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DETERMINATE SYSTEMS

Indefinability in *o*-Minimal Structures of Finite Sets of Matrices Whose Infinite Products Converge and Are Bounded or Unbounded[1]

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Abstract—This paper is concerned with the convergence and boundedness or unboundedness of the set of all possible matrix products with coefficients belonging to some finite set, i.e., the problem to which many problems of control theory and mathematics are reduced. The indefinability of this problem in *o*-minimal structures containing semialgebraic sets, which can be regarded as a characteristic for the complexity of the problem, is demonstrated. The result shows, in particular, that the solution of our problem cannot be found as a finite Boolean combination of conditions containing a finite number of ordinary arithmetical operations of addition, subtraction, and multiplication, as well as exponentiation and application of bounded analytic functions.

1. INTRODUCTION

Many problems of control theory and mathematics are reduced to the study of the properties of infinite matrix products

$$\dots A_n A_{n-1} \dots A_1, \tag{1}$$

where every matrix A_i belongs to some (finite or infinite) matrix set $\mathcal{A}$. Examples of such problems are the estimation of Lyapunov indexes for time-varying linear systems in the absolute stability problem [1, 2], analysis of the convergence of asynchronous parallel computation algorithms [3], the stability of desynchronized control systems [4], computation of the generalized spectral radius of a matrix family [5], etc. In many papers (e.g., [6]), a formal justification is given to demonstrate that it is not a simple matter to study such products if the matrix set $\mathcal{A}$ contains more than one matrix. In particular, according to [4], even sets of two-matrix sets $\mathcal{A}$ for which all products (1) tend to zero or bounded, are not semialgebraic, i.e., cannot be described by combinations of a finite number of algebraic equalities and inequalities consisting of the elements of the matrices belonging to $\mathcal{A}$. In this sense, the convergence of products of the type (1) or boundedness of the set of all possible matrix products with coefficients from $\mathcal{A}$ cannot be "formally" resolved.

Semialgebraic sets have certain good properties (see Section 2). Therefore, there is an innate desire to inquire whether there exist any wider classes of sets having similar properties. Formally, an axiom set that equips semialgebraic sets with "good" properties is successfully constructed in [7–9] and the corresponding theory is called the *o*-minimality theory (see the brief review and references cited in Section 2). In particular, new classes of sets that can be described constructively and

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quite natural for applications and possessing many key properties of semialgebraic sets have been designed. The desire to extend the results of [4] to these classes of sets has stimulated me to write this paper.

Another driving force is the desire to describe new analytical constructs (unlike the geometric constructs of [4]) that may aid in proving the semialgebraic insolubility of our problem or its indefinability in o-minimal structures. The need for such constructs arise, because, on the one hand, their use is technically simpler and, on the other hand, precisely they (but not the fact of insolubility) may be of help in further investigations.

2. SEMIALGEBRAIC SETS AND THE MAIN CONCEPTS OF THE o-MINIMALITY THEORY

In the classical algebraic geometry of closed algebraic fields, the image of an affine algebraic manifold under projection onto an affine space of reduced dimension is a Boolean combination of algebraic manifolds. The situation is rather complicated in the study of algebraic manifolds over fields of reals. For example, the image of the circle $x^2 + y^2 = 1$ under projection onto the axis x is the interval $[-1, 1]$, but not a Boolean combination of algebraic manifolds.

A subset of the space $\mathbb{R}^n$ is said to be *semialgebraic* if it is a finite Boolean combination of sets of solutions of the polynomial equations

$$p(x_1, \dots, x_n) = 0 \tag{2}$$

and polynomial inequalities

$$q(x_1, \dots, x_n) > 0. \tag{3}$$

Semialgebraic sets are attractive for two reasons. On the one hand, the set of all semialgebraic sets is rather rich and semialgebraic sets exhibit diverse properties. On the other hand, semialgebraic sets admit a simple description—to verify whether a point belongs to a given semialgebraic set or not, it suffices to be able to implement the operations of addition, subtraction, multiplication, and congruence of numbers. Therefore, semialgebraic sets are usually identified as ‘tame” objects that admit a finite (algorithmic) description. Nevertheless, it is not a simple matter in concrete situations to express the polynomials defining a semialgebraic set.

In many problems, there is no need to know the concrete type of the polynomials defining a semialgebraic set—but what matters is the semialgebraicity. A construct that aids in many situations in ascertaining the semialgebraicity of a set follows from the Seidenberg–Tarski principle, which asserts that *the image $f(X)$ of a semialgebraic set $X \subset \mathbb{R}^n$ under a polynomial mapping $f : \mathbb{R}^n \to \mathbb{R}^m$ is also a semialgebraic set.*

The semialgebraicity of the complete preimage of a semialgebraic set under a polynomial mapping is self-evident. But the semialgebraicity of the image of a semialgebraic set is an abyssal fact. Note that the preimage of an algebraic set under polynomial mapping is an algebraic set, but the image, in general, is not an algebraic set. We give some examples.

Example 1. The quadratic polynomial $p(x) = x^2 + p_1 x + p_2$ is uniquely defined by its coefficients $\{p_1, p_2\}$ in $\mathbb{R}^2$. Is the set P of pairs $\{p_1, p_2\} \in \mathbb{R}^2$ for which the polynomial $p(x)$ has real roots x_1 and x_2 lying in the interval $[-1, 1]$ algebraic or not? To answer this question, note that $p_1 = -(x_1 + x_2)$ and $p_2 = x_1 x_2$ by the Vieta theorem. Therefore, the set P is the image of the semialgebraic set (square) $-1 \le x_1 \le 1$, $-1 \le x_2 \le 1$ under the polynomial mapping $\{x_1, x_2\} \mapsto \{-(x_1 + x_2), x_1 x_2\}$. Hence the set P is semialgebraic by the Seidenberg–Tarski principle. $\square$

Example 2. Let A be a square matrix of order N with real elements. The matrix A can be identified with a point of the coordinate space $\mathbb{R}^{N^2}$, numbering its elements in some order. Consequently, there is meaning in the algebraicity or nonalgebraicity of a set of matrices. The following statement is true: sets of square matrices A of order N satisfying the condition $\rho(A) \leq 1$ or $\rho(A) < 1$ are semialgebraic. This assertion can be demonstrated, using the well-known Routh–Hurwitz criterion to express in explicit form the algebraic relations between the elements of the matrix A that are equivalent to the condition $\rho(A) \leq 1$ or $\rho(A) < 1$. Another approach to prove this assertion is to apply the idea of Example 1 and the Seidenberg–Tarski principle; it only reveals the semialgebraicity of the respective sets, but does not give their explicit description. □

The Seidenberg–Tarski principle is a powerful tool to demonstrate the semialgebraicity of sets. But sometimes there is a need to demonstrate the opposite, namely, a given set is not semialgebraic. As is known (see, e.g., [10]), *the complement of a real algebraic set to another set contains not more than a finite number of linear connectivity components.*[2] This fact implies that every semialgebraic set contains not more than a finite number of the linear connectivity components. Applying this fact, we find that the graph of the function $y = \sin(1/x)$, $x > 0$, in $\mathbb{R}^2$ is not a semialgebraic set since its intersection with the semialgebraic set $y = 0$ contains infinitely many isolated points.

