Single parameter dissipativity and attractors in discrete time asynchronous systems*

P. E. Kloeden

FB Mathematik, Johann Wolfgang Goethe Universität,

D-60054 Frankfurt am Main, Germany

E-mail: `kloeden@math.uni-frankfurt.de`

V. S. Kozyakin†

Institute for Information Transmission Problems

Russian Academy of Sciences

Bolshoj Karetny lane, 19, 101447 Moscow, Russia

E-mail: `kozyakin@iitp.ru`

Abstract *Discrete time nonautonomous dynamical systems generated by nonautonomous difference equations are formulated as discrete time skew-product systems consisting of cocycle state mappings that are driven by discrete time autonomous dynamical systems. Forwards and pullback attractors are two possible generalizations of autonomous attractors to such systems. Their existence follows from appropriate forwards or pullback dissipativity conditions. For discrete time nonautonomous dynamical systems generated by asynchronous systems with frequency updating components such a dissipativity condition is usually known for a single starting parameter value of the driving system. Additional conditions that then ensure the existence of a forwards or pullback attractor for such an asynchronous system are investigated here.*

AMS Subject Classification: 34C35, 93D09

Key words: cocycle mapping, forwards and pullback attractors, forwards and pullback dissipativity conditions, asynchronous systems

1 Introduction

A discrete time nonautonomous dynamical system on a state space $\mathbb{R}^d$ can be formulated in terms of a parameterized state space system that is driven by an autonomous dynamical

*This work was supported by the DFG Forschungsschwerpunkt "Ergodentheorie, Analysis und effiziente Simulation dynamischer Systeme".

†The author was partially supported by grants No. 00-15-96116 and 0001-00571 of the Russian Foundation for Basic Research.

system on a parameter space P, specifically as a triangularly coupled pair of difference equations

$$x_{n+1} = f(p_n, x_n), \quad p_{n+1} = \Theta(p_n), \qquad n \in \mathbb{Z} \tag{1}$$

on $\mathbb{R}^d \times P$, where P is a topological space and $\Theta : P \to P$ is a homeomorphism. A trivial example is a general nonautonomous difference equation

$$x_{n+1} = g(n, x_n), \qquad n \in \mathbb{Z}, \tag{2}$$

which can be rewritten in the form (1) with $P = \mathbb{Z}$, $f(p, x) := g(n, x)$ and $\Theta(p) := p$. However, examples that involve a compact parameter space P are much richer dynamically and mathematically more interesting [3].

In this paper we focus attention on a class of such systems known as discrete time asynchronous systems [1]. These consist of several subcomponents that are updated at discrete time instants according to the reading of an internal clock.

The system (1) here has the specific form

$$x_{n+1} = f(p_n, x_n), \qquad p_{n+1} = p_n + \eta, \quad \mod 1, \tag{3}$$

for a given $\eta \in (0, 1)$, where $f : P \times \mathbb{R}^d \to \mathbb{R}^d$ satisfies

$$f(p, x) = \begin{cases} F(x), & \text{if} \quad p = 0, \\ G(x), & \text{if} \quad p \in (0, \eta), \\ H(x), & \text{if} \quad p \in [\eta, 1). \end{cases} \tag{4}$$

for three particular functions $F, G, H : \mathbb{R}^d \to \mathbb{R}^d$, while the "clock" $\Theta : P \to P$ with $P = [0, 1]$ is defined by

$$\Theta(p) := p + \eta, \quad \mod 1. \tag{5}$$

Such situations arise in parallel computing, control theory or telecommunication applications, where one is interested to know if some dynamical property for a particular initial value p_0^* of (5) implies the same dynamical property for all initial values $p_0 \in P$. This question has been extensively investigated for the asymptotical stability of an equilibrium state $\bar{x}$, i.e. with $f(p, \bar{x}) = \bar{x}$ for all $p \in P$. See [1, 5, 6], where an affirmative answer is given when η is irrational and the mapping $f(p, x)$ is homeomorphic in x for each $p \in [0, 1)$ in some neighborhood of $\bar{x}$.

The paper is structured as follows. The cocycle formalism of nonautonomous dynamics is recalled in the next section, along with the definitions of forwards and pullback attractors and of forwards and pullback dissipativity conditions. Finally, sufficient conditions ensuring the validity of such a dissipativity condition along the entire parameter trajectory of the driving system when it holds for a single parameter value are then given. These issues are discussed in detail and applied in the third section in the context of discrete time asynchronous systems, which have a very special kind of driving system or clock (5). There are two appendices, the first containing a symbolic analysis of the driving system (5) and its effects on the dynamics of the asynchronous system, while the second explains the origin of the difference equations formalism of the asynchronous systems that are used earlier in the paper.

2 Cocycle dynamics and attractors

The solution mapping $\Phi : \mathbb{Z}^+ \times P \times \mathbb{R}^d \to \mathbb{R}^d$ of (1), which is defined directly as

$$\Phi(n + 1, p, x) = f(\theta_n p, , \cdot) \circ \cdots \circ f(\theta_1 p, \cdot) \circ f(p, x), \qquad x \in \mathbb{R}^d, \, p \in P, \, n \in \mathbb{Z}^+, \tag{6}$$

where we write $\theta_n p := \Theta^n(p)$ for $n \in \mathbb{Z}^+$ for brevity, or recursively as

$$\Phi(n+1, p, x) = f\left(\theta_n p, \Phi(n, p, x)\right), \qquad x \in \mathbb{R}^d,\ p \in P,\ n \in \mathbb{Z}^+,$$

with the *initial condition property*

$$\Phi(0, p, x) = x, \qquad x \in \mathbb{R}^d,\ p \in P,$$

satisfies the *cocycle property*

$$\Phi(m+n, p, x) = \Phi(m, \theta_n p, \Phi(n, p, x)), \qquad x \in \mathbb{R}^d,\ p \in P,\ m, n \in \mathbb{Z}^+.$$

The solution mapping Φ is a *cocycle mapping* on $\mathbb{R}^d$ with respect to the autonomous dynamical system Θ on P and we call the pair (Φ, Θ) a cocycle or nonautonomous dynamical system (NDS).

We make the following assumptions on the mappings f and Θ in (1).

Assumption 1 *The mapping $x \mapsto f(p, x)$ is continuous in $x \in \mathbb{R}^d$ for each $p \in P$.*

Assumption 2 *The mapping $\Theta : P \to P$ is a homeomorphism, i.e. the mapping $p \mapsto \theta_n p := \Theta^n(p)$ exists and is continuous in $p \in P$ for each $n \in \mathbb{Z}$.*

It follows then that the mapping $x \mapsto \Phi(n, p, x)$ is continuous in $x \in \mathbb{R}^d$ for each $n \in \mathbb{Z}^+$ and $p \in P$. Note that in [3] the mapping $f : P \times \mathbb{R}^d \to \mathbb{R}^d$ was assumed to be continuous in (p, x), in which case the mapping $(p, x) \mapsto \Phi(n, p, x)$ would also be is continuous in (p, x) for each $n \in \mathbb{Z}^+$, but this is not essential here and certainly does not apply to the asynchronous system (3)–(5).

