On the spectrum of possible parameters of symmetric configurations

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Abstract

The spectrum of possible parameters of symmetric configurations [11] is considered. New construction methods for configurations are proposed. Many new parameters are obtained, in particular for cyclic symmetric configurations equivalent to deficient cyclic difference sets. Golomb rulers and modular Golomb rulers [15] are used. The minimum integer $E(k)$ such that for each $v \geq E(k)$ there exists a symmetric v_k-configuration is investigated, and several new upper bounds on $E(k)$ are provided. The results obtained extend the range of possible parameters of LDPC codes, generalized LDPC codes, and quasi-cyclic LDPC codes.

I. Introduction

Throughout the paper, q is a prime power and p is a prime.

Definition 1: [11] *i)* A (v_r, b_k)-configuration is an incidence structure of v points and b lines such that each line contains k points, each point lies on r lines, and two distinct points are connected by *at most* one line.

ii) If $v = b$ and, hence, $r = k$, the configuration is *symmetric*, and it is referred to as a v_k-configuration.

iii) A configuration is called *cyclic* if one of its incidence matrix is circulant.

iv) The *deficiency* d of a (v_r, b_k)-configuration is the value $d = v - r(k-1) - 1$.

The deficiency of a symmetric v_k-configuration is equal to $v - (k^2 - k + 1)$.

Any incidence matrix $M(v,k)$ of a symmetric v_k-configuration is a $v \times v$ binary 01-matrix with k units in every row and column; also, the 2×2 matrix J_4 consisting of all units is not a submatrix of $M(v,k)$. Two incidence matrices of the same configuration may differ by a permutation on the rows and the columns. Any matrix $M(v,k)$ can also be considered as an adjacency matrix of a k-regular bipartite graph of girth at least 6, i.e. without 4-cycles. Such graphs are useful for the construction of bipartite-graph codes that can be treated as low-density parity-check (LDPC) codes or generalized LDPC codes [2] - [5], [14]. If $M(v,k)$ is circulant, then the corresponding code is quasi-cyclic; it can be encoded with the help of shift-registers with relatively small complexity, see [4], [5] and the references therein.

Definition 2: [15] *i)* A *Golomb ruler* GR_k of *order* k is an ordered set of k integers $(a_1, a_2, \ldots, a_k)$ such that $0 = a_1 < a_2 < \ldots < a_k$ and all the differences $\{a_i - a_j \,|\, 1 \le j < i \le k\}$ are distinct. The *length* $L_{\mathrm{GR}}(k)$ of GR_k is equal to $a_k - a_1$.

ii) A ruler GR_k is an *optimal Golomb ruler* OGR_k if no shorter Golomb ruler of the same order k exists. Let $L_{\mathrm{OGR}}(k)$ and $L_{\overline{\mathrm{GR}}}(k)$, respectively, be the length of an optimal ruler OGR_k and of the *shortest known* ruler $\overline{\mathrm{GR}}_k$.

iii) A (v, k) *modular Golomb ruler* is an ordered set of k integers $(a_1, a_2, \ldots, a_k)$ such that $0 = a_1 < a_2 < \ldots < a_k$ and all the differences $\{a_i - a_j \,|\, 1 \le i \ne j \le k\}$ are distinct and nonzero modulo v.

iv) A (v, k) modular Golomb ruler is called also a *deficient cyclic difference set* with deficiency $d = v - (k^2 - k + 1)$.

Clearly, $L_{\overline{\mathrm{GR}}}(k) \ge L_{\mathrm{OGR}}(k)$. For $k \le 25$, it is known that $L_{\overline{\mathrm{GR}}}(k) = L_{\mathrm{OGR}}(k)$. By [6],

$$L_{\mathrm{OGR}}(k) > k^2 - 2k\sqrt{k} + \sqrt{k} - 2; \;\; L_{\mathrm{OGR}}(k) < k^2 \text{ for } k < 65000. \tag{1}$$

Theorem 3: [10] Any Golomb ruler GR_k of length $L_{\mathrm{GR}}(k)$ is a (v, k) modular Golomb ruler for all v such that $v \ge 2L_{\mathrm{GR}}(k) + 1$.

