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Complete $(q^2 + q + 8)/2$-caps in the spaces $PG(3, q)$, $q \equiv 2 \pmod 3$ an odd prime, and a complete 20-cap in $PG(3, 5)$

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Abstract An infinite family of complete $(q^2 + q + 8)/2$-caps is constructed in $PG(3, q)$ where q is an odd prime $\equiv 2 \pmod 3$, $q \geq 11$. This yields a new lower bound on the second largest size of complete caps. A variant of our construction also produces one of the two previously known complete 20-caps in $PG(3, 5)$. The associated code weight distribution and other combinatorial properties of the new $(q^2 + q + 8)/2$-caps and the 20-cap in $PG(3, 5)$ are investigated. The updated table of the known sizes of the complete caps in $PG(3, q)$ is given. As a byproduct, we have found that the unique complete 14-arc in $PG(2, 17)$ contains 10 points on a conic. Actually, this shows that an earlier general result dating back to the Seventies fails for $q = 17$.

Keywords Complete caps · Projective spaces of the dimension three · Projective planes

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1 Introduction

Let $PG(n, q)$ be the projective space of dimension n over the Galois field F_q and let $F_q^* = F_q \setminus \{0\}$. A k-cap in $PG(n, q)$ is a set of k points, no three of which are collinear. A k-cap in

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$PG(n,q)$ is called complete if it is not contained in a $(k+1)$-cap of $PG(n,q)$. If $n=2$, then a k-cap is usually called a k-arc. A survey of results on caps and arcs is found in [16, 17], see also [18, 25, 27].

The main questions on caps in $PG(n,q)$ are also of interest in Coding theory and are about the size of very large caps, especially about $m_2(n,q)$, the size of the largest complete cap, and about $m'_2(n,q)$, the size of the second largest complete cap. On the other hand, very small complete caps have also been investigated, especially the problem of determining $t_2(n,q)$, the size of the smallest complete cap.

The known constructions of caps mostly start with a given cap of size $T_q=(q^2+q+4)/2$ containing all but just one point from an elliptic quadric of $PG(3,q)$. As a matter of fact, T_q is an important number in the study of complete caps since any cap containing at least T_q points from an elliptic quadric is entirely contained in it. This was first pointed out by B. Segre in [25], see also [1, 17, 23], and gave a motivation for the study of M_q, the maximum size of complete k-caps other than elliptic quadrics containing as many as T_q-1 points of an elliptic quadric.

It is well known that $m_2(3,q)=q^2+1$ for all $q\geq 3$, but the exact value of $m'_2(3,q)$ has been determined so far only for $q\leq 7$, namely

$$m'_2(3,3)=T_3,\ \ m'_2(3,4)=T_4+2,\ \ m'_2(3,5)=T_5+3,\ \ m'_2(3,7)=T_7+2,$$

see [3, 11, 12, 18, Table 4.2(i)]. For $q\geq 8$, the best known upper bound is approximately cq^2 with $c=1$ for $q>2$ even and for q odd but not prime, while $c=2641/2700$ for q odd prime, see [1, 8, 13–15, 17, 18, Theorem 4.2, Table 4.2(ii)], [23–25, 27]. These and other results give a motivation for the construction of caps whose size is close to $m'_2(3,q)$ which is the subject of the present paper.

Our main result is the construction of an infinite family of complete caps of size T_q+2 for all $q\geq 11$ odd prime such that $q\equiv 2 \pmod 3$. This gives an improvement on the best previous bound on $m'_2(3,q)$:

$$(q^2+q+8)/2=T_q+2\leq m'_2(3,q) \text{ for odd prime } q\equiv 2 \ (\mathrm{mod}\ 3),\ \ q\geq 11. \tag{1}$$

Furthermore, a variant of our construction produces one of the two complete $m'_2(3,5)$-caps in $PG(3,5)$. This is of interest, since $m'_2(3,5)=20$ and there exist only two projectively non-equivalent complete 20-caps in $PG(3,5)$, see [3]. The points of both complete $m'_2(3,5)$-caps in $PG(3,5)$ are listed in [3], and we give a synthetic construction for one of them.

The (T_q+2)-caps produced by our construction share T_q-1 points with an elliptic quadric. Therefore,

$$(q^2+q+8)/2=T_q+2\leq M_q \text{ for odd prime } q\equiv 2 \ (\mathrm{mod}\ 3). \tag{2}$$

The known bounds for these q are $T_q+1\leq M_q\leq T_q+q+1$ [25, Theorem V, p. 73]. For $q=5$, from the above mentioned construction of 20-caps we obtain the exact value

$$M_5=T_5+3=20. \tag{3}$$

For q odd, the best known lower bound on $m'_2(3,q)$ is T_q+1, and is known to be attained for infinite values of q. From [23, Theorems IV,V], there exist complete k-caps for

$$k=T_q+1,\ \ q\equiv 1 \ (\mathrm{mod}\ 4) \text{ or } q=2p-1,\ \ p \text{ prime},$$

and for

$$T_q+1\leq k,\ \ q \text{ odd},\ \ q\neq 2^r-1.$$

Also, in [25, Theorem V, p. 73] complete k-caps in $PG(3, q)$ are constructed for

$$T_q + 1 \leq k \leq T_q + q + 1,\ q \text{ odd},\ q \equiv 2 \pmod 3 \text{ or } q = 3^h \geq 9.$$

Subsets of our $(T_q + 2)$-caps can be adopted as a starter for the greedy algorithms often used for producing complete caps, see [8, Sect. 2]. This has allowed us to obtain many new entries in the spectrum of sizes of complete caps in $PG(3, q)$, $q \leq 23$, and to update the table of the known complete cap sizes in $PG(3, q)$ as given in [8].

The essential idea of our construction of (T_q+2)-caps, as many other known constructions, goes back to Segre, see [25, Theorem V, p. 73], see also [17, Sect. 18.2]. Given an elliptic quadric $\mathcal{Q}$, in $PG(3, q)$, our $(T_q + 2)$-cap comprises a point $P \notin \mathcal{Q}$, the $q + 1$ common points of $\mathcal{Q}$ with the polar plane of P together with one point from each bisecant through P, chosen one of its two common points with $\mathcal{Q}$. Segre proved that either the resulting T_q-cap K is complete or K can be completed by adding to it at most $q + 1$ points. Since the former case occurs with probability 1, a careful special choice of the points from the bisecants is needed in order to avoid that K be itself complete. Here we present such a special choice that allows us to add to K two more points obtaining a complete $(T_q + 2)$-cap. Although our procedure for the choice is not of geometric nature, it gives a good geometrical result.

The points of a k-cap K in $PG(n, q)$ can be viewed as columns of the generating matrix of the *associated code* $\mathcal{A}(K)$ with length k and dimension $n + 1$. Taking into consideration the well known connection between secant hyperplanes of caps and the weight distribution of the associated code, [5, Theorem 16.1], [10, 15, Theorem 4.1], [20], our study of the code associated to the above $(T_q + 2)$-cap K is carried out using the combinatorial properties of K, the orbit structure of the collineation group preserving K, the plane sections of K, and the conics lying in K.

