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THE PERTURBATION OF ATTRACTORS OF SKEW-PRODUCT FLOWS WITH A SHADOWING DRIVING SYSTEM

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Abstract. The influence of the driving system on a skew-product flow generated by a triangular system of differential equations can be perturbed in two ways, directly by perturbing the vector field of the driving system component itself or indirectly by perturbing its input variable in the vector field of the coupled component. The effect of such perturbations on a nonautonomous attractor of the driven component is investigated here. In particular, it is shown that a perturbed nonautonomous attractor with nearby components exists in the indirect case if the driven system has an inflated nonautonomous attractor and that the direct case can be reduced to this case if the driving system is shadowing.

1. Introduction. As in the theory of autonomous dynamical systems, a major issue in the theory of nonautonomous cocycle dynamical systems or skew-product flows concerns the robustness or persistence of certain dynamical properties of the system (e.g., the existence of trajectories of a specific type such as equilibria or of attractors) under various kinds of perturbations. There are already a number of publications devoted to this matter in one context or another, e.g., the persistence of nonautonomous (forwards or pullback) attractors under uniform perturbation or numerical discretization [12, 13], the stability of asynchronous systems under uniform perturbation [17], and the "lifting problem" in skew-product flows [19], as well as the robustness of control sets [3] or random bifurcations [8].

Consider a system of autonomous differential equations

$$\dot{x} = f(x, p), \tag{1.1}$$

$$\dot{p} = g(p), \tag{1.2}$$

where the *p*-component is decoupled, so the system (1.1)-(1.2) generates a skewproduct flow. The *p*-component here may be considered to represent an independent

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system that drives the x-component system in the sense that

$$\dot{x} = f(x, p(t)) \tag{1.3}$$

for any given solution p(t) of (1.2).

Suppose that the resulting dynamics has an attractor of some kind, e.g., a global autonomous attractor for the autonomous skew-product flow or a nonautonomous attractor (to be defined below) for the nonautonomous x-component system driven by the p-component system as in equation (1.3). The issues to be considered in this paper are:

What happens when the driving component is slightly perturbed?

What conditions on the equations (1.1)-(1.2) ensure the persistence of such an attractor under perturbation of the driving component?

The answers to these questions are not as obvious or easy as it might at first seem. Indeed, even an infinitesimal uniform perturbation of the vector field g in (1.2) may cause substantial changes to the individual trajectories p(t) of (1.2), as a result of which the solutions of the "driven" equation (1.3) may then behave in a very different way to the those of the original unperturbed system. This effect will be shown in Example 2.1 of Section 2, where the perturbations to the influence of the driving component will be classified as strong and weak depending on whether just the whole vector field g of the driving equation in (1.2) or just the p-component in the vector field f of the driven equation (1.1) is perturbed. Weak perturbations will be investigated in Section 4 in terms of the "inflation" of the vector field in (1.1) and the existence of inflated nonautonomous attractors. In Section 5 it will be shown that the investigation of the effect of strong perturbations can be reduced to the weak case if the driving system (1.2) satisfies a shadowing property. Background material on skew-product flows, nonautonomous cocycle dynamical systems and their attractors will be given in Section 3.

The following notation and definitions will be used. $H^*(A, B)$ denotes the Hausdorff separation or semi-metric between nonempty compact subsets A and B of \mathbb{R}^d , and is defined by

$$H^*(A,B) := \max_{a \in A} \operatorname{dist}(a,B)$$

where $\operatorname{dist}(a, B) := \min_{b \in B} ||a - b||$. For a nonempty compact subset A of \mathbb{R}^d and r > 0, the open and closed balls about A of radius r are defined, respectively, by

$$B(A;r) := \{ x \in \mathbb{R}^d : \text{ dist}(x,A) < r \}, \qquad B[A;r] := \{ x \in \mathbb{R}^d : \text{ dist}(x,A) \le r \}.$$

For a metric space X other than \mathbb{R}^d the Hausdorff semi-metric will be denoted by H_X^* .

2. Perturbation of the driving system. The following example illustrates how small changes to the vector field g of the driving system (1.2) can lead to substantial changes in the behaviour of the solutions of the driven equation (1.3).