Theorem 4: [10] A circulant $v \times v$ matrix of weight k is an incidence matrix $M(v, k)$ of a cyclic symmetric v_k-configuration if and only if the first row of the matrix corresponds to a (v, k) modular Golomb ruler.

Corollary 5: For any $v \ge 2L_{\overline{\mathrm{GR}}}(k) + 1$, there exists a cyclic symmetric v_k-configuration.

We call the value $G(k) = 2L_{\overline{\mathrm{GR}}}(k) + 1$ the *Golomb bound*. Also, we call the value $P(k) = k^2 - k + 1$ the *projective plane bound*. It is known that $v_k \ge k^2 - k + 1$ and equality is possible if and only if there exists a projective plane of the order $k - 1$. Finally, we introduce an *existence bound* $E(k)$ such that for any $v \ge E(k)$, there exists a symmetric v_k-configuration. The corresponding bound $E_c(k)$ for cyclic symmetric v_k-configurations is called the *cyclic existence bound*. Clearly, $E(k) \le E_c(k) \le G(k)$.

The goal of this work is the investigation of the spectrum of possible parameters of symmetric v_k-configurations in the region $P(k) \le v < G(k)$. We both collect known parameters and obtain new ones. In particular, we improved the known upper bounds on $E(k)$ and $E_c(k)$ and obtained many new parameters for cyclic configurations and, hence, for deficient cyclic difference sets.

II. The known parameters of v_k-configurations with $P(k) \le v < G(k)$

Theorem 6: If a (cyclic) v_k-configuration exists, then for each δ with $0 \le \delta < k$ there exists a family of (cyclic) $v_{k-\delta}$-configurations as well [13].

The following infinite families of v_k-configurations are considered in [1, Constructions (i),(ii), Conjecture 4.4, Remark 4.5, Example 4.6], [2], [4], [5], [7], [8], [9, Constructions 3.2, 3.3, 3.7, Remark 3.5, Theorem 3.8], [10] - [13], [15]:

$$\begin{aligned}
v &= q^2 + q + 1, & k &= q + 1 - \delta, & & q + 1 > \delta \ge 0;\\
v &= q^2 - 1, & k &= q - \delta, & & q > \delta \ge 0;\\
v &= p^2 - p, & k &= p - 1 - \delta, & & p - 1 > \delta \ge 0;\\
v &= q^2 - qs, & k &= q - s - \delta, & & q > s \ge 0,\; q - s > \delta \ge 0;\\
v &= q^2 - (q-1)s - 1, & k &= q - s - \delta, & & q > s \ge 0,\; q - s > \delta \ge 0;\\
v &= c(q^2 + q + 1), & k &= q + c - \delta, & & c = 2, 3, \ldots, q^2 - q,\; q + c > \delta \ge 0;\\
v &= 2p^2, & k &= p + s - \delta, & & 0 < s \le q + 1,\, q^2 + q + 1 \le p,\; p + s > \delta \ge 0.
\end{aligned} \tag{2}$$

The first three families of (2) consist of cyclic configurations.

In [2], the rows corresponding to q parallel lines are dismissed from an incidence matrix of the affine plane $AG(2,q)$. The resulting $q^2 \times q^2$ matrix M is represented as a superposition of permutation matrices of the order q. Also, in [2] the construction methods q-cancellation, Δ-cancellation, and θ-extension are proposed and applied to M. As a result, a family of non-cyclic v_k-configurations is obtained with the following parameters:

$$v = q^2 - qs + \theta,\ k = q - s - \Delta,\ q > s \geq 0,\ q - s > \Delta \geq 0,\ \theta = 0, 1, \ldots, q - s + 1. \quad (3)$$

III. Construction methods for symmetric configurations with new parameters

A. Using permutations of the set of integers $\{0, 1, \ldots, v-1\}$

Throughout this section, $(a_1, a_2, \ldots, a_k)$ is a (v,k) modular Golomb ruler, s is a divisor of v, and $t = v/s$. Also, for every $h = 0, 1, \ldots, s-1$, let

$$A_h = \{a_i \mid a_i = h \pmod{s}\},\ w_h = |A_h|,\ B_h = \{\frac{a_i - h}{s} \mid a_i \in A_h\}. \quad (4)$$

Theorem 7: For every $h = 0, \ldots, s-1$, B_h is a (t, w_h) modular Golomb ruler.