Considering secant planes of our $(T_{17} + 2)$-cap we point out that the unique complete 14-arc of $PG(2, 17)$ must contain 10 points from a conic. Actually, this disproves a result of [22, Theorem 1] where it is claimed that for $q \equiv 1 \pmod 4$ in the plane $PG(2, q)$ a complete arc which has $(q + 3)/2$ points on a conic can contain at most two points outside the conic.

Some results of this work were presented without proofs in [9].

2 An infinite family of $(q^2 + q + 8)/2$-caps in $PG(3, q)$ with odd prime $q \equiv 2 \pmod 3$

Let q be odd prime, $q \equiv 2 \pmod 3$. Since F_q is isomorphic to Z_q, the elements of F_q are identified by the integers $0, 1, \ldots, q - 1$ and the operations in *quantitative* relations are carried out modulo q.

Our aim is to construct an infinite family of $(T_q + 2)$-caps in $PG(3, q)$ containing $T_q - 1$ points from an elliptic quadric.

Let (x_0, x_1, x_2, x_3) be a point of $PG(3, q)$ with $x_i \in F_q$, $x_0 \in \{0, 1\}$. Choose a projective frame in $PG(3, q)$ such that the elliptic quadric Q has equation

$$x_2^2 - \alpha x_1^2 - x_0 x_3 = 0,\ \alpha \in F_q,\ \alpha \text{ is a non square.} \tag{4}$$

Remark 1 In $PG(3, q)$ through every point $V \notin \mathcal{Q}$ there are $q+1$ tangents to $\mathcal{Q}$ and $q(q-1)/2$ bisecants of $\mathcal{Q}$, see [17, 25]. We denote by π_V the polar plane of the point V. The intersection of $\mathcal{Q}$ and π_V is the conic $\mathcal{C}_V = \mathcal{Q} \cap \pi_V$ which is also the set of the tangency points of the tangents through V. For a point $D = (d_0, d_1, d_2, d_3)$ with $d_0 \in \{0, 1\}$, $d_1, d_2, d_3 \in F_q$, the equation of the polar plane π_D can be written as $F_{x_0}(D)x_0+F_{x_1}(D)x_1+F_{x_2}(D)x_2+F_{x_3}(D)x_3=0$ where F_{x_i} is the corresponding partial derivative of the polynomial associated to the quadric.

Our construction can be viewed as a variant of Segre's construction recalled in Introduction.

Construction A1 Let q be an odd prime, $q \equiv 2 \pmod 3$. Then -3 is a non square element in F_q [16, Sect. 1.5]. In $PG(3,q)$ a (T_q+2)-set $\mathcal{K}$ having T_q-1 points on the quadric $\mathcal{Q}$ of equation (4) is obtained as follows. Take the *basic* point $P=(0,1,\overline{x}_2,\overline{x}_3)\notin \mathcal{Q}$ where $\overline{x}_2=\pm\sqrt{-3\alpha}$ and $\overline{x}_3$ is an arbitrary element of F_q. Every bisecant through P contains two points from $\mathcal{Q}$, namely $A'=(1,x_1',x_2',x_3')$ and $A''=(1,x_1'',x_2'',x_3'')$. Here $x_1'\neq x_1''$, and we choose either A' or A'' according as $x_1'<x_1''$ or $x_1'>x_1''$. Following Segre's construction, the set of size T_q comprising P, the polar conic π_P and the above chosen points on the bisecants is a T_q-cap, let be $\mathcal{K}_1$. Such a cap turns out to be incomplete as both points $P_1=(0,1,0,0)$, and $P_2=(0,1,-\overline{x}_2,4\alpha-\overline{x}_3)$ can be added to obtain a (T_q+2)-cap. The latter claim is our Theorem 1. It may be noted that the points P and P_2 are *dual* to each other in the sense that P_2 can be used as the basic point and then P plays the role of the point P_2.

Although P is not uniquely taken, the (T_q+2)-sets produced by Construction A1 seem to be projectively equivalent. This has been proven so far when $5\le q\le 29$ by a computer aided exhaustive search.

Proposition 1 *Let* $5\le q\le 29$. *For all non square elements* α *in (4) and for all* $\overline{x}_3$, *the resulting* (T_q+2)*-sets by Construction A1 are projectively equivalent.*

This suggests to specify the condition of the choice of P in Construction A1 by setting $\alpha=-3$ in the relation (4), and choosing P as follows.

Condition 1 Let q be an odd prime and let $q\equiv 2 \pmod 3$. Let the quadric $\mathcal{Q}$ be given by the equation

$$3x_1^2+x_2^2=x_0x_3. \tag{5}$$

Take $\overline{x}_2=\sqrt{-3\alpha}=3$ and $\overline{x}_3=0$ in the coordinates of P, so that the points outside $\mathcal{Q}$ in Construction A1 are $P=(0,1,3,0)$, $P_1=(0,1,0,0)$, and $P_2=(0,1,-3,-12)$.

For the quadric of equation (5), the polar plane π_D of a point D has equation

$$\pi_D:\quad -d_3x_0+6d_1x_1+2d_2x_2-d_0x_3=0,\quad D=(d_0,d_1,d_2,d_3). \tag{6}$$

A straightforward computation proves the following result.

Lemma 1 *Under Condition 1, the polar planes* π_P, π_{P_1}, *and* π_{P_2} *intersect the quadric* $\mathcal{Q}$ *of (5) in the conics* $\mathcal{C}_P$, $\mathcal{C}_{P_1}$, *and* $\mathcal{C}_{P_2}$, *respectively,*

$$\mathcal{C}_P=\{(1,v,-v,4v^2)|v\in F_q\}\cup\{(0,0,0,1)\}; \tag{7}$$

$$\mathcal{C}_{P_1}=\{(1,0,d,d^2)|d\in F_q\}\cup\{(0,0,0,1)\}; \tag{8}$$

$$\mathcal{C}_{P_2}=\{(1,v,v+2,4v^2+4v+4)|v\in F_q\}\cup\{(0,0,0,1)\}. \tag{9}$$

Lemma 2 *Under Condition 1, the* T_q*-cap* $\mathcal{K}_1$ *in Construction A1 is as follows:*

$$\begin{aligned}\mathcal{K}_1&=\{(1,v,d,3v^2+d^2)|\ v\in F_q,\ d=v-2i,\ i\in\{v,v+1,\dots,q-1\}\}\cup\{A_\infty,P\}\\&=\{(1,v,d,3v^2+d^2)|\ v\in F_q,\ d\in\{-v,\ -v-2,\dots,v+4,v+2\}\}\cup\{A_\infty,P\}\end{aligned} \tag{10}$$

where $A_\infty=(0,0,0,1)$, $P=(0,1,3,0)$, *and the list of d values contains* $q-v$ *terms.*