Semialgebraic sets have the following "tame" properties.

- *Stratifiability.* For every semialgebraic set X, there exists a finite number of disjoint semialgebraic sets $X_1, \ldots, X_n$ such that $X = X_1 \cup \ldots \cup X_n$; moreover, every X_i is a connected smooth real manifold, and if $\overline{X}_i \cap X_j \neq \varnothing$ for $i \neq j$, where $\overline{X}_i$ is the closure of the set X_i, then $\overline{X}_i \supseteq X_j$ and $\dim X_i > \dim X_j$. In particular, every semialgebraic set has a finite number of connectivity components and the boundary of a semialgebraic set is a semialgebraic set of much lower dimension.
- *Triangulability.* Every compact semialgebraic set admits semialgebraic triangulation.
- *Finiteness of the topological type.* Let $X \subseteq \mathbb{R}^{n+m}$ be a semialgebraic set. For every $a \in \mathbb{R}^m$, let $X_a = \{x \in \mathbb{R}^n : (x, a) \in X\}$. Then there exists a finite number of elements $a_1, \ldots, a_l \in \mathbb{R}^m$ for which there exists an $i \leq l$ and a semialgebraic homeomorphism $f : X_a \to X_{a_i}$ for every $a \in \mathbb{R}^m$.
- *Piecewise-smoothness of semialgebraic mappings.* A mapping is said to be semialgebraic if its graph is a semialgebraic set. If $f : X \to \mathbb{R}$ is a semialgebraic mapping, then X can be partitioned into a finite number of pairwise disjoint semialgebraic sets $X_1, \ldots, X_n$ such that every restriction $f|X_i$ is an analytical mapping.
- *Validity of the curve choice lemma.* Let $X \subseteq \mathbb{R}^n$ be a semialgebraic set and let $a \in \overline{X}$. Then there exists a real analytic function $f : (0, 1) \to X$ such that $\lim_{t\to 0} f(t) = a$. Moreover, the function f can be chosen such that its graph is an algebraic set.

These properties are discussed in [11, 12].

Since these properties of semialgebraic sets are rather strong and nontrivial, it is worthwhile to find the description of far wider classes of sets possessing these properties. The most probable candidates for this purpose are semianalytic and subanalytic sets.

A set $X \subseteq \mathbb{R}^n$ is said to be *semianalytic* if for every point $a \in \mathbb{R}^n$ there exists an open neighborhood U such that $X \cap U$ is a finite union of the sets

$$\{x \in U : f_1(x) = \ldots = f_m(x) = 0, g_1(x) > 0, \ldots, g_l(x) > 0\},$$

where $f_1, \ldots, f_m, g_1, \ldots, g_l$ are analytic functions in U.

[2] Recall that a set $X \subseteq \mathbb{R}^n$ is said to be linearly connected if for any two points x and y of the set there exists a continuous function $\gamma : [0, 1] \mapsto \mathbb{R}^n$ such that $\gamma(0) = x$, $\gamma(1) = y$ and $\gamma(t) \in X$ for $0 \leq t \leq 1$.

The local properties of semianalytic sets are mostly similar to those of semialgebraic sets. For example, every semianalytic set $X \subseteq \mathbb{R}^n$ with a compact closure is the union of a finite number of semianalytic sets, each of which is a real analytic manifold. Unfortunately, semianalytic sets are unstable to projection. An example is the unbounded semianalytic set

$$\{(1/n, n) : \; n = 1, 2, \ldots\}\,, \tag{4}$$

whose projection onto $\mathbb{R}$ is not semianalytic in the neighborhood of the point 0. Even bounded semianalytic sets may have projections that are not semianalytic sets. For example, the set

$$Y = \{(x, y, z, w) : 0 < x, y, w \leq 1, yw = x, z = ye^w\}$$

is semianalytic, but, as shown by Osgood (see, for example, [13]), its projection onto the space of first three coordinates is not a semianalytic set.

To avoid such difficulties, Lojasiewicz and Hironaka introduced a class of subanalytic sets (see the Introduction to Subanalytic Sets in [14]). A set $X \subseteq \mathbb{R}^n$ is said to be *subanalytic* if for every point $a \in \mathbb{R}^n$ there exists an open neighborhood U and a semianalytic set $Y \subseteq \mathbb{R}^n \times \mathbb{R}^m$ with compact closure such that $X \cap U$ is the projection of Y onto $\mathbb{R}^n$. Subanalytic sets with compact closure exhibit the "good" properties of semialgebraic sets described above (more exactly, subanalytic sets exhibit these properties locally).

Unfortunately, even such a "tame" set as the graph of the function e^{-1/x^2} is not subanalytic. This compelled Grothendieck to undertake a project (see [15]): *"To study classes of sets having the "tame" properties of semialgebraic sets."*

The o-minimality concept formulated below answers our question. Though the o-minimality concept initially emerged in the theory of models (an area in mathematical logic), knowledge of basic concepts of logic is sufficient to understand and prove our main results.

An ordered structure in $\mathbb{R}$ is defined to be a sequence $\mathcal{S} = (S_1, S_2, \ldots)$ of Boolean algebras S_n of subsets of $\mathbb{R}^n$ having the properties

(i) $\varnothing \in S_n$, $\mathbb{R}^n \in S_n$,
(ii) $\{(x, y) : x, y \in \mathbb{R}^n, x = y\} \in S_{2n}$,
(iii) if $a \in \mathbb{R}$, then $\{a\} \in S_1$,
(iv) $\{(x, y) : x, y \in \mathbb{R}, x < y\} \in S_2$,
(v) if $A \in S_n$, then $A \times \mathbb{R} \in S_{n+1}$ and $\mathbb{R} \times A \in S_{n+1}$,
(vi) if $A \in S_{n+1}$ and B is the projection of A onto the first n coordinates, then $B \in S_n$.

If $A \in S_n$, then the set A is said to be *definable* in the structure $\mathcal{S}$; a function is said to be *definable* if its graph is a definable set.

An ordered structure $\mathcal{S}$ is said to be *o-minimal* if

(vii) every set $A \in S_1$ is a finite union of points and intervals (possibly, unbounded) in $\mathbb{R}$.

An example of an ordered structure is the set of all semialgebraic subsets. But the set of all subanalytic subsets is not an ordered structure since the projection of the unbounded subanalytic set (4) is not a subanalytic set.

In practice, it is often convenient to define an ordered set as a minimal ordered structure containing a certain set of sets: for every $n = 1, 2, \ldots$, a set B_n of subsets of $\mathbb{R}^n$ is taken and the least ordered structure $\mathcal{S}$ for which $B_n \subseteq S_n$ for all n is determined.

Let us give a few examples of ordered structures.