Let $\mathcal{B}_0 = (a_1, a_2, \ldots, a_k)$ and let $\mathcal{B}_i = \{a_1 + i \pmod{v}, \ldots, a_k + i \pmod{v}\}$, $i = 1, \ldots, v-1$. Let σ be a permutation of the set $\{0, 1, \ldots, v-1\}$. Consider the following $v \times v$ binary matrix A_σ: the element in position (i,j) is 1 if and only if $\sigma(j) - \sigma(i) \pmod{v} \in \mathcal{B}_0$.

Lemma 8: For every choice of σ, the matrix A_σ is a J_4-free incidence matrix $M(v,k)$ of a symmetric v_k-configuration.

Theorem 9: Let $v = st$ and let σ_s be the permutation of $\{0, 1, \ldots, v-1\}$ such that for an element $i = at + b$ with $0 \leq a \leq s-1$, $0 \leq b \leq t-1$, the image of i by σ_s is $bs + a$. Then

$$A_{\sigma_s} = \begin{bmatrix} M_0 & M_1 & \ldots & M_{s-3} & M_{s-2} & M_{s-1} \\ T_{s-1} & M_0 & \ldots & M_{s-4} & M_{s-3} & M_{s-2} \\ T_{s-2} & T_{s-1} & \ldots & M_{s-5} & M_{s-4} & M_{s-3} \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ T_2 & T_3 & \ldots & T_{s-1} & M_0 & M_1 \\ T_1 & T_2 & \ldots & T_{s-2} & T_{s-1} & M_0 \end{bmatrix} \quad (5)$$

is an incidence matrix of a v_k-configuration. Here, $M_0, \ldots, M_{s-1}$ and $T_1, \ldots, T_{s-1}$ are binary circulant $t \times t$ matrices such that both the weights of M_h and T_h are equal to w_h. The 1st row of the matrix M_h corresponds to the (t, w_h) Golomb ruler B_h of (4). The matrix T_h is obtained from M_h by a cyclic shift of rows to the right by one position. In addition, $\sum_{h=0}^{s-1} w_h = k$.

Let $W(A_{\sigma_s}) = (w_0, w_1, \ldots, w_{s-1})$ be a *weight vector* of the matrix A_{σ_s} of (5).

Remark 10:

i) Truncating δ_h units in the 1st row of every circulant matrix M_h and T_h of (5), results in an incidence matrix A'_{σ_s} of a $v_{k'}$-configuration with

$$k' = k - \sum_{h=0}^{s-1} \delta_h,\ w_h \geq \delta_h \geq 0,\ W(A'_{\sigma_s}) = (w'_0, w'_1, \ldots, w'_{s-1}),\ w'_h = w_h - \delta_h. \quad (6)$$

ii) In (6), put $w'_1 = w'_2 = \ldots = w'_{s-1} = w$. Then v_k-configurations with the following parameters exist:

$$v = ct,\ k = w'_0 - d + (c-i)w,\ c = 1, 2, \ldots, s,\ i = 1, 2, \ldots, c,\ w'_0 \geq d \geq 0.$$

iii) Assume that s is even. In (6), put $w'_1 = w'_3 = \ldots = w'_{s-1} = w_{\text{od}}$, $w'_2 = w'_4 = \ldots = w'_{s-2} = w_{\text{ev}}$. Then v_k-configurations with the following parameters exist:

$$v = 2ct,\ k = w'_0 - d + cw_{\text{od}} + (c-1)w_{\text{ev}},\ c = 1, 2, \ldots, \frac{s}{2},\ w'_0 \geq d \geq 0.$$

Theorem 11: Let $v = st$, $s \geq k$, $t \geq k-1$. Assume that in (5) $w_h \in \{0, 1\}$ for all h. Then the matrix A_{σ_s} of (5) admits a θ-extension (see [2]) for each $\theta = 1, 2, \ldots, s+1$. This results in $(v+i)_k$-configurations for each $i = 1, 2, \ldots, s+1$.