Proof If the points $A' = (1, f, g, 3f^2+g^2) \in \mathcal{Q}$ and $A'' = (1, v, d, 3v^2+d^2) \in \mathcal{Q}$ are on a bisecant through $P = (0, 1, 3, 0)$, then

$$u^{-1}(1, f, g, 3f^2+g^2) - u^{-1}(1, v, d, 3v^2+d^2) = (0, 1, 3, 0), \ u \in F_q^*,$$

whence

$$f - v = u; \quad g - d = 3u; \quad 3f^2+g^2 = 3v^2+d^2.$$

From this $v - d = 2f$. If $f > v$ then the point $A'' = (1, v, d, 3v^2+d^2)$ is in the cap $\mathcal{K}_1$. So, by our procedure the points A^* of $\mathcal{K}_1$ on the bisecants are

$$A^* = (1, v, d, 3v^2+d^2), \ v - d = 2i, \ i = v+1, v+2, \ldots, q-1. \tag{11}$$

By (7), for points of $\mathcal{C}_P$ we have $v - d = 2v$. Together with (11) it gives (10). □

Example 1 Let $q = 5$. By (10), the points of the T_5-cap $\mathcal{K}_1$ have the form

v	i	d	point	type	v	i	d	point	type	v	i	d	point	type
0	0	0	1000	$\mathcal{C}_P$	1	2	2	1122	A^*	3	3	2	1321	$\mathcal{C}_P$
0	1	3	1034	A^*	1	3	0	1103	A^*	3	4	0	1302	A^*
0	2	1	1011	A^*	1	4	3	1132	A^*	4	4	1	1414	$\mathcal{C}_P$
0	3	4	1041	A^*	2	2	3	1231	$\mathcal{C}_P$				0001	$\mathcal{C}_P(A_\infty)$
0	4	2	1024	A^*	2	3	1	1213	A^*				0130	P
1	1	4	1144	$\mathcal{C}_P$	2	4	4	1243	A^*					

Corollary 1 *Condition 1 implies* $\mathcal{C}_P \cup \mathcal{C}_{P_1} \cup \mathcal{C}_{P_2} \subset \mathcal{K}_1$.

Proof This follows from the definition of Construction A1 and from relations (8)–(10). □

Lemma 3 *Condition 1 implies the point* $P_1 = (0, 1, 0, 0)$ *does not lie on any bisecant of* $\mathcal{K}_1$.

Proof If the points $D' = (1, v, d, 3v^2+d^2) \in \mathcal{Q}$ and $D'' = (1, m, n, 3m^2+n^2) \in \mathcal{Q}$ lie on a bisecant of $\mathcal{Q}$ through $P_1 = (0, 1, 0, 0)$, then

$$u^{-1}(1, v, d, 3v^2+d^2) - u^{-1}(1, m, n, 3m^2+n^2) = (0, 1, 0, 0), \ v, d, m, n \in F_q, \ u \in F_q^*,$$

whence

$$v - m = u; \quad d = n; \quad 3v^2+d^2 = 3m^2+n^2.$$

From this $v = -m, \ u = 2v, \ v \neq 0$. So, P_1 lies on the bisecant through the points D' and D'' where

$$D' = (1, v, d, 3v^2+d^2) \in \mathcal{Q}, \quad D'' = (1, -v, d, 3v^2+d^2) \in \mathcal{Q}, \quad v \neq 0.$$

The points D' and D'' do not simultaneously belong to $\mathcal{K}_1$. Let $D' \in \mathcal{K}_1$. Then, by (10), $d \in \{-v, -v-2, \ldots, v+4, v+2\}$. If $D'' \in \mathcal{K}_1$ then $d \in \{v, v-2, \ldots, -v+4, -v+2\}$. But it is a contradiction since they are complementary subsets of F_q. □

Example 2 Let $q = 11$ and let $D' = (1, 6, d, 9+d^2) \in \mathcal{K}_1$. Then $d \in \{5, 3, 1, -1, -3\}$. If $D'' = (1, -6, d, 9+d^2) \in \mathcal{K}_1$ then $d \in \{6, 4, 2, 0, -2, -4\}$. It is a contradiction.

Lemma 4 *Condition 1 implies the point* $P_2 = (0, 1, -3, -12)$ *does not lie on any bisecant of* $\mathcal{K}_1$.

Proof If the points $V' = (1, v, d, 3v^2 + d^2) \in \mathcal{Q}$ and $V'' = (1, m, n, 3m^2 + n^2) \in \mathcal{Q}$ lie on a bisecant of $\mathcal{Q}$ through $P_2 = (0, 1, -3, -12)$, then

$$u^{-1}(1, v, d, 3v^2 + d^2) - u^{-1}(1, m, n, 3m^2 + n^2)$$
$$= (0, 1, -3, -12), v, d, m, n \in F_q, u \in F_q^*,$$

whence

$$v - m = u; \quad d - n = -3u; \quad 3v^2 + d^2 - 3m^2 - n^2 = -12u. \tag{12}$$

Taking $u \neq 0$ into account, from (12) we obtain

$$v - d = 2u - 2 \neq -2, \quad m - n = -2u - 2 \neq -2. \tag{13}$$

We show that the points V' and V'' do not simultaneously belong to $\mathcal{K}_1$. Let $\{V', V''\} \subset \mathcal{K}_1$. Then, by (10),

$$v - d \in \{2v, 2(v+1), \ldots, 2(q-2)\}, \ v \neq q - 1. \tag{14}$$

The number $2(q-1)$ is not included in (14) as $v - d \neq -2$. By the same reason, $v \neq q - 1$ otherwise $2v = -2$ and $v - d \in \{-2, 0, \ldots\}$. Similarly,

$$m - n \in \{2m, 2(m+1), \ldots, 2(q-2)\}, \ m \neq q - 1. \tag{15}$$

By (12), $m = v - u$. As $m \neq q - 1$ we have $u \neq v + 1$. Now, by (13) and (14),

$$u = \frac{v - d}{2} + 1, \quad u \in \{v + 2, v + 3, \ldots, q - 1\}. \tag{16}$$

From (16),

$$u = v + \Delta, \quad 2 \leq \Delta \leq q - 1 - v. \tag{17}$$

If $v = q - 2$ then the set $\{2v, 2(v+1), \ldots, 2(q-2)\}$ of (14) consists of the only element $2(q-2)$, whence $v - d = 2(q-2)$. It implies $u = \frac{v-d}{2} + 1 = (q-2) + 1 = v + 1$ that contradicts to the second relation of (16). So, $v \neq q - 2$.

Henceforth we represent v as a nonnegative integer, $0 \leq v \leq q - 3$.