- $\mathcal{S}_{\text{lin}}$ (a structure of polyhedral sets) is the minimal ordered structure containing all linear manifolds of the type $\{x \in \mathbb{R}^n : \; \sum r_i x_i = 0, \text{ where } r_1, \ldots, r_n \in \mathbb{R}\}$,

- $\mathcal{S}_{\rm alg}$ (a structure of semialgebraic sets) is the minimal ordered structure containing all algebraic sets,
- $\mathcal{S}_{\rm exp}$ is the minimal ordered structure containing semialgebraic sets and graph of the function $x \mapsto e^x$,
- $\mathcal{S}_{\rm an}$ is the minimal ordered structure containing semialgebraic sets and all subanalytic sets with compact closure,
- $\mathcal{S}_{\rm an,exp}$ is the minimal ordered structure containing semialgebraic sets, graph of the function $x \mapsto e^x$, and all subanalytic sets with compact closure.

The o-minimality of the structure $\mathcal{S}_{\rm lin}$ of polyhedral sets is self-evident. The o-minimality of the structure $\mathcal{S}_{\rm alg}$ of semialgebraic sets is a consequence of the Seidenberg–Tarski principle (in reality, a reformulated version). Van den Dries has shown [16] that the structure $\mathcal{S}_{\rm an}$ of subanalytic sets with compact closure is *o*-minimal [16]. Wilke, who has contributed much to the study of the "tame" properties of sets, has shown [17] that $\mathcal{S}_{\rm exp}$ is an *o*-minimal structure. Finally, Van den Dries, Macintyre, and Marker [18] have shown that the structure $\mathcal{S}_{\rm an,exp}$ is *o*-minimal. Other results on the theory of *o*-minimality are described, for example, in [13].

The question here is which definable sets among ordered structures have properties inherent in semialgebraic sets. Apparently, *if an o-minimal ordered structure $\mathcal{S}$ contains semialgebraic sets, the definable sets in it have all "tame" properties inherent in semialgebraic sets.*

In conclusion, let us describe the sets in $\mathcal{S}_{\rm an}$, $\mathcal{S}_{\rm exp}$, and $\mathcal{S}_{\rm an,exp}$ in less formal terms, but understandable to nonspecialists in mathematical logic and theory of *o*-minimal structures. Recall that semialgebraic sets are defined by sets expressed as Boolean combinations of sets by polynomial formulas (2) and (3). Here the term "polynomial" can be interpreted as an expression resulting from the application of a finite number of operations $\langle +, -, \times \rangle$ to arguments.

Let us begin with the description of sets of the structure $\mathcal{S}_{\rm exp}$. A *formula* corresponding to the structure $\mathcal{S}_{\rm exp}$ is an expression obtained from the application of a finite number of operations $\mathcal{O}_{\rm exp} = \langle +, -, \times, \exp \rangle$ to arguments. An example of a formula corresponding to the structure $\mathcal{S}_{\rm exp}$ is the expression

$$e^{e^{y^2+1}} - ze^{x-2y} + 3xy - 7.$$

Then definable sets in the structure $\mathcal{S}_{\rm exp}$ are precisely the sets that are finite Boolean combinations of the sets of solutions of Eqs. (2) and (3), where $p(x_1, \ldots, x_n)$ and $q(x_1, \ldots, x_n)$ are now not polynomials, but formulas corresponding to the structure $\mathcal{S}_{\rm exp}$.

We now describe sets of the structure $\mathcal{S}_{\rm an}$. Let f be a real analytic function defined in some neighborhood of the cube $[-1, 1]^n \subseteq \mathbb{R}^n$. A function $\widehat{f} : \mathbb{R}^n \to \mathbb{R}^n$ defined by the formula

$$\widehat{f}(x) = \begin{cases} f(x), & x \in [-1, 1]^n \\ 0, & \text{otherwise,} \end{cases}$$

is called a *bounded analytic function.* The set of all bounded analytic functions is denoted by $\{\widehat{f}\,\}$. In this case, formulas corresponding to the structure $\mathcal{S}_{\rm an}$ are expressions resulting from the application of a finite number of operations belonging to the operation set $\mathcal{O}_{\rm an} = \langle +, -, \times, \{\widehat{f}\,\} \rangle$ to arguments. The definable sets in the structure $\mathcal{S}_{\rm an}$ are sets that are finite Boolean combinations of sets of solutions of Eqs. (2) and (3), in which $p(x_1, \ldots, x_n)$ and $q(x_1, \ldots, x_n)$ are now not polynomials, but formulas corresponding to the structure $\mathcal{S}_{\rm an}$.

Finally, the sets of the structure $\mathcal{S}_{\rm an,exp}$ are derived by the same scheme as for the sets of the structure $\mathcal{S}_{\rm an}$, but with the difference that the operation set $\mathcal{O}_{\rm an,exp}$ is obtained by complementing the operation set $\mathcal{O}_{\rm an}$ with the exponentiation operation $\mathcal{O}_{\rm an,exp} = \langle +, -, \times, \exp, \{\widehat{f}\,\} \rangle$.

3. THE MAIN RESULT: MATRIX PRODUCTS

Let $\mathcal{M}(p,q)$ denote a set of ordered sets $\mathcal{L} = \{L_1, L_2, \dots, L_q\}$ of $p \times p$ real[3] matrices L_i, $i = 1, 2, \dots, q$. Obviously, the set $\mathcal{M}(p,q)$ can be identified with the space $\mathbb{R}^{qp^2}$ if the coordinates of the element $\mathcal{L} \in \mathcal{M}(p,q)$ are defined by the elements of the matrix $L_1, L_2, \dots, L_q$, ordering them in some manner (for example, row-wise).

With every set of matrices $\mathcal{L} \in \mathcal{M}(p,q)$ let us associate the corresponding set $\mathcal{P}(\mathcal{L})$ of all finite products of matrices in $\mathcal{L}$:

$$\mathcal{P}(\mathcal{L}) := \{A_n A_{n-1} \dots A_1 : \ A_i \in \mathcal{L}, \ n = 1, 2, \dots\}.$$

The matrix set $\mathcal{L}$ is said to be *convergent* if every matrix sequence $\{A_n\}$, $A_n \in \mathcal{L}$, satisfies the relation

$$A_n A_{n-1} \dots A_1 \to 0 \text{ as } n \to \infty. \tag{5}$$

The set of all convergent sets of matrices belonging to $\mathcal{M}(p,q)$ is denoted by $\mathcal{C}(p,q)$. As is known (see, for example, [4]), if the matrix set $\mathcal{L}$ is convergent, then the matrix set $\mathcal{P}(\mathcal{L})$ is bounded.

A matrix set $\mathcal{L}$ is said to be *bounded* if the matrix set $\mathcal{P}(\mathcal{L})$ is bounded, but relation (5) is not satisfied for at least one of the matrix sequences $\{A_n\}$, $A_n \in \mathcal{L}$. The set of all bounded sets of matrices belonging to $\mathcal{M}(p,q)$ is denoted by $\mathcal{B}(p,q)$.