A family $\widehat{A} = \{A_p;\, p \in P\}$ of nonempty compact subsets of $\mathbb{R}^d$ is called Φ-*invariant* if

$$\Phi(n, p, A_p) = A_{\theta_n p}, \qquad p \in P,\ n \in \mathbb{Z}^+$$

We say that $\widehat{A}$ *forwards attracting* if

$$H^*(\Phi(n, p, D), A_{\theta_n p}) \to 0 \quad \text{as} \quad n \to \infty, \tag{7}$$

and *pullback attracting* if

$$H^*(\Phi(n, \theta_{-n} p, D), A_p) \to 0 \quad \text{as} \quad n \to \infty, \tag{8}$$

for all $p \in P$ and nonempty compact subsets D of $\mathbb{R}^d$, where H^* is the Hausdorff semi-metric. Although pullback convergence (8) may seem less natural than forwards convergence (7), it is convenient as it ensures convergence to a specific component set A_p for a fixed p.

Definition 1 *A family $\widehat{A} = \{A_p;\, p \in P\}$ of nonempty compact subsets of $\mathbb{R}^d$ is called a* forwards attractor *for the NDS (Φ, Θ) if it*
(i) is Φ-invariant,
(i) forwards attracts nonempty compact subsets D of $\mathbb{R}^d$, and
(iii) is the minimal family under inclusion that satisfies (i) and (ii).

Definition 2 *A family $\widehat{A} = \{A_p;\, p \in P\}$ of nonempty compact subsets $\mathbb{R}^d$ is called a* pullback attractor *for the NDS (Φ, Θ) if it*
(i) is Φ-invariant,
(ii) pullback attracts nonempty bounded subsets of $\mathbb{R}^d$, and
(iii) is the minimal family under inclusion that satisfies (i) and (ii).

The two types of convergence (7) and (8) are generally not equivalent, but if one of them holds uniformly in $p \in P$, then so does the other, see [2].

2.1 Construction of forwards and pullback attractors

For an autonomous dynamical system the existence of an attractor follows from that of a simpler absorbing set, the omega limit points of which enable one to construct the attractor. The situation is more complicated for an NDS, but as in [3] an analogous construction of the components of a pullback attractor is possible in terms of a pullback absorbing set (or family of sets). However, such a "construction" is more of theoretical than practical usage, in particular as it must be repeated for every possible starting (or ending) parameter value $p_0 \in P$ of the driving system Θ.

In practice, it is much easier to verify "limiting" properties such as the *forwards dissipativity* property

$$\overline{\lim_{n\to\infty}} H^*(\Phi(n, p_0, D), \{0\}) \leq c < \infty, \tag{9}$$

or the *pullback dissipativity* property

$$\overline{\lim_{n\to\infty}} H^*(\Phi(n, \theta_{-n}p_0, D), \{0\}) \leq c < \infty \tag{10}$$

(these should be satisfied for all compact subset D of $\mathbb{R}^d$) for just a single given starting or ending parameter value $p_0 = p_0^* \in P$ rather than for all possible $p_0 \in P$. The question then is: what structural properties should the mapping f in (1) possess in addition to such a single parameter dissipativity condition in order to guarantee the existence of a forwards or pullback attractor.

Once we have somehow found or constructed just a single component subset $A_{\theta_n p_0}$ of a possible forwards or pullback attractor (i.e. for a specific p_0), then we can use the required Φ-invariance property to determine the corresponding A_p subsets for all future $p = \theta_n p_0$ with $n \geq 1$, namely

$$A_{\theta_n p_0} := \Phi(n, p_0, A_{p_0}) \quad \text{for} \quad n \geq 1$$

If the pullback dissipativity condition (10) also holds for this p_0, then we might be tempted to construct the corresponding A_p subsets for all past $p = \theta_n p_0$ with $n \leq -1$ through the pullback limit

$$A_{\theta_n p_0} := \overline{\bigcup_D \bigcap_{q\geq 0} \overline{\bigcup_{m\geq q} \Phi(m, \theta_{n-m}p_0, D)}} \quad \text{for} \quad n \leq -1 \tag{11}$$

where the outer union is taken over all compact subsets D of $\mathbb{R}^d$. (It is not clear even how to construct a subset A_{p_0} when the forwards dissipativity condition (9) holds). However, there is no guarantee here that resulting component subset in (11) will be compact without some additional assumptions. This failure, which is illustrated in Figure 1, is due to the unbounded growth of the reachable sets as the starting time approaches $-\infty$.

Even if we are successful in constructing compact subsets $A_{\theta_n p_0}$ for all $n \in \mathbb{Z}$, we still need to determine the A_p subsets for $p \in P \setminus \{\theta_n p_0;\ n \in \mathbb{Z}\}$. Some kind of "continuity" argument may seem appropriate here, but in general the mapping $p \mapsto A_p$ is only upper semi continuous, even when the mapping f in (1) (and hence the solution mapping Φ) is continuous in both p and x [2]. However, there is some hope of progress when the trajectory $\{\theta_n p_0;\ n \in \mathbb{Z}\}$ is dense in the parameter space P, which is the case for the asynchronous system (3)–(5) when the constant η in (5) is irrational.

2.2 Dissipativity along a single parameter trajectory

The following Lemmata, the proofs of which are obvious, list several sufficient conditions for the forwards dissipativity condition (9) or the pullback dissipativity condition (10), respec-

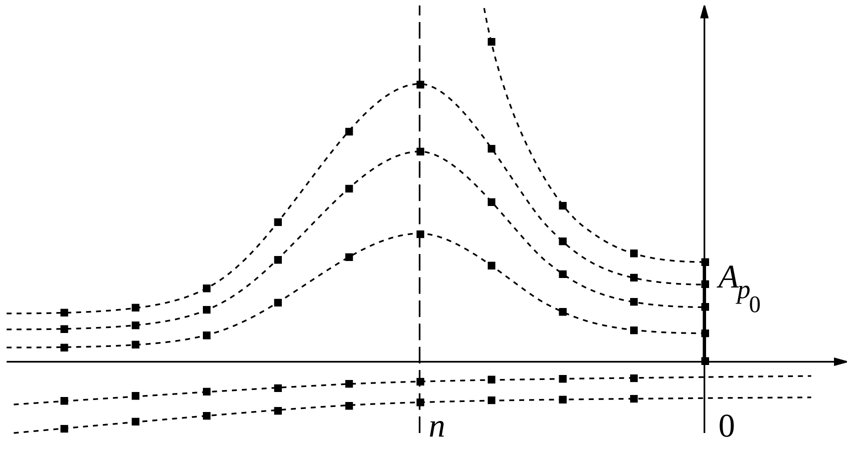

Figure 1:

tively, to hold for all $p_0 \in \{\theta_n p_0^*;\ n \in \mathbb{Z}\}$ when the condition is valid for the single parameter p_0^*.

Lemma 1 *Suppose that the forwards dissipativity condition (9) holds for some $p_0 = p_0^* \in P$. Then (9) holds for any $p_0 \in \{\theta_k p_0^*; k \in \mathbb{Z}\}$ if one of the following is satisfied:*

(i) the preimage $\Phi^{-1}(n, p_0^*, D) := \{x \in \mathbb{R}^d; \Phi(n, p_0^*, x) \in D\}$ is nonempty and bounded for every bounded subset D of $\mathbb{R}^d$ and $n \in \mathbb{Z}^+$;

(ii) the preimage $\Phi^{-1}(1, \theta_n p_0^*, D) := f^{-1}(\theta_n p_0^*, D)$ is nonempty and bounded for every bounded subset D of $\mathbb{R}^d$ and $n \in \mathbb{Z}^+$.