Example 2.1. Consider the system of differential equations

$$\dot{x} = p_1 x - \sqrt{x}, \tag{2.1}$$

$$\dot{p}_1 = -p_2 \gamma(\lambda, p), \quad \dot{p}_2 = p_1 \gamma(\lambda, p), \tag{2.2}$$

where $x \in \mathbb{R}^+$, $p := (p_1, p_2) \in \mathbb{S} := \{p : p_1^2 + p_2^2 = 1\}$ and

$$\gamma(\lambda, p) := 1 + \frac{\lambda^2}{1+2\lambda} + \frac{\lambda(1+\lambda)}{1+2\lambda}\operatorname{sign}(p_1)$$
(2.3)

depending on a parameter $\lambda \in \mathbb{R}$. The function $\gamma(\lambda, p)$ is close to 1 for small λ and provides the "perturbation" of the driving system (2.2) for λ different from 0. (The discontinuity in $\gamma(\lambda, p)$ and hence in the vector field of equation (2.2) at $p_1 = 0$ can be avoided, but at the expense of the convenient explicit solutions used below).

Equations (2.1)–(2.2) determine a skew-product system on $\mathbb{R}^+ \times \mathbb{S}$ with the driving component being defined by (2.2) on \mathbb{S} . The p_1 -component of the solution $p(t) = (p_1(t), p_2(t))$ for any small value of parameter λ is 2π -periodic with the mean value

$$\bar{p}_{1,\lambda} := \lim_{t \to \infty} \frac{1}{t} \int_0^t p_1(s) \, ds = \frac{1}{2\pi} \int_0^{2\pi} p_1(s) \, ds = \frac{2}{\pi} \frac{\lambda}{1+\lambda}.$$

The solution $x(t) = \phi(t, x_0, p_0)$ of the forced equation (2.1) is given explicitly by

$$x(t) = \begin{cases} e^{\int_0^t p_1(s) \, ds} w^2(t) & \text{if } w(t) \ge 0, \\ 0 & \text{if } w(t) < 0, \end{cases}$$
(2.4)

where

$$w(t) := \sqrt{x_0} - \frac{1}{2} \int_0^t e^{-\frac{1}{2} \int_0^s p_1(u) \, du} \, ds$$

(for brevity the appropriate initial values have been omitted here).

Now $\bar{p}_{1,\lambda} \leq 0$ for small $\lambda \leq 0$ and the zero solution $\bar{x}(t) \equiv 0$ of the driven system (2.1) is globally asymptotically stable uniformly in $p \in \mathbb{S}$; see Fig. 1. (In fact, it has a nonautonomous (both forwards and pullback) attractor $\hat{A} := \{A_p = \{0\} : p \in \mathbb{S}\}$, while the corresponding autonomous product system generated by equations (2.1)–(2.2) has the global attractor $\mathcal{A} := \{0\} \times \mathbb{S}$).



FIGURE 1

On the other hand, for small $\lambda > 0$, the behaviour of the driven system (2.1) changes dramatically. Indeed, the solution x(t) for sufficiently small initial value x_0 still tends to zero (thin solid lines in Fig. 2), but for a large initial value x_0 it tends to infinity (dotted lines in Fig. 2). These two kinds of solutions are separated by a unique periodic solution $x_{per}(t)$ (thick solid line in Fig. 2) corresponding to

the initial value $x_{per}(0)$ satisfying



FIGURE 2

The periodic solution $x_{per}(t)$ bifurcates from infinity when $\lambda \to 0$ with $\lambda > 0$ (see [6, 18] for bifurcations at infinity) and is obviously unstable. Specifically, it is the bifurcation from infinity of this unstable periodic solution that destroys the global attractor of the unperturbed system.

In the preceding example the driving system was modified through a perturbation of its vector field g, thus changing the dynamical behaviour of the driving system and consequently also that of the driven system. There is another way in which the influence of the driving system on the driven system can be modified without actually modifying the driving system itself, namely by perturbing its input in the vector field f of the driven system. These two types of perturbations will be called strong and weak perturbations, respectively. They lead to the following close but nevertheless different formulations of the "persistency" problem. Assume that the original system (1.1)-(1.2) has an attractor of some kind.