If a matrix set $\mathcal{P}(\mathcal{L})$ is not bounded, then the matrix set $\mathcal{L}$ is said to be *unbounded*; the set of all unbounded sets of matrices belonging to $\mathcal{M}(p,q)$ is denoted by $\mathcal{U}(p,q)$. Clearly, the matrix sets $\mathcal{C}(p,q)$, $\mathcal{B}(p,q)$, and $\mathcal{U}(p,q)$ are pairwise disjoint and form the whole set $\mathcal{M}(p,q)$ in aggregate:

$$\mathcal{M}(p,q) = \mathcal{C}(p,q) \cup \mathcal{B}(p,q) \cup \mathcal{U}(p,q).$$

Theorem 1. *If $p, q \geq 2$, then none of the sets $\mathcal{C}(p,q)$, $\mathcal{B}(p,q)$, and $\mathcal{U}(p,q)$ is definable in o-ordered structures containing semialgebraic sets.*

The proof of Theorem 1 and all necessary auxiliary constructs are given in the Appendix.

A weaker variant of Theorem 1 formulated in [4] asserts that none of the sets $\mathcal{C}(p,q)$, $\mathcal{B}(p,q)$, and $\mathcal{U}(p,q)$ is semialgebraic. Theorem 1 and examples on o-ordered structures given in Section 2 imply, for example, that none of the sets $\mathcal{C}(p,q)$, $\mathcal{B}(p,q)$, and $\mathcal{U}(p,q)$ can be described by a finite Boolean combination of formulas, each of which, in turn, contains only a finite number of arithmetical operations and exponentiation operation or application of bounded analytic functions.

4. CONCLUSIONS

A traditional characteristic of the complexity of a problem is its algorithmic complexity; algorithmically a complex problem is also said to be NP-hard. As is known (see, e.g., review [6]), the convergence, boundedness or unboundedness of infinite matrix products are NP-hard problems. Furthermore, even an apparently simple problem, viz., whether a finite matrix product with coefficients from a given set of matrices with integral elements vanishes or not, is NP-hard.

My results may also be regarded as an alternative characteristic for the complexity of the class of problems studied in this paper.

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[3] The study of complex matrices does not add anything to the generality and, therefore, we restrict ourselves only to real matrices.

APPENDIX

A.1. PRODUCT OF TWO MATRICES, OF WHICH ONE IS DEGENERATE

Let us describe certain properties of 2×2 matrices that are simple, but nonetheless play a pivotal role in the sequel. Let there be a set $\mathcal{L} = \{L, M\}$ of 2×2 matrices. Let us study the convergence of products of the type

$$A_n A_{n-1} \dots A_1, \tag{A.1}$$

where $A_i = L$ or $A_i = M$ for every $i = 1, 2, \dots, n$.

In the general case in which L and M are not assumed to be commuting matrices, it is not a simple matter to investigate the convergence.[4] To simplify the situation such that it yields to analysis, and at the same time interesting effects in the behavior of products (A.1) do not disappear, we may assume that one of the matrices L or M is degenerate. In what follows, we take that the matrix M is degenerate. The implication of such an assumption is that any matrix product of the type $M^m L^n M^p$ in which $m, p \geq 1$ can be represented as γM with some constant γ, which can be easily determined in many cases in explicit form.

By this remark, we can simply and constructively study the asymptotic properties of (A.1).

Lemma A.1. *Let* $L = \begin{Vmatrix} p & q \\ r & s \end{Vmatrix}$ *and* $M = \begin{Vmatrix} a & b \\ c & d \end{Vmatrix}$, *where* $\det M = 0$. *Then*

$$M^n = (\operatorname{tr} M)^{n-1} M \equiv \rho^{n-1}(M) M, \tag{A.2}$$

$$MLM = \eta(L, M) M, \tag{A.3}$$

where $\eta(L, M) := (ap + br + cq + ds)$.

Proof. If the matrix M is degenerate, then the relations $c = \alpha a$ and $d = \alpha b$ hold for some α, i.e., $M = \begin{Vmatrix} a & b \\ \alpha a & \alpha b \end{Vmatrix}$. Therefore, $M^2 = \begin{Vmatrix} a & b \\ \alpha a & \alpha b \end{Vmatrix} \times \begin{Vmatrix} a & b \\ \alpha a & \alpha b \end{Vmatrix} = (a + \alpha b) M = (\operatorname{tr} M) M$. Hence we obtain (A.2). To demonstrate (A.3), it suffices to note that

$$MLM = \begin{Vmatrix} a & b \\ \alpha a & \alpha b \end{Vmatrix} \times \begin{Vmatrix} p & q \\ r & s \end{Vmatrix} \times \begin{Vmatrix} a & b \\ \alpha a & \alpha b \end{Vmatrix} = (ap + br + \alpha aq + \alpha bs) L = (ap + br + cq + ds) L.$$

Hence we obtain (A.3). □

Now we consider an arbitrary matrix product (A.1), in which $A_i = L$ or $A_i = M$ for every $i = 1, 2, \dots, n$. Grouping like coefficients in (A.1), we find that (A.1) admits one of the representations

$$A_n A_{n-1} \dots A_1 = L^{n_p} M^{n_{p-1}} \dots M^{n_1} L^{n_0},$$
$$A_n A_{n-1} \dots A_1 = M^{n_p} L^{n_{p-1}} \dots M^{n_1} L^{n_0},$$
$$A_n A_{n-1} \dots A_1 = L^{n_p} M^{n_{p-1}} \dots L^{n_1} M^{n_0},$$
$$A_n A_{n-1} \dots A_1 = M^{n_p} L^{n_{p-1}} \dots L^{n_1} M^{n_0},$$

where $n_0 + n_1 + \dots + n_{p-1} + n_p = n$, $n_i \geq 1$.

[4] Furthermore, if L and M are commuting matrices, the convergence of product (A.1) is a trivial problem: $\mathcal{L} \in \mathcal{C}(2, 2)$ if and only if $\rho(L) < 1$ and $\rho(M) < 1$. Hence interesting effects can be expected only for noncommuting matrices.

Without loss of generality, we take $A_nA_{n-1}\dots A_1 = L^{n_p}M^{n_{p-1}}\dots M^{n_1}L^{n_0}$, where n_0 and n_p may be zero. Then, by Lemma A.1, the matrix product $A_nA_{n-1}\dots A_1$ admits the representation

$$A_nA_{n-1}\dots A_1 = \gamma L^{n_p}ML^{n_0}, \tag{A.4}$$

in which the constant γ can be determined in explicit form:

$$\begin{aligned}\gamma &= (\operatorname{tr} M)^{n_{p-1}-1}\eta(L^{n_{p-2}},M)\dots\eta(L^{n_2},M)(\operatorname{tr} M)^{n_1-1}\\ &= (\operatorname{tr} M)^{n_{p-1}+\dots+n_1-[p/2]}\eta(L^{n_{p-2}},M)\dots\eta(L^{n_2},M).\end{aligned} \tag{A.5}$$

From representations (A.4) and (A.5) we obtain the following conditions for the convergence, boundedness, and unboundedness of a matrix set $\{L, M\}$.