From (12) and (17), $m = v - u = -\Delta$. Now, by (15), $(m-n)/2 + \Delta \in \{0, 1, \ldots, \Delta - 2\}$. In the other hand, $(m-n)/2 + \Delta = -u - 1 + \Delta = -v - 1$, see (13),(17). Therefore $-v - 1 \in \{0, 1, \ldots, \Delta - 2\}$. As $v \notin \{q-1, q-2\}$, it holds that $v \in \{q-3, q-4, \ldots, q+1-\Delta\}$. So, $v \geq q + 1 - \Delta$. But $\Delta \leq q - 1 - v$, see (17). So, $v \geq v + 2$; a contradiction. □

Example 3 Let $q = 17$, $v = 4$. By (14), $v - d \in \{8, 10, 12, \ldots, 30\}$. By (16), $u \in \{6, \ldots, 16\}$. By (17), $u = v + \Delta$, $2 \leq \Delta \leq 12$. As $v \in \{q-3, q-4, \ldots, q+1-\Delta\}$, we have $v \in \{14, 13, \ldots, 18 - \Delta\}$. So, $v \geq 18 - \Delta \geq 6$. It is a contradiction to $v = 4$.

Theorem 1 *Under Condition 1, where q is odd prime and $q \equiv 2 \pmod 3$, the set $\mathcal{K}$ in Construction A1 is a $(q^2 + q + 8)/2$-cap in $PG(3, q)$.*

Proof By Lemmas 3 and 4, the points P_1 and P_2 are uncovered by bisecants of the cap $\mathcal{K}_1$. In addition, the line P_1P_2 is external to the quadric $\mathcal{Q}$ and to the cap $\mathcal{K}_1$. □

Corollary 2 *For odd prime $q \equiv 2 \pmod 3$ the value $(q^2 + q + 8)/2$ is a lower bound on the second greatest size for the complete caps in $PG(3, q)$.*

Remark 2 In Construction A1 the points P_1 and $P_{2,}$ as well as the point P, can be regarded as a basic point for the (T_q+2)-cap $\mathcal{K}$, since every point off the elliptic quadric $\mathcal{Q}$ lies on some $q+1$ tangents to $\mathcal{Q}$ and on $q(q-1)/2$ bisecants of $\mathcal{Q}$, see Remark 1. Moreover, it can be shown that, under Condition 1, using the same procedure for choice of the points A^* in bisecants of $\mathcal{Q}$ with respect to the basic point P_2 we obtain the same cap $\mathcal{K}_1$ as from the basic point P. We remind on the duality of the points P and P_2 pointed out in the description of Construction A1. However, for the basic point P_1 we need another procedure for the choosing of the points in the bisecants.

3 Completeness of the $(q^2+q+8)/2$-caps in $PG(3, q)$

We start with the following result.

Lemma 5 *Let Π_∞ be the tangent plane to the quadric $\mathcal{Q}$ with equation $x_0 = 0$. Under Condition 1, all points of Π_∞ outside the (T_q+2)-cap $\mathcal{K}$ of Construction A1 lie on bisecants of $\mathcal{K}$.*

Proof We relay on [25, Theorem V, p. 73, Proof], [17, Theorem 18.2.4, Proof], and use Corollary 1. Assume that on Π_∞ there are points $P_3,\ P_{4,\dots,}\ P_\Delta \notin \mathcal{Q}$ which are not covered by bisecants of $\mathcal{K}$. The polar plane of every such point, as well as the polar planes of P, P_1, and P_2, meet the quadric $\mathcal{Q}$ in a conic entirely contained in the cap $\mathcal{K}_1$, i.e., $\mathcal{C}_P \cup \bigcup_{i=1}^{\Delta} \mathcal{C}_{P_i} \subset \mathcal{K}_1$. By Construction A1, $|\mathcal{Q} \cap \mathcal{K}_1| = (q+1) + q(q-1)/2$. If the conic $\mathcal{C}_{P_i}$ is not entirely contained in $\mathcal{K}_1$ then at least one bisecant of $\mathcal{Q}$ through P_i must contain two points on $\mathcal{K}_1$, see Remark 1. From [17, 25], every point of $\mathcal{K}_1$ belongs to at most three conics mentioned above. In particular, this yields that any tangent plane Π contains at most three points of the complete cap $\mathcal{K}$ which are not on $\mathcal{Q}$, see [1, p. 7], as the tangency point of Π is contained in each conics cut out by the polar planes of other points of Π. We have $\{P, P_1, P_2\} \subset \Pi_\infty$ and the tangency point is $A_\infty = (0, 0, 0, 1) \in \mathcal{C}_P \cap \mathcal{C}_{P_1} \cap \mathcal{C}_{P_2}$, see Lemma 1. □

Now consider points in the affine space $AG(3, q) = PG(3, q) \setminus \Pi_\infty$ with $x_0 = 1$. Let $\overline{\mathcal{Q}} = \mathcal{Q} \setminus (\mathcal{K} \cap \mathcal{Q})$ be the set of points of the quadric $\mathcal{Q}$ not contained in the cap $\mathcal{K}$. Let $\mathcal{S} \subset PG(3, q)$ be a cap. We say that a point $H \notin \mathcal{S}$ *is covered by* $\mathcal{S}$, if H lies on a bisecant of $\mathcal{S}$. Interestingly, if a point $H \notin \mathcal{K} \cup \overline{\mathcal{Q}}$ is covered by $\overline{\mathcal{Q}}$ then this point is covered by $\mathcal{K}$ too. This will be shown later in Lemma 6. Condition A plays a central role in the proof of completeness of $\mathcal{K}$. Condition B relays on Condition A and in turn Conditions C, D, and E are based on B.

Hereinafter the following facts [16, 17, 25] are useful. In $PG(3, q)$, q odd, every plane Φ which is not a tangent plane meets an elliptic quadric in some conic $\mathcal{C}(\Phi)$. Every point of $\Phi \setminus \mathcal{C}(\Phi)$ is covered by any subset of $(q+5)/2$ points of $\mathcal{C}(\Phi)$.

Lemma 6 *Let $\mathcal{R} = \{(q-1)/2,\ (q+1)/2,\ (q+3)/2\}$. Under Condition 1, a point $H = (1, b, c, d) \notin \mathcal{K} \cup \overline{\mathcal{Q}}$ is covered by bisecants of the cap $\mathcal{K}$ if any of the following **sufficient conditions** holds:*

- *(A) The point H lies on a bisecant $\overline{H}_1\overline{H}_2$ through the points $\overline{H}_1, \overline{H}_2 \in \overline{\mathcal{Q}}$.*
- *(B) The point H lies on a plane Φ meeting the quadric $\mathcal{Q}$ in a conic $\mathcal{C}(\Phi)$ with $|\mathcal{C}(\Phi) \cap \mathcal{K}_1| \notin \mathcal{R}$.*
- *(C) $b \notin \mathcal{R}$.*
- *(D) For $q \geq 11,\ c - b \notin \{1, 3, 5\}$.*
- *(E) $c + b \notin \{1, q-3, q-1\}$.*

Proof (A) The point H lies on $q+1$ tangents and $q(q-1)/2$ bisecants of $\mathcal{Q}$, see Remark 1. By Construction A1, $|\mathcal{Q}\cap\mathcal{K}_1| = (q+1)+q(q-1)/2$. Hence, we prove that if Condition A holds, H lies on at least one bisecant of $\mathcal{Q}$ containing two points of $\mathcal{K}_1$.