Question 2.2 (weak perturbations). Will the system

$$\dot{x} = f(x, p + q(t, p)),$$
 (2.5)

$$\dot{p} = g(p), \tag{2.6}$$

possess an attractor when $||q(t,p)|| \leq \epsilon$ for sufficiently small ϵ ? What conditions ensure the existence of an attractor of the weakly perturbed system (2.5)–(2.6)?

Question 2.3 (strong perturbations). Will the system

$$\dot{x} = f(x, p), \tag{2.7}$$

$$\dot{p} = g(p) + h(t, p),$$
 (2.8)

possess an attractor when $||h(t,p)|| \leq \epsilon$ for sufficiently small ϵ ? What conditions ensure the existence of an attractor of the strongly perturbed system (2.7)–(2.8)?

It follows from the example above that the answer to the first part of Questions 2.2 and 2.3 is generally "no".

Finding an answer to the second part of each question seems to be easier for Question 2.2 than for Question 2.3. Indeed, this part of Questions 2.2 will be reformulated below in terms of inflated attractors for the "inflation" of the vector field f of the driven system [11, 12]. The main difficulty for Question 2.3 is that a trajectory of the perturbed driving system (2.8) may have very different long term behaviour to the corresponding trajectory of the unperturbed driving system (1.2), no matter how small perturbation term h(t, p). However, this can be overcome by assuming that the unperturbed driving system (1.2) is "shadowing" (e.g., see [9] and the references therein). Then for any trajectory of the perturbed driving system there will exist a trajectory of the unperturbed driving system, which remains close to it for all time, provided the magnitude of the perturbation is sufficiently small. This will allow the problem formulated in the second part of Question 2.3 to be reduced to that of Question 2.2.

3. Skew-product flows and their attractors. The system of differential equations (1.1)-(1.2) is autonomous. However, the *x*-component often represents the state variable that is visible while the decoupled *p*-component represents an independent driving system that often remains hidden (in fact, this driving system need not even be generated by a differential equation, e.g., as in [3, 8]). In this case the dynamics of the *x*-component appears to be nonautonomously as in the driven equation (1.3) and can be formulated as a nonautonomous cocycle dynamical system or a skew-product flow, for which several kinds of nonautonomous attractors can be defined.

Assumption 3.1. The vector field functions $f : \mathbb{R}^d \times P \to \mathbb{R}^d$ in (1.1) and $g : P \to P$ in (1.2), where P is a compact manifold, satisfy Lipschitz continuity and bounded growth or dissipativity conditions which ensure the forwards existence and uniqueness of solutions of (1.1)–(1.2) and the global existence and uniqueness of solutions of (1.2).

Then the solution mapping $(t, p_0) \mapsto p(t, p_0)$ of (1.2), which is continuous, generates a group $\theta = \{\theta_t : t \in \mathbb{R}\}$ of mappings $\theta_t : P \to P$ defined by $\theta_t p_0 = p(t, p_0)$ for each $t \in \mathbb{R}$ and $p_0 \in P$. This autonomous dynamical system θ on P acts as a driving mechanism responsible for the time variation of the vector field of (1.1) as in the driven differential equation (1.3). The resulting solution mapping $\phi : \mathbb{R}^+ \times \mathbb{R}^d \times P \to \mathbb{R}^d$, which is continuous in all of its variables, satisfies

$$\frac{d}{dt}\phi(t,x_0,p_0) = f\left(\phi(t,x_0,p_0),\theta_t p_0\right), \qquad x_0 \in \mathbb{R}^d, \, p_0 \in P, \, t \in \mathbb{R}^+ \tag{3.1}$$

with the *initial condition property*

$$\phi(0, x_0, p_0) = x_0, \qquad x_0 \in \mathbb{R}^d, \, p_0 \in P \tag{3.2}$$

and the cocycle evolution property

$$\phi(s+t, x_0, p_0) = \phi(s, \phi(t, x_0, p_0), \theta_t p_0), \qquad x_0 \in \mathbb{R}^d, \ p_0 \in P, \ s, t \in \mathbb{R}^+.$$
(3.3)

Consequently, the cartesian product mapping $\pi = (\phi, \theta)$ forms an autonomous semi-dynamical system, which is called a skew-product flow, on the product space $\mathbb{R}^d \times P$ [19]. The mapping ϕ is called a *cocycle mapping* on \mathbb{R}^d with respect to the autonomous dynamical system θ on P.