Lemma A.2. *For $\rho(L)$ and $\rho(M) < 1$, the matrix set $\{L, M\} \in \mathcal{M}(2, 2)$ converges, is bounded, or unbounded if and only if*

$$\eta_*(L, M) := \sup_n |\eta(L^n, M)|$$

is less than, equal to, or greater than 1*, respectively.*

Proof. First assuming that $\eta_*(L, M) < 1$, consider a matrix sequence $\{A_n\}$ in which $A_n = L$ or $A_n = M$. If this sequence contains only a finite number of matrices of the type L or M, then, beginning from some $n = n_0$, all elements if the sequence $\{A_n\}$ are identical. In this case, $A_n = L$ for $n \ge n_0$, or $A_n = M$ for $n \ge n_0$. In either case, the conditions $\rho(L) < 1$ and $\rho(M) \equiv |\operatorname{tr} M| < 1$ imply the relation $A_nA_{n-1}\dots A_1 \to 0$.

If the sequence $\{A_n\}$ contains infinitely many elements of the type L as well as of the type M, then representation (A.4), (A.5) holds. In this case, by the conditions $\rho(L) < 1$ and $\rho(M) < 1$, the factors L^{n_i} and M in (A.4) are uniformly upper bounded in matrix norm. Furthermore, the factor

$$(\operatorname{tr} M)^{n_{p-1}+\dots+n_1-[p/2]} \equiv \rho(M)^{n_{p-1}+\dots+n_1-[p/2]} \tag{A.6}$$

in (A.5), by the condition $\rho(M) < 1$, is not greater than 1, and factors of the type $\eta(L^{n_i}, M)$ in (A.5) are not greater that $\eta_*(L, M) < 1$ in modulus. Since, by assumption, the sequence $\{A_n\}$ contains infinitely many elements of the types L and M, factors of the type $\eta(L^{n_i}, M)$ in (A.5) unboundedly increase in number with n. Hence the estimates given above imply the relation $A_nA_{n-1}\dots A_1 \to 0$.

Now take $\eta_*(L, M) = 1$. Then, as in the previous case, by the conditions $\rho(L) < 1$ and $\rho(M) < 1$, the factors L^{n_i} and M in (A.4) are uniformly upper bounded in matrix norm, factor (A.6) in (A.5) is not greater than 1, and every factor of the type $\eta(L^{n_i}, M)$ in (A.5) is also not greater than $\eta_*(L, M) = 1$ in modulus. This implies the boundedness of the matrix set $\mathcal{P}(\{L, M\})$. Let us demonstrate the existence of a sequence of matrices $A_n \in \{L, M\}$ for which the relation $A_nA_{n-1}\dots A_1 \to 0$ is not satisfied under this situation.

Since, by assumption, $\eta_*(L, M) = \sup_n |\eta(L^n, M)| = 1$, there exists a sequence of indexes n_i, $i = 1, 2, \dots$, for which

$$\prod_i \eta(L^{n_i}, M) > 0. \tag{A.7}$$

In this case, let us construct a matrix sequence $\{A_n\}$, $n \ge 0$, in which the first element is M, then the next n_1 elements are L, the succeeding element is again M, and the next n_2 elements are L and so on. In other words, the matrix sequence $\{A_n\}$ consists of groups of $n_1, n_2, \dots, n_k \dots$ elements of the type L and each group is separated from the other by an element of the type M. In this case, according to representation (A.4), (A.5), every n of the type $n = n_i + \ldots + n_1 + i + 1$ admits the representation

$$A_n A_{n-1} \ldots A_1 = \eta(L^{n_i}, M) \ldots \eta(L^{n_2}, M)\eta(L^{n_1}, M)M$$

and, consequently, by virtue of (A.7), the product of matrices A_n does not tend to zero. This completes the proof for the boundedness of the matrix set $\{L, M\}$.

Finally, let the condition $\eta_*(L, M) > 1$ hold. Then, the inequality $|\eta(L^{n_0}, M)| > 1$ holds for some $n = n_0$ and representation (A.4), (A.5) implies that the periodic sequence $\{A_n\}$, in which the first $n_0 + 1$ elements are of the type $A_1 = M$, $A_2 = A_3 = \ldots = A_{n_0+1} = L$ and other elements are repeated with period $n_0 + 1$, satisfies the relation

$$\overline{\lim_{n\to\infty}} \|A_n A_{n-1} \ldots A_1\| = \infty.$$

This completes the proof for the sufficiency of the conditions $\eta_*(L, M) < 1$, $\eta_*(L, M) = 1$, and $\eta_*(L, M) > 1$ for the convergence, boundedness, and unboundedness of the matrix $\{L, M\}$, respectively. Since these three conditions contain all possible combinations of relations between the numbers $\eta_*(L, M)$ and 1, they are also the necessary conditions. This completes the proof. □

According to Lemmas A.1 and A.2, to study the convergence of the matrix set $\{L, M\}$ under the above assumptions, we must compute $\eta_*(L, M)$, and, consequently, $\eta(L, M)$.

At first sight, the function $\eta(L^n, M)$ admits an explicit expression through the elements of the matrices L and M only in certain particular cases: either for $n = 1$ or if the elements of the matrix L^n is expressed explicitly through the elements of the matrix L. Fortunately, applying some suitable coordinate substitution, we can always reduce the matrix L to normal Jordan form. Hence, without loss of generality, we can assume that the matrix L is defined by one of the equalities[5]

$$L = \left\| \begin{matrix} \lambda & 0 \\ 0 & \mu \end{matrix} \right\|, \quad \text{or} \quad L = \left\| \begin{matrix} \lambda & 1 \\ 0 & \lambda \end{matrix} \right\|, \quad \text{or} \quad L = \lambda \left\| \begin{matrix} \cos\varphi & -\sin\varphi \\ \sin\varphi & \cos\varphi \end{matrix} \right\|.$$

Now explicit expressions for the respective constants $\eta(L^n, M)$ can be found with Lemma A.1; they are given in the next lemma.

Lemma A.3. *Let* $M = \left\| \begin{matrix} a & b \\ c & d \end{matrix} \right\|$, *where* $\det M = 0$. *Then*

$$\eta(L^n, M) = a\lambda^n + d\mu^n \quad \textit{for} \quad L = \left\| \begin{matrix} \lambda & 0 \\ 0 & \mu \end{matrix} \right\|, \tag{A.8}$$

$$\eta(L^n, M) = a\lambda^n + (n-1)c\lambda^{n-1} + d\lambda^n \quad \textit{for} \quad L = \left\| \begin{matrix} \lambda & 1 \\ 0 & \lambda \end{matrix} \right\|, \tag{A.9}$$

$$\eta(L^n, M) = (a+d)\lambda^n \cos n\varphi + (b-c)\lambda^n \sin n\varphi \quad \textit{for} \quad L = \lambda \left\| \begin{matrix} \cos\varphi & -\sin\varphi \\ \sin\varphi & \cos\varphi \end{matrix} \right\|. \tag{A.10}$$

As has already been mentioned, by Lemmas A.1 and A.2, to study the convergence of the matrix set $\{L, M\}$, we must compute $\eta_*(L, M) := \sup\limits_n |\eta(L^n, M)|$, which consists in determining the largest value of the function $|\eta(L^n, M)|$. The corresponding analysis for a diagonal matrix L is given in the next lemma.