In the pullback case we express the corresponding conditions slightly differently for later convenience and include some additional ones. Here $B[0; r]$ denotes the closed ball in $\mathbb{R}^d$ of radius r that is centered at the origin.

Lemma 2 *Suppose that the pullback dissipativity condition (10) holds for some $p_0 = p_0^* \in P$. Then (10) holds for any $p_0 \in \{\theta_k p_0^*; k \in \mathbb{Z}\}$ if one of the following is satisfied:*

(i) the preimage $\Phi^{-1}(n, \theta_{-n} p_0^*, x_0) := \{x \in \mathbb{R}^d; \Phi(n, \theta_{-n} p_0^*, x) = x_0\}$ is nonempty and bounded for every $x_0 \in B[0; c]$, where c is the constant in (10), and $n \in \mathbb{Z}^+$;

(ii) the preimage $\Phi^{-1}(n, \theta_{-n} p_0^*, x_0)$ is nonempty and bounded for every $x_0 \in \mathbb{R}^d$ and $n \in \mathbb{Z}^+$;

(iii) for every $n \in \mathbb{Z}^+$

$$\|\Phi(1, \theta_{-n} p_0^*, x)\| = \|f(\theta_{-n} p_0^*, x)\| \to \infty \qquad \text{as} \quad \|x\| \to \infty.$$

(iv) for every $p \in P$

$$\|\Phi(1, p, x)\| := \|f(p, x)\| \to \infty \qquad \text{as} \quad \|x\| \to \infty.$$

Condition (i) in Lemma 2 is the least demanding, but the most difficult to verify, conditions (ii) and (iii) are equivalent and easier to verify than (i), while condition (iv) is the most demanding, but the easiest to verify. None of the conditions is necessary for the stated result.

3 Attractors of two-component asynchronous systems

The asynchronous systems described by equations (3)–(5) correspond to a system with two components and a particular switching sequence between these components, see [1]. Our aim is to determine conditions to supplement the forwards dissipativity condition (9) or the pullback dissipativity condition (10) for a single p_0^* that ensure the existence of a forwards or pullback attractor, respectively, for such an asynchronous system. We begin by investigating more closely the dissipativity conditions (9) and (10) for systems of the form (3)–(5).

3.1 Uniform boundedness conditions

The forwards dissipativity condition (9) for a given $p_0 \in P$ implies that for every nonempty bounded subset D of $\mathbb{R}^d$ there exists a positive number $\rho_f(p_0, D)$ such that

$$H^*\left(\Phi(n, p_0, D), \{0\}\right) \leq \rho_f(p_0, D), \qquad \forall n \in \mathbb{Z}^+, \tag{12}$$

which means the entire set-valued trajectory $\Phi(n, p_0, D)$ corresponding to the set-valued initial condition D is bounded for any bounded subset D. However, in general, the boundedness condition (12) does not imply the uniform boundedness condition

$$H^*\left(\Phi(n, \theta_m p_0, D), \{0\}\right) \leq \rho_f(p_0, D), \qquad \forall\, n, m \in \mathbb{Z}^+, \tag{13}$$

i.e. uniform with respect to the starting parameter values $\theta_m p_0$ over $m \in \mathbb{Z}^+$

Similarly, the backwards dissipativity condition (10) for a given $p_0 \in P$ does not imply that an analogous uniform boundedness condition holds for set-valued pullback trajectories, i.e. the existence a positive number $\rho_{pb}(p_0, D)$ for every nonempty bounded subset D of $\mathbb{R}^d$ such that

$$H^*\left(\Phi(n, \theta_{-m} p_0, D), \{0\}\right) \leq \rho_{pb}(p_0, D), \qquad \forall\, m \in \mathbb{Z}^+, \quad 0 \leq n \leq m. \tag{14}$$

For an asynchronous system (3)–(5) with irrational η the forwards and pullback uniform boundedness conditions (13) and (14) are in fact equivalent.

Lemma 3 *Let η in (5) be irrational and $p_0 \in [0, 1)$. Then conditions(13) and (14) for an asynchronous system (3)–(5) are equivalent to each other as well as to the following condition: for every bounded subset D of $\mathbb{R}^d$ there exists a constant $\rho(D) < \infty$ such that*

$$H^*\left(\Phi(n, p, D), \{0\}\right) \leq \rho(D), \qquad \forall\, n \in \mathbb{Z}^+, \quad \forall\, p \in [0, 1). \tag{15}$$

Proof. Condition (15) obviously implies both of the conditions (13) and (14), so the lemma will be proved if we show that each of conditions (13) and (14) implies (15).

Let condition (13) hold and assume first that the pair $(n, p) \in \mathbb{N} \times [0, 1)$ is such that $0, \eta \notin \{\theta_k p; k = 0, \ldots, n-1\}$. Then by Lemma 7 (see Appendix 1) there exists an $\varepsilon_0 > 0$ such that $\Phi(n, p', \cdot) \equiv \Phi(n, p, \cdot)$ for all p' with $|p' - p| < \varepsilon_0$. Now by the irrationality of η the trajectory $\{\theta_k p_0; k \in \mathbb{Z}^+\}$ is dense in $[0, 1)$, so there exists a $k_0 \in \mathbb{Z}^+$ such that $|p - \theta_{k_0} p_0| < \varepsilon_0$. Hence by Lemma 7

$$\Phi(n, \theta_{k_0} p_0, \cdot)) \equiv \Phi(n, p, \cdot)$$

while by (13)

$$H^*(\Phi(n, \theta_{k_0} p_0, D), \{0\}) \leq \rho_f(p_0, D).$$

Hence

$$H^*(\Phi(n,p,D),\{0\}) \le \rho_f(p_0,D), \qquad \forall\,(n,p):\ 0,\eta \notin \{\theta_k p; k=0,\ldots,n-1\}. \tag{16}$$

Consider now an arbitrary pair (n,p). Again, due to the irrationality of η, there exist at most one integer $k=k_0 \in [0,n-1]$ such that $\theta_{k_0}p=0$ and at most one integer $k=k_1 \in [0,n-1]$ such that $\theta_{k_1}p=\eta$. Moreover, if both of the numbers k_0 and k_1 belong to the interval $[0,n-1]$, then $k_1=k_0+1$. By the definition (6), in either case, the solution mapping $\Phi(n,p,\cdot)$ can be represented in the form

$$\Phi(n,p,\cdot)=\Phi(n-k_0-2,\theta_{k_0+1}p,\cdot)\circ f(\theta_{k_0+1}p,\cdot)\circ f(\theta_{k_0}p,\cdot)\circ\Phi(k_0,p,\cdot).$$

The first and the last factors here satisfy the uniform bound (16) proved above, while the intermediate factors remain bounded on bounded sets due to piecewise structure in p and continuity in x of the mapping $f(p,x)$ defined in (4). The required estimate (15) follows immediately, that is, condition (13) implies (15).

The proof that (14) implies (15) is analogous, so will be omitted. □

We shall now show that single parameter trajectory attractivity conditions (9) and (10) together with single parameter trajectory uniform boundedness conditions (13) and (14) imply the uniform attractivity of the asynchronous system (3)–(5).