The definition of a global attractor for an autonomous semi-dynamical system π on a metric state space X is well known [9]. Specifically, a nonempty compact

subset \mathcal{A} of X which is π -invariant, i.e., with $\pi(t, \mathcal{A}) = \mathcal{A}$ for all $t \in \mathbb{R}^+$ (in fact, for all $t \in \mathbb{R}$), is called a *global attractor* for π if

$$\lim_{t \to \infty} H_X^*(\pi(t, D), \mathcal{A}) = 0$$

for every nonempty compact subset D of X. For the skew-product flow $\pi = (\phi, \theta)$ above the state space $X = \mathbb{R}^d \times P$.

However, such a definition is often too restrictive in the "nonautonomous" context of a driven differential equation (1.3) in \mathbb{R}^d . Instead, it is often more useful to say that a family $\widehat{A} = \{A_p; p \in P\}$ of nonempty compact subsets of \mathbb{R}^d is *invariant* under ϕ , or ϕ -invariant, if

$$\phi(t, A_p, p) = A_{\theta_t p}, \qquad t \in \mathbb{R}^+, \ p \in P.$$

The natural generalization of convergence then seems to be the *forwards conver*gence defined by

$$H^*(\phi(t, D, p), A_{\theta_t p}) \to 0 \quad \text{as} \quad t \to \infty$$

$$(3.4)$$

for all nonempty compact subsets D of \mathbb{R}^d , but this does not ensure convergence to a specific component set A_p for a fixed p. For that one needs to start "progressively earlier" at $\theta_{-t}p$ in order to "finish" at p, which leads to the concept of *pullback* convergence defined by

$$H^*(\phi(t, D, \theta_{-t}p), A_p) \to 0 \quad \text{as} \ t \to \infty.$$
 (3.5)

The ϕ -invariant family \widehat{A} is then called a *pullback attractor* in the case of pullback convergence and a *forwards attractor* in the case of forwards convergence. To ensure uniqueness in both cases it is usually also required that \widehat{A} be minimal under componentwise set inclusion for all ϕ -invariant families which are pullback attracting, see [5, 11].

The concepts of forwards and pullback attractors are usually independent of each other [10]. However, if one of the limits (3.5) or (3.4) holds uniformly in $p \in P$, then so does the other one and the family \hat{A} is then both a forwards and pullback attractor [2].

Note that for a forwards or pullback attractor \widehat{A} the mapping $t \mapsto A_{\theta_t p}$ is continuous for each fixed $p \in P$ due to the continuity of ϕ in t and the ϕ -invariance of \widehat{A} . However, the mapping $p \mapsto A_p$ is usually only upper semi continuous [2, 16]. Thus, if $\widehat{A} = \{A_p; p \in P\}$ is a forwards or pullback attractor, then the set $\mathcal{A} = \bigcup_{p \in P} (A_p \times \{p\})$ is a compact subset of $\mathbb{R}^d \times P$ and is invariant for the skew-product flow $\pi = (\phi, \theta)$ on $\mathbb{R}^d \times P$. It is a global attractor for the autonomous semi-dynamical system π when \widehat{A} is a forwards attractor, but need not be when \widehat{A} is a pullback attractor.

A family $\widehat{B} = \{B_p ; p \in P\}$ of nonempty compact subsets B_p of \mathbb{R}^d is called a *pullback absorbing neighbourhood system* if it *pullback absorbs* all nonempty compact subsets D of \mathbb{R}^d , i.e., for each such D and $p \in P$ there exists a $T(D,p) \in \mathbb{R}^+$ such that

 $\phi(t, D, \theta_{-t}p) \subset B_p$ for all $t \ge T(D, p)$.

The existence of pullback absorbing neighbourhood system $\widehat{B} = \{B_p ; p \in P\}$ ensures the existence of a unique pullback attractor $\widehat{A} = \{A_p ; p \in P\}$ for which each component subset A_p is determined by

$$A_p = \bigcap_{\tau \ge 0} \bigcup_{t \ge \tau} \phi(t, B_{\theta_{-t}p}, \theta_{-t}p) \quad \text{for each} \quad p \in P.$$
(3.6)

See [4, 7, 14, 15]. Conversely, given a pullback attractor \widehat{A} there always exists a pullback absorbing neighbourhood system \widehat{B} containing \widehat{A} , see [10]. In this case \widehat{B} is ϕ -positively invariant, i.e., satisfies

$$\phi(t, B_p, p) \subset B_{\theta_t p}$$
 for all $t \ge 0, p \in P$.

