\mathcal{A}}\geqslant(\rho(\mathcal{A})-C\cdot H(\mathcal{A},\mathcal{B}))\,\|x\|_{\mathcal{A}},\quad\forall x\in\mathbb{K}^d,$$

from which by Lemma 2 we obtain (13).

Inequalities (12), (13) with $C=\operatorname{ecc}(\|\cdot\|_{\mathcal{A}},\|\cdot\|)$ imply (9) which finalizes the first step of the proof of Theorem 1.

4.2. Proof of estimate (10)

By definition of the eccentricity, the quantity $\mathrm{ecc}(\|\cdot\|_{\mathcal{A}},\|\cdot\|)$ is defined as follows

$$\mathrm{ecc}(\|\cdot\|_{\mathcal{A}},\|\cdot\|)=\frac{e^{+}(\|\cdot\|_{\mathcal{A}},\|\cdot\|)}{e^{-}(\|\cdot\|_{\mathcal{A}},\|\cdot\|)}.$$

Here, by the definition (7) of the quantities $e^{-}(\cdot)$ and $e^{+}(\cdot)$,

$$e^{-}(\|\cdot\|_{\mathcal{A}},\|\cdot\|)=\|x^{-}\|_{\mathcal{A}},\quad e^{+}(\|\cdot\|_{\mathcal{A}},\|\cdot\|)=\|x^{+}\|_{\mathcal{A}}$$

for some elements x^{-} and x^{+} satisfying $\|x^{-}\|=1$, $\|x^{+}\|=1$. Hence

$$\mathrm{ecc}(\|\cdot\|_{\mathcal{A}},\|\cdot\|)=\frac{\|x^{+}\|_{\mathcal{A}}}{\|x^{-}\|_{\mathcal{A}}}. \tag{15}$$

By definition of the measure of irreducibility $\chi_p(\mathcal{A})$, for elements x^{-} and x^{+} there are a natural number m, matrices $\widetilde{A}_i\in\mathcal{A}_p$, $i=1,2,\ldots,m$, and real numbers λ_i, $i=1,2,\ldots,m$, such that

$$\chi_p(\mathcal{A})x^{+}=\sum_{i=1}^{m}\lambda_i\widetilde{A}_i x^{-},\quad \sum_{i=1}^{m}|\lambda_i|\leqslant 1. \tag{16}$$

Here each matrix $\widetilde{A}_i$ is either the identity matrix or a product of no more than p factors from $\mathcal{A}$, that is $\widetilde{A}_i=A_{i_k}\cdots A_{i_1}$ with some $k\leqslant p$ and $A_{i_j}\in\mathcal{A}$. If $\widetilde{A}_i=I$ then $\|\widetilde{A}_i\|_{\mathcal{A}}=1$. If $\widetilde{A}_i=A_{i_k}\cdots A_{i_1}$ then $\|\widetilde{A}_i\|_{\mathcal{A}}\leqslant(\rho(\mathcal{A}))^k$ because, by definition of the Barabanov norm, $\|\widetilde{A}_{i_j}\|_{\mathcal{A}}\leqslant\rho(\mathcal{A})$ for any matrix $A_{i_j}\in\mathcal{A}$. Therefore

$$\|\widetilde{A}_i\|_{\mathcal{A}}\leqslant\max\left\{1,(\rho(\mathcal{A}))^k\right\}\leqslant\max\left\{1,(\rho(\mathcal{A}))^p\right\},$$

and (16) implies

$$\chi_p(\mathcal{A})\|x^{+}\|_{\mathcal{A}}\leqslant\max\left\{1,(\rho(\mathcal{A}))^p\right\}\|x^{-}\|_{\mathcal{A}}.$$

From here and from (15)

$$\mathrm{ecc}(\|\cdot\|_{\mathcal{A}},\|\cdot\|)\leqslant\frac{\max\{1,(\rho(\mathcal{A}))^p\}}{\chi_p(\mathcal{A})},$$

and, since $\rho(\mathcal{A})\leqslant\|\mathcal{A}\|$ by (2), this last inequality implies the estimate (10), which finalizes the second step of the proof.

The proof of Theorem 1 is completed.

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