Lemma 4 *Let η in (5) be irrational and let $p_0 \in [0,1)$ be such that one of the following conditions holds:*

(i) $0,\eta \notin \{\theta_k p_0; k\in\mathbb{Z}^+\}$ and (9) and (13) hold for every bounded subset D of $\mathbb{R}^d$.

(ii) $0,\eta \notin \{\theta_k p_0; k\in\mathbb{Z}^-\}$ and (10) and (14) hold for every bounded subset D of $\mathbb{R}^d$.

Then there exists a constant $C<\infty$ such that for every bounded subset D of $\mathbb{R}^d$ a number $N(D)$ can be found for which

$$H^*(\Phi(n,p,D),\{0\}) \le C, \qquad \forall\, n\ge N(D), \quad \forall\, p\in[0,1). \tag{17}$$

Proof. Since the statement of the lemma is proved analogously for both conditions (i) or (ii), we will provide the proof only for the case of condition (ii).

Suppose condition (ii) holds. Then by Lemma 3 there is a function $\rho(\cdot)$ for which the uniform bound (15) holds. Fix an arbitrary $\varepsilon_0>0$ and set

$$C := \rho(B[0;c_*])$$

where $c_* := c+\varepsilon_0$ and c is the constant from (10). Consider the ball $D_* := B[0;\rho(D)]$ for a given bounded subset D of $\mathbb{R}^d$. Then by (10) (which is a part of supposition (ii)) there is an integer $n(D_*)$ such that

$$H^*\left(\Phi(n,\theta_{-n}p_0,D_*),\{0\}\right) \le c_* \qquad \text{for} \quad n\ge n(D_*),$$

or, what is the same

$$\Phi\left(n,\theta_{-n}p_0,D_*\right) \subseteq B[0;c_*] \qquad \text{for} \quad n\ge n(D_*).$$

Finally, set $N(D) := 6N_*(D)+2$, where $N_*(D) := \max\{r,s\}$ and $r,s \ge n(D_*) = n(B[0;\rho(D)])$ are the pair of numbers given in Lemma 6 (see Appendix 1). The constants C and $N(D)$ so defined are the ones that we need in the assertion of the Lemma under proof.

Fix $n \geq N(D)$ and arbitrary $p \in [0,1)$. Since the relations $0, \eta \in \{\theta_k p; k = -n, \ldots, 0\}$ can be satisfied only for at most two consecutive values of k, then either $0, \eta \notin \{\theta_k p; k = -n, \ldots, -n+3N_*(D)\}$ or $0, \eta \notin \{\theta_k p; k = -3N_*(D), \ldots, 0\}$. In either case an integer m_0 can be found such that $[-m_0, -m_0 + 3N_*(D)] \subset [-n, 0]$ and $0, \eta \notin \{\theta_k p; k = -m_0, \ldots, -m_0 + 3N_*(D)\}$, and so that the solution mapping $\Phi(n, \theta_{-n}p, \cdot)$ can be represented in the form

$$\Phi(n, \theta_{-n}p, \cdot) = \Phi(m_0 - 3N_*(D), \theta_{-m_0+3N_*(D)}, \cdot) \circ \Phi(3N_*(D), \theta_{-m_0}p, \cdot) \circ \Phi(n - m_0, \theta_{-n}p, \cdot) \quad (18)$$

Further, by Lemma 9 (see Appendix 1) there exist integers n_0 and n_* (where $n_* = r$ or $n_* = s$) such that $[-n_0, -n_0 + n_*] \subset [-m_0, -m_0 + 3N_*(D)]$ and so that the solution mapping $\Phi(3N_*(D), \theta_{-m_0}p, \cdot)$ can be represented in the form

$$\begin{aligned} \Phi(3N_*(D), \theta_{-m_0}p, \cdot) \quad = \quad & \Phi(-m_0 + 3N_*(D) + n_0 - n_*, \theta_{-n_0+n_*}p, \cdot) \circ \\ & \Phi(n_*, \theta_{-n_0}p, \cdot) \circ \Phi(m_0 - n_0, \theta_{-m_0}p, \cdot) \end{aligned} \quad (19)$$

where

$$\Phi(n_*, \theta_{-n_0}p, \cdot) \equiv \Phi(n_*, \theta_{-n_*}p_0, \cdot). \quad (20)$$

It thus follows from (18) and (19) follows that we can write

$$\Phi(n, \theta_{-n}p, \cdot) = \Phi(n_0 - n_*, \theta_{n_0+n_*}p, \cdot) \circ \Phi(n_*, \theta_{-n_0}p, \cdot) \circ \Phi(n - n_0, \theta_{-n}p, \cdot)$$

or, taking into account (20),

$$\Phi(n, \theta_{-n}, \cdot) = \Phi(n_0 - n_*, \theta_{-n_0+n_*}p, \cdot) \circ \Phi(n_*, \theta_{-n_*}p_0, \cdot) \circ \Phi(n - n_0, \theta_{-n}p, \cdot) \quad (21)$$

where the middle factor is expressed through the solution mapping with parameter p_0.

Now, by Lemma 3

$$\Phi(n - n_0, \theta_{-n}p, D) \subseteq B[0; \rho(D)] = D_*,$$

then by choice of the number n_*

$$\Phi(n_*, \theta_{-n_*}p_0, D_*) \subseteq B[0; c_*]$$

and, at last, by definition of the constant C

$$\Phi(n_0 - n_*, \theta_{-n_0+n_*}p, B[0; c_*]) \subseteq B[0; C].$$

The statement of the lemma then follows from these inclusions and from (21) (see Figure 2).

The statement of the lemma under supposition (i) is proved analogously with the exception that Lemma 8 (see Appendix 1) on the "forwards decomposition" of solution mapping should be used instead of the "pullback decomposition" in Lemma 9. $\square$

3.2 Main result

Our main result, Theorem 3 to be stated at the end of the subsection, concerns the existence of forwards and pullback attractors of the asynchronous system (3)–(5). Its formulation and proof are based on the following two theorems.

Theorem 1 *Let η in (5) be irrational and let $p_0 \in [0,1)$ be such that (10) and (14) hold for every bounded subset D of $\mathbb{R}^d$. Suppose also that the preimage $f^{-1}(p, D) := \{x \in \mathbb{R}^d; f(p,x) \in D\}$, where f is defined by (4), is bounded for every bounded subset D of $\mathbb{R}^d$ and $p \in [0,1)$.*

Then there exists a finite constant C such that for every bounded subset D of $\mathbb{R}^d$ a number $N(D)$ can be found for which the uniform bound (17) is satisfied.