[5] Along with the reduction of the matrix L to Jordan form by some transformation, the matrix M also must be reduced. Since the choice of M is arbitrary, the reduction of L to the normal Jordan form is not restrictive.

Lemma A.4. *For $0 < \lambda,\ \mu < 1$, and $|a+d| < 1$, the function $\varphi(u) := |a\lambda^u + d\mu^u|,\ u \geq 0,\ u \in \mathbb{R}$, is maximal at the point*

$$u_* = \frac{\ln\left(-\dfrac{a\ln\lambda}{d\ln\mu}\right)}{\ln\mu - \ln\lambda} \tag{A.11}$$

if

$$ad < 0, \quad (\mu - \lambda)\,(a|\ln\lambda| - d|\ln\mu|) > 0, \tag{A.12}$$

or at the point $u_* = 0$ otherwise.

Proof. A function $\varphi(u)$ of the form $\varphi'(u) = a\lambda^u \ln\lambda + d\mu^u \ln\mu$ vanishes only if

$$\left(\frac{\mu}{\lambda}\right)^u = -\frac{a\ln\lambda}{d\ln\mu}. \tag{A.13}$$

Hence Eq. (A.13) has exactly one positive solution (A.11) or no solution if condition (A.12) holds or not. Consequently, the function $|\varphi(u)|$ steadily decreases for $u \geq 0$ and has a unique maximum at zero. $\square$

Corollary. *If $a = -d > 0$ and $\mu = \lambda^2 < 1$, then a function $\varphi(u)$ of the form $\varphi(u) = a(\lambda^u - \lambda^{2u})$ is maximal at the point $u_* = -\ln 2/\ln\lambda$ and*

$$\sup_{u\geq 0}\varphi(u) = \varphi(u_*) = \frac{a}{4}. \tag{A.14}$$

Lemma A.5. *If the conditions of Lemma* A.1 *hold and $\rho(L) < 1$, then $\eta_*(L,M)$ continuously depends on $L, M \in \mathcal{M}(2,2)$.*

Proof. Let $L_k, M_k \in \mathcal{M}(2,2)$. Let $L_k \to L_0$ and let $M_k \to M_0$. Then the relations

$$\lim_{k\to\infty} \eta_*(L_k, M_k) := \lim_{k\to\infty} \sup_m \eta(L_k^m, M_k) \geq \lim_{k\to\infty} \eta(L_k^n, M_k) = \eta(L_0^n, M_0)$$

hold for every fixed n. Hence, taking the supremum in the right and left sides of the resulting chain of inequalities, we obtain

$$\liminf_{k\to\infty} \eta_*(L_k, M_k) \geq \sup_n \eta(L_0^n, M_0) = \eta_*(L_0, M_0).$$

What now remains to conclude the proof is to show that

$$\overline{\lim_{k\to\infty}}\, \eta_*(L_k, M_k) \leq \eta_*(L_0, M_0). \tag{A.15}$$

Let $\|\cdot\|$ be some norm in the space of matrices $\mathcal{M}(2,2)$. Then, by the condition $\rho(L_0) < 1$, there exist numbers $q \in (0,1)$, $\sigma > 0$, and $c < \infty$ such that the uniform estimates $\|L^n\| \leq cq^n$ hold for all matrices L in the σ-neighborhood of the matrix L_0 (i.e., if $\|L - L_0\| < \sigma$). Indeed, the spectral radius of the matrix L_0 is given by the formula [19]

$$\rho(L_0) = \lim_{n\to\infty} \|L_0^n\|^{1/n}. \tag{A.16}$$

For some positive number $\nu < 1 - \rho(L_0)$, let

$$\|x\|_\nu = \sup_{n\geq 0}\,(\rho(L_0) + \nu)^{-n}\,\|L_0^n x\|.$$

Formula (A.16) shows that the numbers $(\rho(L_0)+\nu)^{-n}\,\|L_0^n\|$ are bounded above by some constant $c<\infty$. Therefore, $\|x\|\le\|x\|_\nu\le c\|x\|$ and, consequently, $\|\cdot\|_\nu$ is the norm. Simple computations yield

$$\|L_0x\|_\nu=\sup_{n\ge0}\,(\rho(L_0)+\nu)^{-n}\,\|L_0^{n+1}x\|\le(\rho(L_0)+\nu)\,\|x\|_\nu.$$

Hence $\|L_0\|_\nu\le\rho(L_0)+\nu$. Now we choose a number q such that $\rho(L_0)+\nu<q<1$. Then the estimate $\|L\|_\nu\le q$ holds for all matrices L satisfying the relation $\|L-L_0\|_\nu\le q-\rho(L_0)-\nu$. Consequently, such matrices L also satisfy the relation $\|L^n\|_\nu\le q^n$ for $n=1,2\ldots$. Therefore $\|L^n\|\le cq^n$, $n=1,2\ldots$, for all matrices L satisfying the relation $\|L-L_0\|\le\sigma=(q-\rho(L_0)-\nu)/c$.

Thus, without loss of generality, we can assume that the estimates $\|L_k^n\|\le cq^n$ hold for all n and k. Therefore, for every $\varepsilon>0$, there exists a number $n(\varepsilon)$ such that

$$|\eta(L_k^n,M_k)|\le\varepsilon\ \forall n\ge n(\varepsilon)\ \forall k. \tag{A.17}$$

Now expressing $\eta_*(L_k,M)$ as

$$\eta_*(L_k,M):=\sup_n|\eta(L_k^n,M_k)|=\max\left\{\max_{n<n(\varepsilon)}|\eta(L_k^n,M_k)|,\quad\sup_{n\ge n(\varepsilon)}|\eta(L_k^n,M_k)|\right\},$$

we obtain

$$\eta_*(L_k,M)\le\max\left\{\max_{n<n(\varepsilon)}|\eta(L_k^n,M_k)|,\ \varepsilon\right\},$$

by virtue of (A.17). In the estimate, taking the limit as $k\to\infty$ (this is possible, because the inner maximum contains a finite number of terms not dependent on k), we obtain

$$\overline{\lim_{k\to\infty}}\,\eta_*(L_k,M_k)\le\max\left\{\max_{n<n(\varepsilon)}|\eta(L_0^n,M_0)|,\ \varepsilon\right\}\le\max\left\{\eta_*(L_0,M_0),\ \varepsilon\right\}.$$

Since the choice of ε is arbitrary, we have

$$\overline{\lim_{k\to\infty}}\,\eta_*(L_k,M_k)\le\eta_*(L_0,M_0).$$

This completes the proof of inequality (A.15) and, along with it, the proof of the lemma. □

A.2. PROOF OF THEOREM 1

The underlying idea of the proof of Theorem 1 is similar to the idea underlying the proof of a weaker assertion of [4] based on the fact that semialgebraic sets only contain a finite number of connectivity components. Since, as mentioned in Section 2, definable sets in *o*-minimal structures also contain only a finite number of connectivity components, to prove Theorem 1, we can apply the constructs and reasoning of [4]. Nevertheless, not only the indefinability of the sets $\mathcal{C}(p,q)$, $\mathcal{B}(p,q)$, and $\mathcal{U}(p,q)$ in *o*-ordered structures, but also the constructs used in demonstrating this fact may be of interest in application. Therefore, below we give an analytical proof for Theorem 1, which is quite different from that given in [4].