Let t_i and s_i denote, respectively, the number of tangents and bisecants to $\mathcal{Q}$ through H containing i points of $\mathcal{K}_1$, $i \geq 0$. Obviously, $t_0+t_1 = q+1,\ s_0+s_1+s_2 = q(q-1)/2,\ t_1+s_1+2s_2 = |\mathcal{Q}\cap\mathcal{K}_1| = (q+1)+q(q-1)/2$. If $s_2 > 0$ the point H is covered by $\mathcal{K}$.

Condition A implies $s_0 > 0$. Hence, $s_1 < q(q-1)/2,\ t_1+2s_2 > q+1$, and $s_2 > 0$.

(B) As $\mathcal{C}(\Phi)\subset\mathcal{Q}\subset\mathcal{K}\cup\overline{\mathcal{Q}}$ we have $H\in\Phi\setminus\mathcal{C}(\Phi)$. Condition $|\mathcal{C}(\Phi)\cap\mathcal{K}_1|\notin\mathcal{R}$ means either $|\mathcal{C}(\Phi)\cap\mathcal{K}_1|\geq(q+5)/2$ or $|\mathcal{C}(\Phi)\cap\mathcal{K}_1|\leq(q-3)/2$.

If $|\mathcal{C}(\Phi)\cap\mathcal{K}_1|\geq(q+5)/2$ all points of $\Phi\setminus\mathcal{C}(\Phi)$ are covered by $\mathcal{C}(\Phi)\cap\mathcal{K}_1$.

If $|\mathcal{C}(\Phi)\cap\mathcal{K}_1|\leq(q-3)/2$ then $|\mathcal{C}(\Phi)\cap\overline{\mathcal{Q}}|\geq(q+5)/2$ and all points of $\Phi\setminus\mathcal{C}(\Phi)$ are covered by $\mathcal{C}(\Phi)\cap\overline{\mathcal{Q}}$. So, we may use Condition A.

(C) We consider the plane Φ_b with equation $x_1 = bx_0$. Obviously, $H\in\Phi_b$. The plane Φ_b meets $\mathcal{Q}$ in the conic $\mathcal{C}(\Phi_b) = \{(1,b,d,3b^2+d^2)|\ d\in F_q\}\cup A_\infty$ with $A_\infty = (0,0,0,1)$. By (10), $|\mathcal{C}(\Phi_b)\cap\mathcal{K}_1| = q-b+1$. If $b\notin\mathcal{R}$ then $q-b+1\notin\mathcal{R}$ and we may use Condition B.

(D) Let Φ_-^Δ be the plane with equation $x_2-x_1 = \Delta x_0$, $\Delta = c-b$. Clearly, $H\in\Phi_-^\Delta$. The plane Φ_-^Δ intersects $\mathcal{Q}$ in the conic $\mathcal{C}(\Phi_-^\Delta) = \{(1,v,d,3v^2+d^2)|v\in F_q,\ d-v=\Delta\}\cup A_\infty$.

Let $q\geq 11$. We show that $|\mathcal{C}(\Phi_-^\Delta)\cap\mathcal{K}_1|\notin\mathcal{R}$ if $\Delta\notin\{1,3,5\}$.

The relation (10) can be rewritten as follows.

$$\mathcal{K}_1 = \{(1,0,d,d^2)|d\in F_q\}\cup\{(1,v,d,3v^2+d^2)|v\in F_q^*,\ d-v\in M_v\}\cup\{A_\infty,P\},\quad (18)$$

$$M_v = \begin{cases}\{2,4,6,\ldots,q-1\}\cup\{1,3,5,\ldots,q-2v\} & \text{if } 1\leq v\leq(q-1)/2\\ \{2,4,6,\ldots,2(q-v)\} & \text{if } (q+1)/2\leq v\leq q-1\end{cases}.$$

Really, by (10), $d-v = -2i\in\{-2v,-2v-2,-2v-4,\ldots,4,2\}$. If $v=0$ then $d\in\{0\}\cup\{q-2,q-4,\ldots,1\}\cup\{q-1,q-3,\ldots,2\} = F_q$. If $1\leq v\leq(q-1)/2$ then $2v<q,\ q-2v$ is odd, and $d-v\in\{q-2v,q-2v-2,\ldots,1\}\cup\{q-1,q-3,\ldots,2\}$. Finally, if $(q+1)/2\leq v\leq q-1$ then $2v>q,\ 2q-2v$ is even, and $d-v\in\{2q-2v,2q-2v-2,\ldots,4,2\}$.

Let $q-1\geq d-v=2t\geq 2$. Then $d-v\in M_v$ if $1\leq v\leq(q-1)/2$ or $2(q-v)\geq 2t$. The last inequality gives $v\leq q-t$. As $q-1\geq 2t$, we have $(q+1)/2\leq q-t$. Therefore $d-v\in M_v$ if and only if $1\leq v\leq q-t$. Also, by (18), $(1,0,2t,4t^2)\in\mathcal{K}_1$. So, the cap $\mathcal{K}_1$ contains $q-t+1$ points $(1,v,d,3v^2+d^2)$ with $d-v=2t\geq 2$. Hence, $|\mathcal{C}(\Phi_-^{2t})\cap\mathcal{K}_1| = (q-t+1)+1$ where the last summand "+1" comes from A_∞. As $2t\leq q-1$, it follows that $|\mathcal{C}(\Phi_-^{2t})\cap\ \mathcal{K}_1|\geq(q+5)/2$.

Let $q-2\geq d-v = 2u-1$. Then $d-v\in M_v$ if and only if $1\leq v\leq(q+1)/2-u$ that implies $q-2v\geq 2u-1$. Also, $(1,0,2u-1,(2u-1)^2)\in\mathcal{K}_1$. So, $\mathcal{K}_1$ contains $(q+1)/2-u+1$ points $(1,v,d,3v^2+d^2)$ with $d-v=2u-1$. Hence, $|\mathcal{C}(\Phi_-^{2u-1})\cap\mathcal{K}_1| = (q+3)/2-u+1$ where "+1" is connected with A_∞. If $d-v\notin\{1,3,5\}$ then $2u-1\geq 7$ and $|\mathcal{C}(\Phi_-^{2u-1})\cap\mathcal{K}_1|\leq(q-3)/2$.

Finally, $d-v=0$ only for the point $(1,0,0,0)$. So, $|\mathcal{C}(\Phi_-^0)\cap\mathcal{K}_1| = 1+1 < (q-3)/2$.