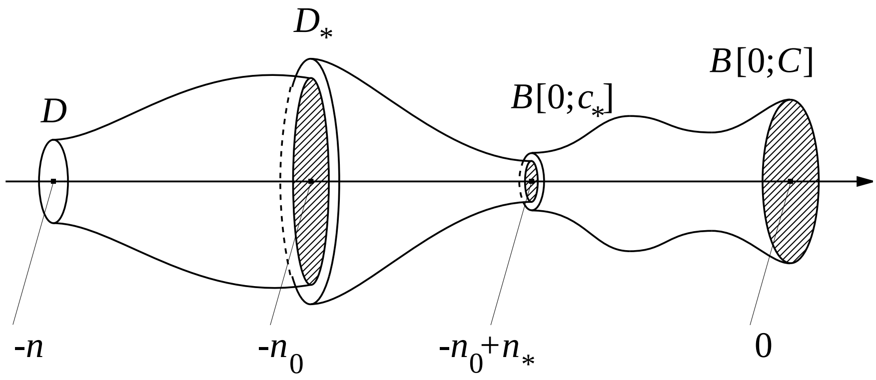

Figure 2:

Proof. The assertion of the theorem immediately follows from statement (ii) of Lemma 4 when $0, \eta \notin \{\theta_k p_0; k \in \mathbb{Z}^-\}$. If one of the points 0 or η belongs to the set $\{\theta_k p_0; k \in \mathbb{Z}^-\}$, we can find a $p_* \in \{\theta_k p_0; k \in \mathbb{Z}^-\}$ such that $0, \eta \notin \{\theta_k p_0; k \in \mathbb{Z}^-\}$. Then, from the condition that the preimage $f^{-1}(p, D)$ is bounded for any bounded subset D of $\mathbb{R}^d$ and any $p \in [0, 1)$, it follows by Lemma 2 that (10) and (14) hold with $p_0 = p_*$ for every bounded subset D of $\mathbb{R}^d$. The assertion of the theorem follows from this, again by the statement (ii) of Lemma 4. □

Theorem 2 *Let η in (5) be irrational and let $p_0 \in [0, 1)$ be such that (9) and (13) hold for every bounded subset D of $\mathbb{R}^d$. Suppose also that the preimage $f^{-1}(p, D)$, where f is defined by (4), is bounded for any bounded subset D of $\mathbb{R}^d$ and any $p \in [0, 1)$.*

Then there exists a finite constant C such that for every bounded subset D of $\mathbb{R}^d$ a number $N(D)$ can be found for which the uniform bound (17) is satisfied.

Proof. The proof is a word by word repetition of the proof of Theorem 1 with the exception that Lemma 1 should be used instead of Lemma 2 and statement (i) of Lemma 4 should be used instead of the statement (ii). □

Remark 1 The condition in Theorem 1 that *the preimage $f^{-1}(p, D)$, where f is defined by (4), is bounded for every bounded subset D of $\mathbb{R}^d$ and any $p \in [0, 1)$* can be replaced by less demanding condition (i) of Lemma 1. Analogously, the same in Theorem 2 condition can be replaced by any one of the conditions (i)–(iv) of Lemma 2.

We now formulate our main result.

Theorem 3 *Under the conditions of Theorem 1 the asynchronous system (3)–(5) has a pullback attractor, while under the conditions of Theorem 2 it has a forwards attractor.*

Proof. The assertions follow immediately from the uniform bound (17), the validity of which is assured by Theorems 1 or 2 and by Theorems 2.8 or 2.9 of [2], respectively. □

Theorems 1, 2 and 3 are similar to the stability investigations of asynchronous systems mentioned in the Introduction in the sense that they are also concerned with determining if some dynamical property will hold for all initial values p_0 of the "clock" (5) when it holds for a particular initial value p_0^*. However, they differ in the sense that they are concerned with global properties, the existence of uniform bounds and attractors, rather than with a local property such as determining the asymptotic stability of an equilibrium point that is already known to exist.

4 Appendix 1: Symbolic analysis of the driving component

The dynamics of the asynchronous system (3)–(5) is strongly affected by the properties of the driving system (5), that is, the shift mapping Θ of the interval $[0,1)$ onto itself, which was defined by $\Theta(p) := p + \eta, \mod 1$, and is obviously invertible on $[0,1)$, so

$$\theta_n p := \Theta^n(p) = p + n\eta, \quad \mod 1, \qquad n \in \mathbb{Z}.$$

Thus the sequence $\{p_n; n \in \mathbb{Z}^+\}$ defined by the recursion

$$p_{n+1} = \Theta(p_n) := p_n + \eta, \quad \mod 1, \qquad n \in \mathbb{Z}^+,$$

for an initial value $p_0 \in [0,1)$ satisfies $p_n = p_0 + n\eta$ for each $n \in \mathbb{Z}^+$. However, the dynamics of this sequence is not easily described unless η is rational, in which case it is periodic. This sequence, in fact, determines the corresponding state space dynamics because the solution mapping $\Phi(n, p_0, \cdot)$ is explicitly represented by

$$\Phi(n, p_0, \cdot) = f(p_{n-1}, \cdot) \circ \cdots \circ f(p_1, \cdot) \circ f(p_0, \cdot). \tag{22}$$

and thus, in view of the piecewise constant structure of the mapping f defined by (4), consists of compositions of the mappings F, G and H depending on whether the p_j belong to $\{0\}$, $(0,\eta)$ or $[\eta, 1)$.

We can thus associate the abstract symbols, say, "a", "b" and "c", with each of these possible parameter states $\{0\}$, $(0,\eta)$ or $[\eta,1)$ and then represent the parameter sequence $\{p_k; k = 0, \ldots, n-1\}$ by the symbolic sequence $\bar{\boldsymbol{\sigma}}(\theta) := \{\bar{\sigma}_k(p_0); k = 0, \ldots, n-1\}$, where $\bar{\sigma}_k(p_0) := \chi(p_k)$ with the mapping $\chi : [0,1) \mapsto \{a, b, c\}$ defined by

$$\chi(\theta) = \begin{cases} a, & \text{if} \quad \theta \in (0,\eta), \\ b, & \text{if} \quad \theta \in [\eta, 1), \\ c, & \text{if} \quad \theta = 0. \end{cases} \tag{23}$$

Specifically,

Lemma 5 *The representation (22) of the solution mapping $\Phi(n, p_0, \cdot)$ is completely determined by the symbolic sequence $\bar{\boldsymbol{\sigma}}(p_0) := \{\chi(p_k); k = 0, \ldots, n-1\}$ in the sense that*

$$f(p, x) = \begin{cases} F(x), & \textit{iff} \quad \chi(p) = c \quad \Leftrightarrow \quad p = 0, \\ G(x), & \textit{iff} \quad \chi(\theta) = a \quad \Leftrightarrow \quad p \in (0,\eta), \\ H(x), & \textit{iff} \quad \chi(p) = b \quad \Leftrightarrow \quad p \in [\eta, 1) \end{cases}$$

for each $p = p_k$, $k = 0, 1, \ldots, n-1$.

We need some auxiliary terminology [9]. In particular, elements in symbolic sequences will be not separated by commas. Let $\mathcal{A}$ be a fixed *alphabet*, that is, a set of elements called *letters* or *symbols*. A finite cortege $\boldsymbol{\nu} = \nu_1 \ldots \nu_n$ of letters from $\mathcal{A}$ is called *a word*. The *product* of any words $\boldsymbol{\nu}^1 = \nu_1^1 \ldots \nu_{n_1}^1$ and $\boldsymbol{\nu}^2 = \nu_1^2 \ldots \nu_{n_2}^2$ is the word $\boldsymbol{\nu}^1\boldsymbol{\nu}^2 = \nu_1^1 \ldots \nu_{n_1}^1 \nu_1^2 \ldots \nu_{n_2}^2$ and the *left factor* (of length $j \le n$) of a word $\boldsymbol{\nu} = \nu_1 \ldots \nu_n$ is the initial fragment $\boldsymbol{\nu}_j = \nu_1 \ldots \nu_j$ of $\boldsymbol{\nu}$. An infinite sequence $\boldsymbol{\sigma} = \sigma_1\sigma_2\ldots$ from the alphabet $\mathcal{A}$ is called an *infinite word* or *text*. For such a text $\boldsymbol{\sigma}$, the word $\boldsymbol{\sigma}_n = \sigma_1\sigma_2\ldots\sigma_n$ is called its *left factor* (of length n) and the text $\boldsymbol{\sigma}^n = \sigma_{n+1}\sigma_{n+2}\ldots$ is called its *right factor* (of the colength n), while any word $\sigma_i \ldots \sigma_j$ with $i \le j$ is called a *factor* of $\boldsymbol{\sigma}$.