Example 12: i) The cyclic $(q^4 + q^2 + 1)_{q^2}$-configuration based on the projective plane $PG(2, q^2)$ [4] - [8], [15] has an incidence matrix A'_{σ_s} with

$$s = q^2 - q + 1,\ t = q^2 + q + 1,\ w'_0 = q + 1 - \delta_0 \geq 0,\ w'_1 = \ldots = w'_{q^2-q} = 1.$$

ii) The Ruzsa's cyclic $(p^2 - p)_{p-1}$-configuration [6], [15] has incidence matrices A_{σ_s} with parameters

$$s = p, \quad t = p - 1, \quad w_0 = 0,\ w_1 = w_2 = \ldots = w_{p-1} = 1.$$
$$s = p - 1, \quad t = p, \quad w_0 = w_1 = w_2 = \ldots = w_{p-2} = 1.$$

iii) The cyclic $(q^2 - 1)_q$-configuration based on the affine plane $AG(2, q)$ [5], [6], [15] has an incidence matrix A_{σ_s} with parameters

$$s = q + 1, \quad t = q - 1, \quad w_0 = 0,\ w_1 = w_2 = \ldots = w_q = 1. \tag{7}$$

If q is an odd square, then the following parameters can be obtained:

$$s = \sqrt{q} + 1, \quad t = (\sqrt{q} - 1)(q + 1), \quad w_0 = 1,\ w_1 = w_3 = \ldots = w_{s-1} = \sqrt{q} - 1,$$
$$w_2 = w_4 = \ldots = w_{s-2} = \sqrt{q} + 1.$$

Applying Remark 10ii and Theorem 11 to (7), we obtain parameters

$$v = q^2 - 1 - s(q-1) + \theta,\ k = q - s - \Delta,\ q > s \geq 0,$$
$$q - s > \Delta \geq 0,\ \theta = 0, 1, \ldots, q - s + 2. \tag{8}$$

B. Extension, reduction and truncating of modular Golomb rulers

A (v, k) modular Golomb ruler, in principle, can be also a $(v \pm \Delta, k)$ modular Golomb ruler with $\Delta > 0$ [10]. This possibility depends on the concrete ruler in object, and can be checked by computer. The following theorem can be useful for obtaining more starting concrete Golomb rulers.

Theorem 13: [15] If $(a_1, a_2, \ldots, a_k)$ is a (v, k) modular Golomb ruler and m and b are integers with $\gcd(m, v) = 1$ then $(ma_1 + b, ma_2 + b, \ldots, ma_k + b)$ is also a (v, k) modular Golomb ruler.

Also, by truncating δ integers from a (v, k) modular Golomb ruler we obtain a $(v, k - \delta)$ modular Golomb ruler. Then one can try to change the module v.

IV. The Spectrum of Parameters of Symmetric Configurations

Using the results of Sections II and III and taking modular Golomb rulers of [16] as the starting sets for computer search, we obtain Tables I and II where the known parameters of symmetric v_k-configurations of [1], [2], [4], [5], [7] - [13], [15], [16] and our new results are reported. New upper bounds on $E(k)$ and $E_c(k)$ are listed. The entry $\overline{v}$ means that the corresponding v_k-configuration does not exist, see [8], [10], [11]. The theorem of Bruck-Ryser-Chowla is used to prove the nonexistence of some configurations connected with planes and biplanes. These configurations have deficiency $d = 0, 1$. Also, $\overline{v^c}$ notes the nonexistence of a cyclic v_k-configuration, see [7], [12].