We have shown that $|\mathcal{C}(\Phi_-^\Delta)\cap\mathcal{K}_1|\notin\mathcal{R}$ if $\Delta\notin\{1,3,5\}$. This allows us to use Condition B.

(E) The proof is similar to that for Condition D. Let Φ_+^δ be the plane with equation $x_2+x_1 = \delta x_0$ where $\delta = c+b$. We have $H\in\Phi_+^\delta,\ \mathcal{C}(\Phi_+^\delta) = \{(1,v,d,3v^2+d^2)|v\in F_q,\ d+v=\delta\}\cup A_\infty$. The relation (10) can be rewritten as follows.

$$\mathcal{K}_1 = \{(1,0,d,d^2)|d\in F_q\}\cup\{(1,v,d,3v^2+d^2)|v\in F_q^*,\ d+v\in T_v\}\cup\{A_\infty,P\},\quad (19)$$

$$T_v = \begin{cases} \{0\} \cup \{2v+2, 2v+4, \ldots, q-1\} \cup \{1,3,5,\ldots,q-2\} & \text{if } 1 \le v \le (q-3)/2 \\ \{0\} \cup \{2v+2-q, 2v+4-q, 2v+6-q, \ldots, q-2\} & \text{if } (q-1)/2 \le v \le q-2 \\ \{0\} & \text{if } v = q-1 \end{cases}.$$

Really, by the second row of (10), $d+v \in \{0,-2,-4,\ldots,2v+4,2v+2\}$. If $v=0$ then $d \in \{0\} \cup \{q-2, q-4, \ldots, 1\} \cup \{q-1, q-3, \ldots, 2\} = F_q$. If $1 \le v \le (q-3)/2$ then $4 \le 2v+2 \le q-1$, $2v+2$ is even, and $d+v \in \{0\} \cup \{q-2, q-4, \ldots, 1\} \cup \{q-1, q-3, \ldots, 2v+2\}$. If $(q-1)/2 \le v \le q-2$ then $q+1 \le 2v+2 \le 2q-2$, $2v+2$ is odd modulo q, and $d+v \in \{0\} \cup \{2v+2-q, 2v+4-q, \ldots, q-2\}$. Finally, if $v=q-1$ then $2v+2=0$.

Let $d+v=2t \ge 2$. Then $d+v \in T_v$ if and only if $1 \le v \le t-1$ that implies $2v+2 \le 2t$. Taking into account the points $(1,0,2t,4t^2)$ and A_∞, we have $|\mathcal{C}(\Phi_+^{2t}) \cap \mathcal{K}_1| = t+1$. By Condition E, $d+v \notin \{q-3, q-1\}$. So, $2t \le q-5$ and $|\mathcal{C}(\Phi_+^{2t}) \cap \mathcal{K}_1| \le (q-3)/2$.

Let $q-2 \ge d+v = 2u-1$. Then $d+v \in T_v$ if $1 \le v \le (q-3)/2$ or $2v+2-q \le 2u-1$. The last inequality implies $v \le (q-3)/2+u$. So, $d+v \in T_v$ if and only if $1 \le v \le (q-3)/2+u$. Due to the points $(1,0,2u-1,(2u-1)^2)$ and A_∞, we have $|\mathcal{C}(\Phi_+^{2u-1}) \cap \mathcal{K}_1| = (q+1)/2+u$. By condition E, $d+v \notin \{1\}$. So, $2u-1 \ge 3$, $u \ge 2$, and $|\mathcal{C}(\Phi_+^{2u-1}) \cap \mathcal{K}_1| \ge (q+5)/2$.

At last, $|\mathcal{C}(\Phi_+^0) \cap \mathcal{K}_1| = q+1$ as there is $d+v=0$ for every v.

We have shown that $|\mathcal{C}(\Phi_+^\delta) \cap \mathcal{K}_1| \notin \mathcal{R}$ if $\delta \notin \{1, q-3, q-1\}$. Finally, Condition B is applied. □

Lemma 7 *Under Condition 1, for $q \ge 11$ all points of the form $H=(1,b,c,d) \notin \mathcal{K} \cup \overline{\mathcal{Q}}$ are covered by bisecants of the (T_q+2)-cap $\mathcal{K}$ in Construction A1.*

Proof If either $b \notin \mathcal{R}$ or $c-b \notin \{1,3,5\}$ the point H is covered due to Sufficient Condition C or D. Let $b \in \mathcal{R}$ and $c-b \in \{1,3,5\}$. Then the pair $(b,c) \in \{((q-1)/2,(q+1)/2), ((q-1)/2,(q+5)/2), ((q-1)/2,(q+9)/2), ((q+1)/2,(q+3)/2), ((q+1)/2,(q+7)/2), ((q+1)/2,(q+11)/2), ((q+3)/2,(q+5)/2), ((q+3)/2,(q+9)/2), ((q+3)/2,(q+13)/2)\}$. So, the sum $b+c \in \{0,2,4,6,8\}$. If $q>11$ then $\{0,2,4,6,8\} \cap \{1,q-3,q-1\} = \emptyset$ and the point H is covered due to Sufficient Condition E.

Let $q=11$. Then $\{0,2,4,6,8\} \cap \{1,q-3,q-1\} = q-3 = 8$ and we should consider the points $H=(1,(q+3)/2,(q+13)/2,d)=(1,7,1,d)$. By (10), $(1,\bar{b},1,d) \in \mathcal{K}_1$ if $d=1,2,4,5,6,10$; for these values of d the point $(1,7,1,d)$ is collinear with $(1,\bar{b},1,d)$ and A_∞ since we have $(1,\bar{b},1,d)+(7-\bar{b})(0,1,0,0)=(1,7,1,d)$. On the other hand, for the remaining values of d we have $(1,7,1,0)=3(1,6,3,7)-2(1,0,4,5)$, $(1,7,1,3)=(1,3,8,3)+(1,0,5,3)$, $(1,7,1,7)=2(1,1,2,7)-(1,6,3,7)$, $(1,7,1,8)=3(1,3,5,8)-2(1,1,7,8)$, and $(1,7,1,9)=(1,4,7,9)+2(1,3,9,9)$, where, by (10), all points of the linear combinations belong to $\mathcal{K}_1$. □

Theorem 2 *For $q \ge 11$, under Condition 1, the $(q^2+q+8)/2$-cap $\mathcal{K}$ in Construction A1 is complete.*

Proof By Construction A1, all points $H \in \overline{\mathcal{Q}}$ are covered by bisecants of $\mathcal{K}$ through the point P. Points of the form $H=(0,b,c,d) \in \Pi_\infty$ and $H=(1,b,c,d) \notin \mathcal{K} \cup \overline{\mathcal{Q}}$ are covered by Lemmas 5 and 7, respectively. □

4 The associated code weight distribution and other combinatorial properties of the $(q^2+q+8)/2$-caps

For any cap K in $PG(n,q)$, w_i stands for the number of codewords with weight i in the associated code $\mathcal{A}(K)$, $d_{\min}(K)$ is the minimum distance of this code, and h_j denotes the number of hyperplanes in $PG(n,q)$ meeting K in $j \geq 0$ points. From previous works, see [15, Theorem 4.1], [10], [5, Theorem 16.1], [20], see also the references therein, we have the following result.