Let us outline the scheme of proof for the case $p=q=2$. We construct a family of pairs of 2×2 matrices $\mathcal{L}(t)=\{L(t),M(t)\}\in\mathcal{M}(2,2)$ dependent on a parameter $t\in[0,1]$ and having a property, i.e., the set $\mathcal{L}_+=\bigcup\limits_{t\in[0,1]}(\mathcal{L}(t),t)$ is semialgebraic set and, consequently, a definable subset of $\mathcal{M}(2,2)\times\mathbb{R}^1$ in the respective o-ordered structure. Then, assuming that the sets $\mathcal{C}(2,2)$, $\mathcal{B}(2,2)$, are $\mathcal{U}(2,2)$ are definable in the respective o-ordered structure, we find that each of the sets

$$\mathcal{L}_{\mathcal{C}} = \mathcal{L}_+ \cap \Big(\mathcal{C}(2,2)\times\mathbb{R}^1\Big), \quad \mathcal{L}_{\mathcal{B}} = \mathcal{L}_+ \cap \Big(\mathcal{B}(2,2)\times\mathbb{R}^1\Big), \quad \mathcal{L}_{\mathcal{U}} = \mathcal{L}_+ \cap \Big(\mathcal{U}(2,2)\times\mathbb{R}^1\Big),$$

being a Boolean combination of definable sets, must also be definable in the respective o-ordered structure. Therefore, by property (vi) of ordered structures, the projections of the sets $\mathcal{L}_{\mathcal{C}}$, $\mathcal{L}_{\mathcal{B}}$ and $\mathcal{L}_{\mathcal{U}}$ onto the $\mathbb{R}^1$-component of the direct product $\mathcal{M}(2,2)\times\mathbb{R}^1$ must also be definable. But the matrices $L(t)$ and $M(t)$ are constructed such that the corresponding projections consist of an infinite number of connectivity components and, consequently, are not definable in o-ordered structures containing semialgebraic sets. This contradiction implies that the sets $\mathcal{C}(2,2)$, $\mathcal{B}(2,2)$, and $\mathcal{U}(2,2)$ are not definable in o-ordered structures containing semialgebraic sets.

Then to prove the theorem for arbitrary $p,q\ge 2$, it suffices first to construct two $p\times p$ matrices $\widetilde{L}_1(t)$ and $\widetilde{L}_2(t)$ containing predetermined matrices $L(t)$ and $M(t)$, respectively, at the upper left corner and zero at other places. Then $\widetilde{L}_3(t),\ldots,\widetilde{L}_q(t)$ are taken to be zero matrices. The matrix set $\widetilde{\mathcal{L}}(t) := \Big\{\widetilde{L}_1(t),\widetilde{L}_2(t),\widetilde{L}_3(t),\ldots,\widetilde{L}_q(t)\Big\}$ thus constructed belongs to $\mathcal{M}(p,q)$ for every t and is obviously convergent, bounded or unbounded if and only if the set of 2×2 matrices $\{L(t),M(t)\}$ is convergent, bounded or unbounded, respectively. Then the indefinability of the sets $\mathcal{C}(2,2)$, $\mathcal{B}(2,2)$, and $\mathcal{U}(2,2)$ in o-ordered structures containing semialgebraic sets implies that the sets $\mathcal{C}(p,q)$, $\mathcal{B}(p,q)$, and $\mathcal{U}(p,q)$ are undefinable in o-ordered structures containing semialgebraic sets.

We now construct pairs of matrices $\mathcal{L}(t)=\{L(t),M(t)\}\in\mathcal{M}(2,2)$ for which the projections of the sets $\mathcal{L}_{\mathcal{C}}$, $\mathcal{L}_{\mathcal{B}}$, and $\mathcal{L}_{\mathcal{U}}$ on the $\mathbb{R}^1$-component of the direct product $\mathcal{M}(2,2)\times\mathbb{R}^1$ contain an infinite number of connectivity components. Suppose that we have constructed families of matrices $L(t)$ and $M(t)$ satisfying the following conditions.

Δ_1: The elements of the matrices $L(t)$ and $M(t)$ are interrelated with one another and with t by a polynomial dependence:

Δ_2: the inequalities $\rho(L(t))<1$ and $\rho(M(t))<1$ hold for all sufficiently small $t>0$,

Δ_3: there exist $t_n\to 0$, $t_n>0$, for which $\sup_n|\eta(L^n(t_n),M(t_n))|<1$,

Δ_4: there exist $s_n\to 0$, $s_n>0$, for which $\sup_n|\eta(L^n(s_n),M(s_n))|>1$, and

Δ_5: there exist $r_n\to 0$, $r_n>0$, for which $\sup_n|\eta(L^n(r_n),M(r_n))|=1$.

In this case, without loss of generality, we can assume that the elements of the sequences $\{t_n\}$, $\{s_n\}$ and $\{r_n\}$ are interrelated by the expression $t_{n+1}<s_n<r_n<t_n$. Then the projection of each of the *semialgebraic* sets $\mathcal{L}_{\mathcal{C}}$, $\mathcal{L}_{\mathcal{B}}$, and $\mathcal{L}_{\mathcal{U}}$ onto the t-coordinate contains an infinite number of connectivity components (since the elements of $\{t_n\}$, $\{s_n\}$, and $\{r_n\}$ belonging to projections of different sets, by virtue of the relations $t_{n+1}<s_n<r_n<t_n$, alternate). Hence, as mentioned above, we find that the sets $\mathcal{C}(2,2)$, $\mathcal{B}(2,2)$, and $\mathcal{U}(2,2)$ are not definable in o-ordered structures containing semialgebraic sets.

Remark A.1. The proof requires only the existence of the sequences $\{t_n\}$ and $\{s_n\}$ having the above properties. The existence of the sequence $\{r_n\}$ is implied by the existence of the sequences $\{t_n\}$ and $\{s_n\}$. Indeed, by the continuity of the function $\eta_*(L(t),M(t))$ (see Lemma A.5), for every pair of numbers $s_n<t_n$ satisfying conditions Δ_3 and Δ_4, there exists an $r_n\in(s_n,t_n)$ for which condition Δ_5 is satisfied, that means, the sequence $\{r_n\}$ exists.