4. Weak perturbations and inflated attractors. The effects of perturbations of the driving component variable p in the driven equation (2.5) can be investigated in terms of the "inflation" [11, 12] of the vector field of equation (2.5), specifically by replacing the driven equation (1.1) by a differential inclusion

$$\dot{x} \in F_{\epsilon}(x, p), \tag{4.1}$$

while the driving system (1.2) remains unaltered. Here $F_{\epsilon}(x, p)$ is a nonempty compact set, which can be defined either as the *internal* ϵ -preinflation,

$$F_{\epsilon}^{(ip)}(x,p) := \{ y \in \mathbb{R}^d : y = f(x,p+q), \|q\| \le \epsilon \},\$$

or as the internal ϵ -inflation

$$F_{\epsilon}^{(ii)}(x,p) := \{ y \in \mathbb{R}^d : \| y - f(x,p+q) \| \le \rho_{\epsilon}(x,p) \} = B[f(x,p);\rho_{\epsilon}(x,p)], \quad (4.2)$$

where

$$\rho_{\epsilon}(x,p) := \sup\{\|f(x,p+q) - f(x,p)\| : \|q\| \le \epsilon\}.$$

The choice of the ϵ -preinflation allows a more accurate tracing of the properties of the unperturbed system (1.1)–(1.2), while the ϵ -inflation allows direct access to the well developed theory of differential inclusions [1], since the set $F_{\epsilon}^{(ii)}(x,p)$ is convex.

The system (4.1) with the internal ϵ -inflation (4.2) and the unperturbed driving system (1.2) is a straightforward generalization of the ϵ -inflated system introduced in [9, 11], the main difference being that the inflation radius $\rho_{\epsilon}(x,p)$ here may grow unboundedly as $||x|| \to \infty$, while in [11] the corresponding the inflation radius did not depend on either x or p. The set-valued mapping $(\epsilon, x, p) \mapsto F_{\epsilon}(x, p) =$ $F_{\epsilon}^{(ii)}(x,p)$ is continuous in all of its variables with compact convex values. Hence, for any initial value $x(0) = x_0$, the differential inclusion (4.1) has an absolutely continuous solution x(t) satisfying

$$\dot{x}(t) \in F_{\epsilon}(x(t), \theta_t p_0) \tag{4.3}$$

for almost all $t \in [0,T)$ for some maximal $T = T(\epsilon, x_0, p_0) \leq +\infty$, where $\theta_t p_0$ denotes the solution $p(t, p_0)$ of the driving equation (1.2) with the initial condition $p(0) = p_0 \in P$.

Define $\Phi^{\epsilon}(t, x_0, p_0)$ to be the set of all points $y \in \mathbb{R}^d$ for which there exists a solution x(t) of the differential inclusion (4.3) for this $p_0 \in P$ with $x(0) = x_0$ and x(t) = y. The set-valued mapping Φ^{ϵ} will be called the *internal* ϵ -inflation of the single-valued cocycle mapping ϕ generated by unperturbed system (1.1)–(1.2). As in [12], Φ^{ϵ} satisfies:

i) the mapping $t \mapsto \Phi^{\epsilon}(t, x_0, p_0)$ is defined on a maximal interval $[0, T(\epsilon, x_0, p_0))$ for each x_0 and p_0 , where $T(\epsilon, x_0, p_0) \leq +\infty$;

ii) Φ^{ϵ} is continuous in all of its variables (t, x_0, p_0) as well as in $\epsilon \ge 0$;

iii) $\Phi^{\epsilon}(t, C, p_0)$ is compact for any $0 \leq t \leq \inf_{x_0 \in C} T(\epsilon, x_0, p_0)$, compact subset C of \mathbb{R}^d and $p_0 \in P$;

iv) Φ^{ϵ} satisfies the initial value property $\Phi^{\epsilon}(0, x_0, p_0) = x_0$ for all $x_0 \in \mathbb{R}^d$, $p_0 \in P$ and the cocycle property

$$\Phi^{\epsilon}(s+t,C,p_0) = \Phi^{\epsilon}(s,\Phi^{\epsilon}(t,C,p),\theta_t p_0)$$

for all $s, t \ge 0$ such that $s + t < \inf_{x_0 \in C} T(\epsilon, x_0, p_0)$ for any compact subset C of \mathbb{R}^d and $p_0 \in P$.