Theorem 3 *Let K be a k-cap in $PG(n,q)$. Then*

$$w_0 = 1, \quad w_{k-j} = (q-1)h_j \text{ for } j < k, \quad d_{\min}(K) \geq k - m_2(n-1,q), \tag{20}$$

where $h_j \geq 0$ for $m_2(n-1,q) \geq j \geq 0$ and $h_j = 0$ for $j > m_2(n-1,q)$. We have $d_{\min}(K) = k - m_2(n-1,q)$ if and only if the cap K contains an $m_2(n-1,q)$-cap.

Corollary 3 *In $PG(3,q)$, under Condition 1, the (T_q+2)-cap $\mathcal{K}$ in Construction A1 has $d_{\min}(\mathcal{K}) = T_q + 1 - q$. In the weight distribution of the code $\mathcal{A}(\mathcal{K})$ we have $w_i > 0$ for all values i in the region $T_q + 2 \geq i \geq d_{\min}(\mathcal{K})$ and $w_{d_{\min}(\mathcal{K})} \geq 3(q-1)$.*

Proof By Corollary 1, the cap $\mathcal{K}$ contains at least three conics. This shows the relations about $d_{\min}(\mathcal{K})$ and $w_{d_{\min}(\mathcal{K})}$. By the proof of Lemma 6 involving Sufficient Condition C, the plain Φ_b with $x_1 = bx_0$, $b \in F_q$, contains $q-b+1$ points of $\mathcal{K}_1$ and the same holds true for $\mathcal{K}$. So, $h_j > 0$ for $q+1 \geq j \geq 2$. By (6),(10), the plane φ with $6x_1 = 3x_0 + x_3$ is the polar plane of the point $A = (1,1,0,3) \in \mathcal{K}_1 \setminus \{P\} \subset \mathcal{Q}$. Hence φ is tangent to $\mathcal{Q}$ and to $\mathcal{K}_1$ at the tangency point A. As $\{P, P_1, P_2\} \cap \varphi = \emptyset$, the plane φ is tangent to $\mathcal{K}$ as well and $h_1 > 0$. Similarly, the polar plane ψ with equation $12x_1 = 12x_0 + x_3$ of the point $(1,2,0,12) \notin \mathcal{K}_1$ is tangent to $\mathcal{Q}$ but external to $\mathcal{K}$ as $\{P, P_1, P_2\} \cap \psi = \emptyset$. Hence $h_0 > 0$. □

By computer aided arguments valid for $q = 11, 17, 23, 29, 41$ we established some properties of stabilizer group, that is the collineation group preserving the (T_q+2)-cap. For groups we use the notation of [16, Table 2.3].

Proposition 2 *Under Condition 1, for $q = 11, 17, 23, 29, 41$ the $(q^2+q+8)/2$-cap $\mathcal{K}$ obtained by Construction A1 has the following properties:*

- *(i) the stabilizer group of the cap is the symmetric group $\mathbf{S_3}$ of order 6;*
- *(ii) the stabilizer group partitions the cap into one 1-orbit $\{(0,0,0,1)\}$, $(q+3)/2$ 3-orbits, and $(q^2-2q-3)/12$ 6-orbits;*
- *(iii) points of $\mathcal{K}$ outside $\mathcal{Q}$ form the 3-orbit $\{P, P_1, P_2\}$.*

Also, by computer aided arguments, for the (T_q+2)-cap $\mathcal{K}$ under Condition 1 we have computed the numbers h_j of planes intersecting $\mathcal{K}$ in j points, see Table 1. By (20), these values define the weight distribution of the associated code $\mathcal{A}(\mathcal{K})$. The number of conics entirely contained in $\mathcal{K}$ is equal to h_{q+1}. By Corollary 1, $h_{q+1} \geq 3$. The conics $\mathcal{C}_P$, $\mathcal{C}_{P_1}$, and $\mathcal{C}_{P_2}$ form a flat tetrahedral system [17, Lemma 18.4.4]. Each point of $\mathcal{K}_1$ lies on at most three conics. Any two conics are either of disjoint or have two common points. There is not a triple of disjoint conics.

Remark 3 For $q = 17$ in $PG(2,q)$ there is the unique complete $(q+11)/2$-arc [13, Table 2.2]. Considering the secant planes through the line PP_1 for the new $(T_{17}+2)$-cap constructed in Sect. 2 we represented this arc in the form $\{(1,0,6), (1,0,11), (1,1,4),$

Table 1 The number h_j of planes intersecting the (T_q+2)-cap $\mathcal{K} \subset PG(3,q)$ in j points

q	h_0	h_1	h_2	h_3	h_4	h_5	h_6	h_7	h_8	h_9	h_{10}	h_{11}	h_{12}
11	56	36	51	19	103	102	254	297	417	58	57	6	**8**
17	137	105	57	16	61	39	264	153	825	410	1509	855	518
23	253	210	69	7	22	12	154	57	477	222	1200	984	2214
29	406	351	84	6	4	6	39	24	189	46	756	336	1800
41	820	741	120	6	4	3	4	3	21	4	75	18	334

q	h_{13}	h_{14}	h_{15}	h_{16}	h_{17}	h_{18}	h_{19}	h_{20}	h_{21}	h_{22}	h_{23}	h_{24}	h_{25}
17	66	153	18	18	9	**7**							
23	2217	2820	525	753	93	328	24	57	6	9	3	**4**	
29	1083	3687	1981	6057	3222	2569	651	1242	171	399	18	103	12
41	93	1131	508	2556	1239	5043	3000	9951	5691	15132	7605	7185	1905

q	h_{26}	h_{27}	h_{28}	h_{29}	h_{30}	h_{31}	h_{32}	h_{33}	h_{34}	h_{35}	h_{36}	h_{37}	h_{38}
29	6	3	3	3	**3**								
41	3687	827	1737	255	681	33	159	12	33	3	7	3	3

q	h_{39}	h_{40}	h_{41}	h_{42}
41	3	3	3	**3**

(1, 1, 13), (1, 6, 8), (1, 6, 9), (1, 10, 5), (1, 10, 12), (1, 14, 3), (1, 3, 14), (0, 1, 3), (0, 1, 0), (1, 5, 1), (1, 14, 10)}. The first $(q+3)/2 = 10$ points lie on the conic $3x_1^2 + x_2^2 = 2x_0^2$ and the last four points are placed outside the conic. This is a contradiction to [22, Theorem 1] where it is claimed that for $q \equiv 1 \pmod 4$ in the plane $PG(2,q)$ a complete arc which has $(q+3)/2$ points on an irreducible conic can contain at most two points outside the conic. Note that for $q = 13$ the similar contradiction was found in [4]. Note also that for $q = 2p-1$ with odd prime p, plane arcs sharing $(q+3)/2$ points with a conic are studied in [19].