Let $\mathbf{\Sigma} = \{\boldsymbol{\sigma}_1, \ldots, \boldsymbol{\sigma}_k\}$ be a set of words over the alphabet $\mathcal{A}$. A text $\boldsymbol{\sigma}$ is said to be $\mathbf{\Sigma}$-*decomposable* if it can be represented as a product of words belonging to a set of words $\mathbf{\Sigma}$ and is said to be *weakly* $\mathbf{\Sigma}$-*decomposable* if it has a $\mathbf{\Sigma}$-decomposable right factor $\boldsymbol{\sigma}^n$ for which the colength n is strictly less than the maximal length of the words in $\mathbf{\Sigma}$.

In particular, we will investigate the symbolic sequence or text

$$\boldsymbol{\sigma}(p_0) = \sigma_0(p_0)\sigma_1(p_0)\ldots\sigma_n(p_0)\ldots \tag{24}$$

consisting of two letters, say a and b and defined by

$$\sigma_n(p_0) = \begin{cases} a, & \text{if} \quad p_n = \theta_n p_0 \in [0, \eta), \\ b, & \text{if} \quad p_n = \theta_n p_0 \in [\eta, 1). \end{cases} \tag{25}$$

Such texts (24)–(25) are called as *sturmian beams* with a-frequency η, see [8].

When η is rational the texts (24)–(25) are all eventually periodic, but if η is irrational then the structure of texts is considerably more complicated. The following result is a reformulation of Theorem 1 from [6] (see also Theorem 5 from [5]).

Lemma 6 *Let η be irrational and let $p_0 \in [0, 1)$ with $p_0 \neq \eta$. Then there exist arbitrary large integers r and s such that for any $\tilde{p}_0 \in [0, 1)$ the text $\boldsymbol{\sigma}(\tilde{p}_0)$ will be weakly $\mathbf{\Sigma}$-decomposable with respect to the set of words $\mathbf{\Sigma} = \{\boldsymbol{\sigma}_r(p_0), \boldsymbol{\sigma}_s(p_0)\}$.*

Remark 2 The sequences (24)–(25) here are not quite the same as were used in (23) to characterize the structure of the solution mapping $\Phi(n, p, \cdot)$, essentially absorbing the symbol "c" into the symbol "a". We will see below that the symbol "c" rarely occurs and that the dynamics can be completely handled by the sequences (24)–(25)

The solution mapping $\Phi(n, p, \cdot)$ inherits the piecewise constant structure in the variable p from the mapping $f(p, \cdot)$.

Lemma 7 *Let p_0 be such that $0, \eta \notin \{\theta_k p_0; k = 0, \ldots, n-1\}$. Then there exists an $\varepsilon_0 > 0$ such that $\Phi(k, p, \cdot) \equiv \Phi(k, p_0, \cdot)$ for all $k = 1, 2, \ldots, n$ and p satisfying $|p - p_0| < \varepsilon_0$.*

Proof. Since $0, \eta \notin \{\theta_k p_0; k = 0, \ldots, n-1\}$, we have either $\theta_k p_0 \in (0, \eta)$ or $\theta_k p_0 \in (\eta, 1)$ for $k = 0, 1, \ldots, n-1$. By the continuity of the shift mapping Θ on the torus $\mathbb{T}^1 := [0, 1)$ there thus exists an $\varepsilon_0 > 0$ such that the $\theta_k p$ belong to the same intervals $(0, \eta)$ or $(\eta, 1)$ as the corresponding $\theta_k p_0$ when $|p - p_0| < \varepsilon_0$. By definition (23) of the mapping χ, we thus have $\chi(\theta_k p) = \chi(\theta_k p_0)$ for $k = 0, 1, \ldots, n-1$ and $|p - p_0| < \varepsilon_0$. The assertion of the lemma then follows by Lemma 5. $\square$

The next two lemmas show that even when the solution mapping $\Phi(n, p, \cdot)$ is not periodic in the n variable it nevertheless possesses a kind of pseudo periodicity.

Lemma 8 (A forward decomposition of the solution mapping) *Let $p_0 \in [0, 1)$ be such that $0, \eta \notin \{\theta_k p_0; k \in \mathbb{Z}^+\}$ and let $\tilde{p}_0 \in [0, 1)$ be such that $0 \notin \{\theta_k \tilde{p}_0; k = 0, \ldots, n\}$ for some $n > 0$. Then for any $N > 0$ there exist integers $r \geq s \geq N$ and mappings $g_1, g_2, \ldots, g_m$ such that*

$$\Phi(n, \tilde{p}_0, \cdot) = g_m \circ \cdots \circ g_2 \circ g_1(\cdot), \tag{26}$$

where

$$g_i(\cdot) = \begin{cases} \Phi(k_i, \tilde{p}_0, \cdot) & \text{for} \quad i = 1, \\ \Phi(k_i, p_0, \cdot) & \text{for} \quad i > 1, \end{cases}$$

and

$$k_i < \min\{r, s\} \quad \text{for} \quad i = 1, m, \qquad k_i = r \text{ or } s \quad \text{for} \quad i = 2, 3, \ldots, m-1.$$

Remark 3 Lemma 8 means that, given a $p_0 \in [0, 1)$ satisfying the conditions of the lemma, the solution mapping $\Phi(m, \tilde{p}_0, \cdot)$ can be represented as the product of "long factors" $\Phi(r, p_0, \cdot)$ and $\Phi(s, p_0, \cdot)$, apart from initial and ending "short factors" $\Phi(k_1, \tilde{p}_0, \cdot)$ and $\Phi(k_m, p_0, \cdot)$. In this way it "inherits" the limiting properties of the solution mapping $\Phi(n, p_0, \cdot)$.

Remark 4 The condition $0, \eta \notin \{\theta_k p_0; k \in \mathbb{Z}^+\}$ is not restrictive in context of the "limiting" properties of the system since, by the irrationality of η, both points 0 and η can occur only once in the sequence $\{\theta_k p_0); k \in \mathbb{Z}^+\}$. In this case it suffices to replace p_0 by $\theta_k p_0$ with sufficiently large k.

Remark 5 We will see from the proof that the condition $0 \notin \{\theta_k \tilde{p}_0; k \in \mathbb{Z}^+\}$ is also not restrictive since by the irrationality of η the point 0 can occur only once in the sequence $\{\theta_k \tilde{p}_0; k \in \mathbb{Z}^+\}$, say $0 = \theta_{k_0} \tilde{p}_0$. The sequence $\{\theta_k \tilde{p}_0; k \in \mathbb{Z}^+\}$ can then be split into three parts, namely $\{\theta_k \tilde{p}_0; k = 0, \ldots, k_0 - 1\}, \{\theta_{k_0} \tilde{p}_0\}$ and $\{\theta_k \tilde{p}_0; k = k_0 + 1, \ldots, \infty\}$. The first and the third parts here do not contain 0, so Lemma 8 can be applied to these sequences.