Suppose $T(\epsilon, x_0, p_0) = \infty$ for all $x_0 \in \mathbb{R}^d$ and $p_0 \in P$. Then Φ^{ϵ} is a set-valued cocycle dynamical system driven by the autonomous system θ and analogous definitions to the single-valued case apply for a family $\widehat{A}^{\epsilon} = \{A_p^{\epsilon} : p \in P\}$ of nonempty compact subsets of \mathbb{R}^d to be Φ^{ϵ} -invariant, a forwards attractor or a pullback attractor of the set-valued cocycle mapping Φ^{ϵ} . Such an attractor (forwards or pullback) of the internal ϵ -inflated set-valued cocycle Φ^{ϵ} will be called the internal ϵ -inflated attractor (forwards or pullback) of the unperturbed cocycle mapping ϕ . The following theorem is adapted from [11].

Theorem 4.1. Suppose that the single-valued cocycle dynamical system ϕ has an internal ϵ_0 -inflated pullback attractor $\widehat{A}^{\epsilon_0} = \{A_p^{\epsilon_0} : p \in P\}$ for some $\epsilon_0 > 0$. Then ϕ has an internal ϵ -inflated pullback attractor $\widehat{A}^{\epsilon} = \{A_p^{\epsilon} : p \in P\}$ for every $\epsilon \in [0, \epsilon_0]$ and these are related through

$$A_p^{\epsilon} \subset A_p^{\epsilon'}, \qquad A_p^{\epsilon} = \bigcap_{\epsilon < \epsilon'} A_p^{\epsilon'}$$

for any $0 \leq \epsilon < \epsilon' \leq \epsilon_0$ and each $p \in P$.

Dissipativity conditions on the vector field function f of equation (1.1) ensuring the existence of an inflated pullback attractor for some $\epsilon_0 > 0$ are given in [11] in the case of a uniform inflation radius. A related result for the discretization of a uniform (hence both forwards and pullback) attractor is given in [13], where a Lyapunov function characterizing the given attractor is used to construct a pullback absorbing system for the discretized system. Both of these approaches can be adapted without difficulty to the variable inflation context under consideration here.

In such cases it follows from Theorem 4.1 that weak perturbations of the driving system variable in the vector field of the driven equation (1.1) give rise to a pullback attractor of the perturbed system, which is componentwise close to the unperturbed pullback attractor in the sense of upper semi continuous convergence as the perturbation magnitude ϵ approaches zero.

Remark 4.2. There are many situations, like that above or in the proof of Theorem 5.3 below, where the uniformity of a nonautonomous attra ctor may be replaced by a weaker notion of uniformity of the absorbing neighbourhood system. A family $\mathcal{B} = \{B_p ; p \in P\}$ of uniformly bounded nonempty compact subsets $B_p \subset \mathbb{R}^d$ will be called a *uniform absorbing neighbourhood system* for a nonautonomous dynamical system (ϕ, θ) if it is ϕ -positively invariant and *uniformly absorbs* all nonempty compact subsets sets of \mathbb{R}^d , i.e., for each such set D there exists a $T(D) \in \mathbb{R}^+$, which is independent of $p \in P$, such that

$$\phi(t, p, D) \subset B_{\theta_t p}$$
 for all $t \geq T(D), p \in P$.

The existence of a uniform absorbing neighbourhood system ensures the existence of both forwards or pullback attractors (which generally need not coincide nor be uniform), see [4, 7, 14, 15].

5. Strong perturbations and shadowing. The autonomous dynamical system θ generated by equation (1.2) is said to have the *shadowing property* if for any $\epsilon > 0$ there is a $\delta = \delta(\epsilon) > 0$ such that for any absolutely continuous function $q : \mathbb{R} \to P$ satisfying

$$\|\dot{q}(t) - g(q(t))\| < \delta, \quad \text{for almost all } t \in \mathbb{R},$$

$$(5.1)$$

there exists a solution p(t) of unperturbed equation (1.2) such that

$$\|q(t) - p(t)\| < \epsilon, \qquad t \in \mathbb{R}, \tag{5.2}$$

holds.