Thus, our problem now is to construct families of matrices $L(t)$ and $M(t)$ satisfying conditions Δ_1–Δ_5. Let us choose a pair of matrices

$$L(t)=\begin{Vmatrix}\lambda(t) & 0\\ 0 & \mu(t)\end{Vmatrix}, \quad M(t)=\begin{Vmatrix}a(t) & b(t)\\ c(t) & d(t)\end{Vmatrix}, \quad \det M(t)\equiv 0, \tag{A.18}$$

where $\lambda(t) > 0$ and $\mu(t) > 0$. Then the first condition of Δ_2 is equivalent to the inequalities $0 < \lambda(t)$ and $\mu(t) < 1$, and the second condition of Δ_2 takes the form $|a(t) + d(t)| < 1$. We additionally assume that the elements $\lambda(t)$, $\mu(t)$, $a(t)$, and $d(t)$ of the matrices $L(t)$ and $M(t)$ satisfy the relations

$$\mu(t) \equiv \lambda^2(t), \quad a(t) \equiv -d(t) > 0.$$

Then, by Lemmas A.2 and A.4, the convergence of the matrix set $\{L(t), M(t)\}$ for a certain value of the parameter t is determined by

$$\eta_*(L(t), M(t)) := \sup_n |\eta(L^n(t), M(t))| = \sup_{u\geq 0, u\in\mathbb{Z}} \varphi_t(u).$$

At the same time, by the corollary of Lemma A.4,

$$|\eta(L^n(t), M(t))| \equiv \varphi_t(n) = a(t)(\lambda^u(t) - \lambda^{2u}(t)).$$

Moreover,

$$\sup_{u\geq 0, u\in\mathbb{R}} \varphi_t(u) = \varphi_t(u_t) = \frac{a(t)}{4}, \quad \text{where} \quad u_t = -\frac{\ln 2}{\ln \lambda(t)}. \tag{A.19}$$

Note that $\sup\limits_{k\geq 0, k\in\mathbb{Z}} \varphi_t(k)$ coincides with $\sup\limits_{u\geq 0, u\in\mathbb{R}} \varphi_t(u)$ only if u_t is an integer. In other cases, we have the strict inequality

$$\sup_{k\geq 0, k\in\mathbb{Z}} \varphi_t(k) = \max\left\{\varphi_t([u_t]),\ \varphi_t([u_t]+1)\right\} < \sup_{u\geq 0, u\in\mathbb{R}} \varphi_t(u), \tag{A.20}$$

where $[\,\cdot\,]$ is the integral part of the number (i.e., the largest integer that is not greater than the value of the argument). The concluding part of the proof depends precisely on the strictness of the inequality in the right side of (A.20).

Let us find a number sequence $\{t_n\}$ such that $u_{t_n} = n$ for every n. By virtue of (A.19), we have

$$\sup_{k\geq 0, k\in\mathbb{Z}} \varphi_{t_n}(k) = \varphi_{t_n}(n) = \frac{a(t_n)}{4}, \quad \lambda(t_n) := 2^{-\frac{1}{n}}. \tag{A.21}$$

Let us now find a number sequence $\{s_n\}$ such that $u_{s_n} = n + \dfrac{1}{2}$ for every n. Then, by (A.20), we obtain the relations

$$\sup_{k\geq 0, k\in\mathbb{Z}} \varphi_{s_n}(k) = \max\left\{\varphi_{s_n}(n),\ \varphi_{s_n}(n+1)\right\}, \quad \lambda(s_n) := 2^{-\frac{2}{2n+1}}. \tag{A.22}$$

Here, by the definition of the function $\varphi_t(u)$,

$$\varphi_{s_n}(n) = a(s_n)\left(\lambda^n(s_n) - \lambda^{2n}(s_n)\right), \quad \varphi_{s_n}(n+1) = a(s_n)\left(\lambda^{n+1}(s_n) - \lambda^{2n+2}(s_n)\right)$$

and direct computation yields $\lambda^n(s_n) - \lambda^{2n}(s_n) < \lambda^{n+1}(s_n) - \lambda^{2n+2}(s_n)$. Therefore

$$\sup_{k\geq 0, k\in\mathbb{Z}} \varphi_{s_n}(k) = \varphi_{s_n}(n+1) = a(s_n)\left(\lambda^{n+1}(s_n) - \lambda^{2n+2}(s_n)\right),$$

where, by virtue of the relation $\lambda(s_n) := 2^{-\frac{2}{2n+1}}$,

$$\lambda^{n+1}(s_n) - \lambda^{2n+2}(s_n) = \frac{1}{2}\sqrt{\lambda(s_n)} - \frac{1}{4}\lambda(s_n) = \frac{1}{4}\left(1 - \left(1 - \sqrt{\lambda(s_n)}\right)^2\right).$$

Hence we finally obtain

$$\sup_{k\geq 0, k\in\mathbb{Z}} \varphi_{s_n}(k) = \frac{a(s_n)}{4}\left(1-\left(1-\sqrt{\lambda(s_n)}\right)^2\right), \qquad \lambda(s_n) := 2^{-\frac{2}{2n+1}}. \tag{A.23}$$

Now to conclude the proof, it suffices to choose functions $a(t)$ and $\lambda(t)$ polynomially dependent on t for which $\sup_{k\geq 0, k\in\mathbb{Z}} \varphi_{t_n}(k) > 1$ and $\sup_{k\geq 0, k\in\mathbb{Z}} \varphi_{s_n}(k) < 1$, respectively, or, which, by virtue of (A.21) and (A.23), are equivalent to

$$\frac{a(t_n)}{4} > 1, \quad \frac{a(s_n)}{4}\left(1-\left(1-\sqrt{\lambda(s_n)}\right)^2\right) < 1, \tag{A.24}$$

in which t_n and s_n are defined by the equalities $\lambda(t_n) := 2^{-\frac{1}{n}}$, and $\lambda(s_n) := 2^{-\frac{2}{2n+1}}$, respectively.

It is easy matter to verify that relations (A.24) hold if the functions $a(t)$ and $\lambda(t)$ are defined by $a(t) = 4 + t^3$ and $\lambda(t) = 1 - t$, respectively. Indeed, in this case $\frac{a(t_n)}{4} = 1 + \frac{1}{4}t_n^3 > 1$ and

$$\frac{a(s_n)}{4}\left(1-\left(1-\sqrt{\lambda(s_n)}\right)^2\right) \simeq \left(1+\frac{1}{4}s_n^3\right)\left(1-\frac{1}{4}s_n^2\right) \simeq 1 - \frac{1}{4}s_n^2 < 1$$

for all sufficiently large n (and, accordingly, small s_n).

Hence, by virtue (A.24), the conditions Δ_3 and Δ_4 hold for a family of pairs of 2×2 matrices $\mathcal{L}(t) = \{L(t), M(t)\}$ with elements $\lambda(t) = 1 - t$, $\mu(t) = \lambda^2(t)$, and $a(t) = b(t) = -d(t) = -c(t) = 4 + t^3$. Consequently, the projection of each of *semialgebraic* sets $\mathcal{L}_{\mathcal{C}}$, $\mathcal{L}_{\mathcal{B}}$, and $\mathcal{L}_{\mathcal{U}}$ onto the t-coordinate contains an infinite number of connectivity components.

This completes the proof of the theorem. □

Remark A.2. This scheme of the proof of Theorem 1 is also applicable to the case in which the matrix $L(t)$ is not diagonal as in (A.8), but representable as a Jordan block (A.9) or a rotation matrix (A.10). These cases are not considered here, because the proof steps differ only in technical details related to the determination of the largest value of the function $|\eta(L^n, M)|$.

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