5 Construction of a complete 20-cap in $PG(3,5)$

There exist only two projectively non-equivalent complete 20-caps in $PG(3,5)$ where $20 = T_5 + 3 = m'_2(3,5)$, see [3]. These caps are studied in [2, 3] where it is noted that each is preserved by a collineation group acting sharply transitively on it. The points of the caps are listed in [3] although no geometric construction is given. We show that our construction may be slightly modified in such a way that the resulting (T_5+3)-cap is one of these two 20-caps, namely K_2 with the notation used in [2, 3]. We show that this cap contains 16 points from $\mathcal{Q}$ and that these points can be chosen by the same procedure as for our (T_q+2)-caps. However one more point off $\mathcal{Q}$ can be added to the cap $\mathcal{K}$ for $q = 5$, which actually is the unique value of q uncovered in Lemma 7.

Construction A2 Applying Construction A1 for $q = 5$ produces a (T_5+2)-set $\mathcal{K}$; a variant of Construction A1, under Condition 1, produces a (T_5+3)-set $\mathcal{K}^* = \mathcal{K} \cup \{P_3\}$ joining $\mathcal{K}$ with the further point $P_3 = (1, 3, 1, 0)$.

Theorem 4 *The (T_5+3)-set $\mathcal{K}^*$ in Construction A2 is a complete 20-cap in $PG(3,5)$ projectively equivalent to the 20-cap K_2 given in [3, Sect. 3].*

Table 2 The known sizes of the complete k-caps in $PG(3,q)$, $T_q = (q^2+q+4)/2$

q	$t_2(3,q)$	The known sizes k of the complete k-caps with $t_2(3,q) \le k \le m'_2(3,q)$	$m'_2(3,q)$	$m_2(3,q)$	References
3	8	$8 = T_q$	8	10	[12, 13, 25]
4	10	$10, 12, 13, 14 = T_q + 2$	14	17	[12, 13, 25]
5	12	$12 \le k \le 18$ and $k = 20 = T_q + 3$	20	26	[2, 3, 12, 13]
7	17	$17 \le k \le 30$ and $k = 32 = T_q + 2$	32	50	[6, 8, 11, 13, 17, 23–25]
8	≥ 14	$20 \le k \le 41 = T_q + 3$	≤ 60	65	[8, 12, 13, 17]
9	≥ 15	$24 \le k \le 48 = T_q + 1$	≤ 78	82	[8, 12, 13, 23–25]
11	≥ 18	$30 \le k \le \mathbf{70} = T_q + 2$	≤ 116	122	⋆, [8, 12, 13, 17, 23–25]
13	≥ 21	$\mathbf{36} \le k \le 94 = T_q + 1$	≤ 162	170	⋆, [8, 12, 13, 23, 24]
16	≥ 25	$41 \le k \le 138 = T_q$	≤ 242	257	[8, 12–14, 17, 18]
17	≥ 26	$51 \le k \le \mathbf{157} = T_q + 2$	≤ 278	290	⋆, [8, 12, 13, 23–25]
19	≥ 29	$\mathbf{58} \le k \le 192 = T_q$	≤ 348	362	⋆, [8, 12, 13, 17, 23–25]
23	≥ 35	$\mathbf{72} \le k \le \mathbf{280} = T_q + 2$	≤ 512	530	⋆, [8, 12, 13, 17, 21, 23–25]

Proof Construction A1 and Theorem 1 provide a 19-cap $\mathcal{K}$. Since it is known that $m'_2(3,5) = 20$ and there are no complete 19-caps in $PG(3,5)$, see [3, 12, 13, Table 3.1], the cap $\mathcal{K}$ is incomplete and we can add to it just one point. Actually such a point must be P_3; this can be shown by direct calculation. The two planes of equations $x_3 = \pm 2x_0$ are 5-secant planes for the cap $\mathcal{K}^*$. By [3, Tables 1,2], the cap K_2 is the unique 20-cap in $PG(3,5)$ provided with 5-secant planes. □

By Theorem 3 and [3, Table 2], the weight distribution of the code $\mathcal{A}(K_2)$ is as follows: $w_0 = 1,\ w_{14} = 80,\ w_{15} = 240,\ w_{16} = 140,\ w_{18} = 120,\ w_{20} = 44$. This code is *optimal* as a 5-ary linear code with length 20 and dimension 4 has the minimal distance at most 14 [7].

6 On the spectrum of complete caps sizes in $PG(3,q)$

The known sizes of complete caps in $PG(3,q)$ are given in Table 2. We take the sizes of complete caps from the papers [3, 6, 8, 11–13, 17, 21, 23–25], see also the references therein. New entries obtained from our Theorem 2 and by a computer search are marked by asterisk ⋆.

The following *new lower bounds* come from Construction A1 and Theorem 2:

$$70 = T_{11} + 2 \le m'_2(3,11),\ 157 = T_{17} + 2 \le m'_2(3,17),\ 280 = T_{23} + 2 \le m'_2(3,23).$$

New upper bounds $t_2(3,13) \le 36,\ t_2(3,19) \le 58,\ t_2(3,23) \le 72$, are obtained in this paper by the randomized greedy algorithms [8, Sect. 2]. As the starter set for $q = 19$ we have used the first 20 points of the 40-cap of [26, Sect. 4.2].

For $7 \le q \le 19$ the sizes and bounds already known from the literature are collected in [8, Table 1]. The results $t_2(3,7) = 17$ [6] and $m'_2(3,16) \le 242$ [14] are recent. In this paper we have obtained the following *new sizes* of the complete k-caps in $PG(3,q)$: $k = 90, 91, 92$ for $q = 13$, $k = 154$ for $q = 17$, and $k = 188, 190, 191$ for $q = 19$. In $PG(3,23)$ the sizes $k = 73$ [8, Table 3], $k = 120$-122 [24], $k = 186$ [13, p. 90], $k = 246$ [24], $k = 268$

[21], and $k = 278$ [25], have already been known. The remaining sizes for $q = 23$ given in Table 2 are *new* entries. The bounds on $t_2(3, 23)$ and $m'_2(3, 23)$ come from [12, Lemma 2.5] and [17, Theorem 18.4.1].

For $q = 13, 17, 19, 23$, new sizes have been obtained using the greedy algorithms. To get starter caps for the algorithms, we have remove some points from the T_q-cap $\mathcal{K}_1$ in Construction A1.

For $q = 3, 4, 5$, the spectrum of complete caps sizes is complete, see [3, 12, 13, Table 3.1].

Open problem To complete the spectrum of sizes of complete caps in $PG(3, 7)$, it only remains to solve the problem: "Is there a complete 31-cap in $PG(3, 7)$?"

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