The following analog of Lemma 8 is useful in the pullback case.

Lemma 9 (A pullback decomposition of the solution mapping) *Let $p_0 \in [0, 1)$ be such that $0, \eta \notin \{\theta_k \tilde{p}_0; k \in \mathbb{Z}^-\}$ and let $\tilde{p}_0 \in [0, 1)$ be such that $0 \notin \{\theta_k \tilde{p}_0; k = -n, \ldots, 0\}$ for some $n > 0$. Then for any $N > 0$ there exist integers $r \geq s \geq N$ and maps $g_1, g_2, \ldots, g_m$ such that*

$$\Phi(n, \theta_{-n} \tilde{p}_0, \cdot) = g_m \circ \cdots \circ g_2 \circ g_1(\cdot),$$

where

$$g_i = \begin{cases} \Phi(k_i, \theta_{-k_i} \tilde{p}_0, \cdot) & \text{for} \quad i = m, \\ \Phi(k_i, \theta_{-k_i} p_0, \cdot) & \text{for} \quad i < m, \end{cases}$$

and

$$k_i < \min\{r, s\} \quad \text{for} \quad i = 1, m, \qquad k_i = r \text{ or } s \quad \text{for} \quad i = 2, 3, \ldots, m - 1.$$

Appropriately modified analogs of the Remarks 3– 5 also apply to Lemma 9.

Proof of Lemma 8. Consider the symbolic sequences $\bar{\boldsymbol{\sigma}}(p_0)$ and $\bar{\boldsymbol{\sigma}}(\tilde{p}_0)$ defined through (23) by the sequences $\{\theta_k p_0\}$ and $\{\theta_k \tilde{p}_0\}$, respectively. According to Lemma 5 the structure of the solution mappings $\Phi(n, p_0, \cdot)$ and $\Phi(n, \tilde{p}_0, \cdot)$ is completely determined by $\bar{\boldsymbol{\sigma}}(p_0)$ and $\bar{\boldsymbol{\sigma}}(\tilde{p}_0)$.

Since, by assumptions of the lemma, we have $0, \eta \notin \{\theta_k p_0; k \in \mathbb{Z}^+\}$, then according to (23) $c \notin \bar{\boldsymbol{\sigma}}(p_0)$ and so

$$\bar{\boldsymbol{\sigma}}(p_0) \equiv \boldsymbol{\sigma}(p_0). \tag{27}$$

Analogously, from suppositions of the lemma $0 \notin \{\theta_k \tilde{p}_0; k = 0, \ldots, n\}$ for some $n > 0$, it follows that

$$\bar{\boldsymbol{\sigma}}_n(\tilde{p}_0) \equiv \boldsymbol{\sigma}_n(\tilde{p}_0) \tag{28}$$

From relations (27) and (28) it follows the possibility to apply Lemma 6 for investigation of structures of the symbolic sequences $\bar{\boldsymbol{\sigma}}(p_0)$ and $\bar{\boldsymbol{\sigma}}_n(\tilde{p}_0)$.

Fix an arbitrary integer $N > 0$. By Lemma 6 there exist such integers $r, s \geq N$ for which the text $\boldsymbol{\sigma}(\tilde{p}_0)$ will be weakly $\boldsymbol{\Sigma}$-decomposable with respect to the set of words $\boldsymbol{\Sigma} = \{\boldsymbol{\sigma}_r(p_0), \boldsymbol{\sigma}_s(p_0)\}$. This means that

$$\boldsymbol{\sigma}_n(\tilde{p}_0) = \sigma_{r_0}(\tilde{p}_0) \ldots \sigma_{r_1 - 1}(\tilde{p}_0) \sigma_{r_1}(\tilde{p}_0) \ldots \sigma_{r_2 - 1}(\tilde{p}_0) \ldots \sigma_{r_{m-1}}(\tilde{p}_0) \ldots \sigma_{r_m - 1}(\tilde{p}_0),$$

where, by Lemma 6, $r_0 = 0$, $r_k = n$, $r_1 < \min\{r, s\}$ and for $1 \le i < m - 1$ either the equality $\sigma_{r_i}(\tilde{p}_0) \dots \sigma_{r_{i+1}-1}(\tilde{p}_0) = \boldsymbol{\sigma}_r(p_0)$ is valid with $r_{i+1} - r_i = r$ or the equality $\sigma_{r_i}(\tilde{p}_0) \dots \sigma_{r_{i+1}-1}(\tilde{p}_0) = \boldsymbol{\sigma}_s(p_0)$ is valid with $r_{i+1} - r_i = s$, while for $i = k - 1$ the equality $\sigma_{r_{m-1}}(\tilde{p}_0) \dots \sigma_{r_{m-}}(\tilde{p}_0) = \boldsymbol{\sigma}_q(p_0)$ with $q = r_m - r_{m-1} < \min\{r, s\}$ takes place. By definition of the symbolic sequences $\boldsymbol{\sigma}(p)$ from here and from Lemma 5 the relationship (26) follows. □

Proof of Lemma 9 is provided analogously to the presented above proof of Lemma 8 and so is omitted.

5 Appendix 2: Origin of discrete time asynchronous systems

Asynchronous systems with frequency updating components are a natural, yet at the same time non traditional, example of skew product or cocycle systems with rather complicated behaviour. To avoid inessential details we will be restricted our attention here to two-component systems.

Asynchronous systems arise quite naturally in the following setting. Consider a system $\mathcal{W}$ consisting of two components (parts, elements) W_1 and W_2 for which the state of the component W_i is described by a vector $\xi_i \in \mathbb{R}^{d_i}$ for some $d_i \ge 1$. The main assumption is that such a vector can be updated only at some discrete time instants in accordance with the rule:

$$\xi_{i,\text{new}} := \varphi_i(\xi_{\text{old}}), \qquad \xi_{\text{old}} := (\xi_{1,\text{old}}, \xi_{2,\text{old}}), \quad i = 1, 2,$$

where $\varphi_i : \mathbb{R}^{d_1} \times \mathbb{R}^{d_2} \mapsto \mathbb{R}^{d_i}$ is some nonlinear function remaining unchanged at other moments.