Let $\phi^{\{h\}}$ be the cocycle solution mapping of system (1.1) corresponding to the solution mapping $\theta_t^{\{h\}}$ of the perturbed driving system (2.8) with the strong perturbation mapping h(t, p). (These may be set-valued depending on the regularity properties of h, but for convenience will be written here as if they are single-valued).

Lemma 5.1. Suppose that the autonomous system (1.2) has the shadowing property and that the perturbation term h(t, p) in (2.8) satisfies $||h(t, p)|| < \delta$ for $\delta = \delta(\epsilon)$ as given in the shadowing property. Then, for any nonempty compact subset D of \mathbb{R}^d ,

$$\phi^{\{h\}}(t, D, q) \subseteq \bigcup_{\|p-q\| \le \epsilon} \Phi^{\epsilon}(t, D, p), \qquad t \in \mathbb{R}^+, \tag{5.3}$$

and

$$\phi^{\{h\}}(t, D, \theta_{-t}^{\{h\}}q) \subseteq \bigcup_{\|p-q\| \le \epsilon} \Phi^{\epsilon}(t, D, \theta_{-t}p), \qquad t \in \mathbb{R}^+,$$
(5.4)

where Φ^{ϵ} is the internal ϵ -inflated cocycle solution mapping of (1.1)–(1.2).

Proof. Fix an $\epsilon > 0$ and let $||h(t, p)|| < \delta$ with $\delta = \delta(\epsilon)$ as given in the shadowing property for the equation (1.2). Let q(t) be a solution of the perturbed driving equation (2.8), i.e., satisfying

$$\dot{q}(t) = g(q(t)) + h(t, q(t)), \qquad t \in \mathbb{R}.$$

The inequality (5.1) thus holds for q(t), so by the shadowing property there exists a solution p(t) of the unperturbed equation (1.2) such that the inequality (5.2) holds.

Let $x(t) := \phi^{\{h\}}(t, x_0, q(0))$ denote the solution of the driven equation of

$$\dot{x}(t) = f(x(t), q(t)), \qquad x(0) = x_0$$

i.e., $q(t) = \theta_t^{\{h\}} q(0)$ here. Then

$$\dot{x}(t) = f(x(t), p(t) + (q(t) - p(t))), \qquad x(0) = x_0,$$

and so, by the shadowing inequality (5.2),

$$\dot{x}(t) \in F_{\epsilon}(x(t), p(t)), \qquad x(0) = x_0$$

It then follows from the definition of the solution mapping for the internally inflated system that

$$\phi^{\{h\}}(t, x_0, q(0)) = x(t) \in \Phi^{\epsilon}(t, x_0, p(0))$$

and the required forwards inclusion (5.3) is an immediate consequence of the fact that $||p(0) - q(0)|| < \epsilon$. The backwards inclusion (5.4) is proved analogously.

Remark 5.2. In spite of its simplicity, Lemma 5.1 is important because it clearly demonstrates the differing influences of the weak and strong perturbations on the behaviour of the system, the former manifesting itself through the second ϵ in (5.3) and (5.4), i.e., in the Φ^{ϵ} term, and the latter through the first ϵ in (5.3) and (5.4), i.e., under the set union symbol.

Theorem 5.3. Suppose that the system (1.1)-(1.2) satisfies Assumption 3.1, that the driven cocycle system ϕ possesses a uniform internal ϵ -inflated attractor $\widehat{A}^{\epsilon} :=$ $\{A_p^{\epsilon} : p \in P\}$ for each $\epsilon \in [0, \epsilon_0]$ for some $\epsilon_0 > 0$, and that the driving system (1.2) has the shadowing property. In addition, suppose that the perturbation term h(t, p) in (2.8) satisfies $||h(t,p)|| < \delta$ for $\delta = \delta(\epsilon)$ as given in the shadowing property for some $\epsilon \in (0, \epsilon_0]$.