Let $\ldots < T_{i,0} < T_{i,1} < \ldots < T_{i,n} < \ldots$ denote the updating times for the component W_i. Then the updating law for the variable state $\xi_i(t)$ of the component W_i can be described

$$\xi_i(T_{i,n} + 0) := \varphi_i(\xi(T_{i,n} - 0)), \qquad i = 1, 2, \tag{29}$$

with the function $\xi_i(t)$ being constant on each time interval $T_{i,n} < t \le T_{i,n+1}$, i.e.

$$\xi_i(t) = \text{const} \qquad \text{for} \quad T_{i,n} < t \le T_{i,n+1},\ i = 1, 2,\ n \in \mathbb{Z}.$$

For this exposition, we will suppose that each component W_i is updated periodically, i.e. its updating times are given by

$$T_{i,n} = n\tau_i + p_i \qquad i = 1, 2,\ n \in \mathbb{Z},$$

where, without loss of generality, we may assume that

$$\tau_1 = 1, \quad p_1 = 0, \qquad \tau_2 = \eta \in (0, 1], \quad p_2 \in [0, \eta).$$

Let us now reformulate the above continuous time description (29) of an asynchronous system by a more convenient discrete time system. For this we denote by $\omega \subseteq \{1, 2\}$ the set of indices of the components of system $\mathcal{W}$ that can be updated at a given instant. In addition, writing $\xi = (\xi_1, \xi_2)$ and $\xi_i \in \mathbb{R}^{d_i}$ for $i = 1, 2$, we denote by

$$\varphi(\omega, \xi) := (\varphi_1(\omega, \xi), \varphi_2(\omega, \xi)), \qquad \varphi_i(\omega, \xi) \in \mathbb{R}^{d_i},\ i = 1, 2, \tag{30}$$

the mapping which is obtained from the above mapping

$$\varphi(\xi) := (\varphi_1(\xi), \varphi_2(\xi)), \qquad \varphi_i(\xi) \in \mathbb{R}^{d_i},\ i = 1, 2, \tag{31}$$

by replacing its components with indices $i \notin \omega$ on the identity mappings (in corresponding spaces $\mathbb{R}^{d_i}$), obtaining

$$\varphi(\omega, \xi) := \begin{cases} (\varphi_1(\xi), \xi_2), & \text{if} \quad \omega = \{1\}, \\ (\xi_1, \varphi_2(\xi)), & \text{if} \quad \omega = \{2\}, \\ (\varphi_1(\xi), \varphi_2(\xi)) = \varphi(\xi), & \text{if} \quad \omega = \{1, 2\}. \end{cases}$$

The mapping (30) is called *the ω-mixture* of the mapping (31). It allows us to describe quite simply the updating procedure and dynamics of the asynchronous systems. Specifically, the "updating procedure" for the state of the system $\mathcal{W}$ is described here by

$$\xi_{\text{new}} := \varphi(\omega, \xi_{\text{old}}), \tag{32}$$

where ω is the set of indices of updatable (vector-valued) components $\xi_i \in \mathbb{R}^{d_i}$ of ξ.

To describe the updating law for the state (32) as dynamical equations, we first enumerate the totality of updating times $\{T_{i,j}\}$, $i \in \mathbb{Z}$, $j = 1, 2$, in the increasing order as a sequence $\ldots < T_0 < T_1 < \ldots < T_n < \ldots$ and denote by $\omega_n \subseteq \{1, 2\}$ the set of indices of the components $\{W_i\}$ that are updated at time T_n. Then, from (29) and (32), we obtain the following equation for the dynamics of the system $\mathcal{W}$:

$$\xi(T_{n+1} - 0) \equiv \xi(T_n + 0) = \varphi(\omega_n, \xi(T_n - 0)), \qquad n \in \mathbb{Z}. \tag{33}$$

In the case under discussion, it is possible to describe behaviour of the driving component ω in terms of a dynamical system. To do this, we partition the time axis $\mathbb{R}$ into nonoverlapping intervals $(n\eta, (n + 1)\eta]$, $n \in \mathbb{Z}$ and write $x(n)$ for the state vector of the system $\mathcal{W}$ at time instances immediately following n, i.e.

$$x(n) = \xi(n\eta + 0).$$

Each interval $(n\eta, (n + 1)\eta]$ obviously contains no more than one updating time of each component of the system $\mathcal{W}$. It contains exactly one updating time of the second component which coincides with $(n + 1)\eta$, and it contains at most one updating time of the first component. To determine whether the interval $(n\eta, (n+1)\eta]$ contains the updating time for the first component we introduce the number

$$p(n) := (n + 1)\eta - [(n + 1)\eta] \in [0, 1), \tag{34}$$

where $[\cdot]$ denotes the (floor) integer part of a real number, i.e. the largest integer does not exceeding the corresponding real number. Then $[(n + 1)\eta]$ will be exactly the largest updating time of the first component that does not exceed $(n+1)\eta$, so the interval $(n\eta, (n+1)\eta]$ will contain the updating time of the first component if and only if $p(n) \in [0, \eta)$.

It thus follows that the vector $x(n + 1)$ is obtained from the vector $x(n)$ through the difference equation

$$x(n + 1) = f(p(n), x(n)), \tag{35}$$

where the function $f(p, x)$ is defined by

$$f(p, x) = \begin{cases} F(x) := (\varphi_1(x_1, x_2),\ \varphi_2(x_1, x_2)), & \text{if} \quad p = 0, \\ G(x) := (\varphi_1(x_1, x_2),\ \varphi_2(\varphi_1(x_1, x_2), x_2)), & \text{if} \quad p \in (0, \eta), \\ H(x) := (x_1,\ \varphi_2(x_1, x_2)), & \text{if} \quad p \in [\eta, 1). \end{cases} \tag{36}$$

This representation follows from (30) since for $p = p(n) = 0$ both components of the system $\mathcal{W}$ are updated simultaneously, for $p \in (0, \eta)$ the first and second components are updated sequentially, and for $p \in [\eta, 1)$ only the second component is updated. In addition, the $p(n+1)$ and $p(n)$ are related by the "η-shift mapping" on the interval $[0, 1)$, i.e.

$$p(n+1) = p(n) + \eta, \quad \mod 1. \tag{37}$$

The dynamics of the two-component asynchronous system $\mathcal{W}$ can thus be described by the skew product system (35), (36) and (37), which is the same as system (3)–(5) in the Introduction after some minor notational changes.

References

[1] E.A. Asarin, V.S. Kozyakin, M.A. Krasnosel'skii, N.A. Kuznetsov, *Analysis of the Stability of Ssynchronous Discrete Systems*, "Nauka", Moscow, 1992, 408p. (in Russian).

[2] D.N. Cheban, P.E. Kloeden and B. Schmalfuß, *The relationship between pullback, forwards and global attractors of nonautonomous dynamical systems. DANSE*-Preprint 27/00, FU Berlin, 2000

[3] P.E. Kloeden, Pullback attractors in nonautonomous difference equations. *J. Difference Eqns. Applns.* **6** (2000), 33–52.

[4] P.E. Kloeden and V.S. Kozyakin, The inflation of nonautonomous systems and their pullback attractors. *Transactions of the Russian Academy of Natural Sciences, Series MMMIU.* **4**, No. 1–2, (2000), 144–169.

[5] V.S. Kozyakin, Stability of phase-frequency desynchronized systems under component switching time disturbances. *Avtomat. i Telemekh. 1990, OVYR no. 8, 35–41; English translation in Automat. Remote Control.* **51**, No. 8, part 1, (1991), 1034–1040.

[6] V.S. Kozyakin, Analysis of the stability of desynchronized systems by methods of symbolic dynamics. *Dokl. Akad. Nauk SSSR 311 (1990), no. 3, 549–552; English translation in Soviet Phys. Dokl.* **35**, No. 3, (1990), 218–220.

[7] M.A. Krasnosel'skii, *The Operator of Translation along Trajectories of Differential Equations*, Translations of Mathematical Monographs, Volume 19. American Math. Soc., Providence, R.I., 1968.

[8] M. Morse and G.A. Hedlund, Symbolic dynamics II: Sturmian trajectories, *American J. Math.* **62** (1940), 1–42.

[9] M. Lothaire, *Combinatorics of Words,* in Encyclopedia of Mathematics and its Applications, v. 17. London, Addison-Wesley Publishing Company, 1983, 238p.