Then the strongly perturbed system (2.7)–(2.8) has a pullback attractor $\widehat{A}^{\{h\}} := \{A_a^{\{h\}} : q \in P\}$ such that

$$A_q^{\{h\}} \subseteq \bigcup \left\{ A_p^{\epsilon} : \|q - p\| \le \epsilon \right\}, \qquad q \in P.$$
(5.5)

Proof of Theorem 5.3. Let $\epsilon \in (0, \epsilon_0]$ correspond to the $\delta(\epsilon)$ that bounds the perturbation h. Then by the uniformity assumption on the internal ϵ -inflated attractor \widehat{A}^{ϵ} , for any $\sigma > 0$ and nonempty compact subset D of \mathbb{R}^d there exists a $T = T(\epsilon, \sigma, D) \ge 0$ such that

$$\Phi^{\epsilon}(t,D,p) \subset B\left(A^{\epsilon}_{\theta_{t}p};\sigma\right) \qquad \text{for all } t \geq T(\epsilon,\sigma,D), \ p \in P$$

It follows immediately from this inclusion and from Lemma 5.1 that

$$\phi^{\{h\}}(t, D, q) \subseteq B\left(\bigcup\left\{A^{\epsilon}_{\theta_t p} : \|p - q\| \le \epsilon\right\}; \sigma\right)$$
(5.6)

and

$$\phi^{\{h\}}(t, D, \theta_{-t}^{\{h\}}q) \subseteq B\left(\bigcup\left\{A_p^{\epsilon} : \|p-q\| \le \epsilon\right\}; \sigma\right)$$

$$(5.7)$$

for all $t \geq T(\epsilon, \sigma, D)$ and $q \in P$. Now fix an arbitrary $\sigma > 0$ and define

$$B_{\sigma} := B\left[\bigcup\left\{A_{p}^{\epsilon_{0}}: p \in P\right\}; \sigma\right],$$

where $\widehat{A}^{\epsilon_0} := \{A_p^{\epsilon_0} : p \in P\}$ is the uniform internally ϵ_0 -inflated attractor. This set B_{σ} is compact since the set P and the sets $A_p^{\epsilon_0}$ are compact and the mapping $p \mapsto A_p^{\epsilon_0}$ is upper semi-continuous. Moreover, by Theorem 4.1, B_{σ} contains any set A_p^{ϵ} with $\epsilon \in [0, \epsilon_0)$ and $p \in P$. Hence by (5.6) and (5.7), respectively,

$$\phi^{\{h\}}(t,D,q) \subseteq B\left(\bigcup \left\{A_{\theta_t p}^{\epsilon} : \|p-q\| \le \epsilon\right\}; \sigma\right) \subseteq B_{\sigma}$$

and

$$\phi^{\{h\}}(t, D, \theta_{-t}^{\{h\}}q) \subseteq B\left(\bigcup\left\{A_p^{\epsilon} : \|p-q\| \le \epsilon\right\}; \sigma\right) \subseteq B_{\sigma}$$

$$(5.8)$$

for all $t \geq T(\epsilon, \sigma, D)$ and $q \in P$.

The existence of an attractor (forwards and pullback) $\widehat{A}^{\{h\}} := \{A_p^{\{h\}} : p \in P\}$ of the strongly perturbed system (2.7)–(2.8) follows from the above inclusions by Theorems 2.8 or 2.9 of [2]. In particular, (3.6) for $(\phi^{\{h\}}, \theta^{\{h\}})$ instead of (ϕ, θ) with the pullback absorbing system consisting of the same subset B_{σ} gives

$$A_q^{\{h\}} = \bigcap_{\tau \ge 0} \overline{\bigcup_{t \ge \tau} \phi^{\{h\}}(t, B_\sigma, \theta_{-t}^{\{h\}}q)} \subseteq B\left(\bigcup \left\{A_p^\epsilon : \|q - p\| \le \epsilon\right\}; \sigma\right), \qquad q \in P,$$

where the set inclusion follows from (5.8) with $D = B_{\sigma}$. The desired inclusion (5.5) then follows since $\sigma > 0$ can be chosen arbitrarily small.

Remark 5.4. Although the system (1.1)–(1.2) is supposed to have a <u>uniform</u> attractor, it is not clear whether the attractor $\hat{A}^{\{h\}}$ of the perturbed system (2.7)–(2.8) will be uniform or not.

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