TABLE I
THE UPDATED TABLE OF THE KNOWN PARAMETERS OF SYMMETRIC v_k-CONFIGURATIONS

k	$P(k)$	$P(k) \leq v < G(k)$	$G(k)$	$E(k) \leq$
7	43	$\overline{43}$, $\overline{44}$, 45, 48-50	51	48
8	57	57, $\overline{58}$, 63-68	69	63
9	73	73, $\overline{74}$, 78, 80-88	89	80
10	91	91, $\overline{92}$, 98, 107-110	111	107
11	111	$\overline{111}$, $\overline{112}$, 120-133, 135, 137, 139, 142-144	145	142
12	133	133, $\overline{134}$, 135, 156-170	171	156
13	157	$\overline{158}$, 168-183, 189, 197, 201, 203-212	213	203
14	183	183, $\overline{184}$, 210, 224-254	255	224
15	211	$\overline{211}$, $\overline{212}$, 231, 240-302	303	240
16	241	252, 255-354	355	255
17	273	273, $\overline{274}$, 288-307, 323-398	399	323
18	307	307, 342-381, 401, 403, 405-407, 410, 412-432	433	412
19	343	$\overline{344}$, 360-381, 434, 437-492	493	437
20	381	381, $\overline{382}$, 460-566	567	460
21	421	$\overline{422}$, 483-666	667	483
22	463	$\overline{463}$, $\overline{464}$, 506-712	713	506
23	507	$\overline{507}$, $\overline{508}$, 528-553, 558, 575-744	745	575
24	553	553, $\overline{554}$, 589, 600-850	851	600
25	601	620, 624-651, 675-960	961	675
26	651	651, $\overline{652}$, 702-984	985	702
27	703	728-757, 783-1106	1107	783
28	757	757, $\overline{758}$, 812-1170	1171	812
29	813	$\overline{814}$, 840-871, 899-1057, 1073-1246	1247	1073
30	871	871, $\overline{872}$, 930-1057, 1110-1360	1361	1110
31	931	$\overline{931}$, $\overline{932}$, 960-1057, 1147-1494	1495	1147
32	993	993, $\overline{994}$, 1023-1057, 1184-1568	1569	1184
33	1057	1057, $\overline{1058}$, 1221-1718	1719	1221

REFERENCES

[1] M. Abreu, M. Funk, D. Labbate, and V. Napolitano, "On (minimal) regular graphs of girth 6," *Australas. J. Combin.* vol. 35, pp. 119–132, 2006.

[2] V. B. Afanassiev, A. A. Davydov, and V. V. Zyablov, "Low density concatenated codes with Reed-Solomon component codes," in *Proc. XI Int. Symp. on Problems of Redundancy in Inf. and Control Syst.*, St.-Petersburg, Russia, Jul. 2007, pp. 47–51, [Online]. Available: http://k36.org/redundancy2007

[3] A. Barg and G. Zémor, "Distances properties of expander codes," *IEEE Trans. Inf. Theory*, vol. 52, pp. 78–90, 2006.

[4] A. A. Davydov, M. Giulietti, S. Marcugini, and F. Pambianco, "Symmetric configurations for bipartite-graph codes," in *Proc. XI Int. Workshop Algebraic Comb. Coding Theory*, ACCT2008, Pamporovo, Bulgaria, Jun. 2008, pp. 63–69, [Online]. Available: http://www.moi.math.bas.bg/acct2008/b11.pdf

[5] A. A. Davydov, M. Giulietti, S. Marcugini, and F. Pambianco, Some Combinatorial Aspects of Constructing Bipartite-Graph Codes, submitted.

[6] A. Dimitromanolakis, "Analysis of the Golomb Ruler and the Sidon Set Problems, and Determination of Large, Near-Optimal Golomb Rulers," Depart. Electronic Comput. Eng. Techn. University of Crete, 2002, [Online]. Available: http://www.cs.toronto.edu/∼apostol/golomb/main.pdf

[7] M. Funk, "Cyclic Difference Sets of Positive Deficiency," *Bull. Inst. Combin. Appl.*, vol. 53, pp. 47–56, 2008.

[8] M. Funk, D. Labbate, V. Napolitano, "Tactical (de-)compositions of symmetric configurations," *Discr. Math.*, vol. 309, pp. 741–747, 2009.

[9] A. Gács and T. Héger, "On geometric constructions of (k, g)-graphs," *Contrib. Discr. Math.*, vol. 3, pp. 63–80, 2008.

[10] H. Gropp, "On the existence and non-existence of configurations n_k," *J. Combin. Inform. System Sci.*, vol. 15, pp. 34–48, 1990.

[11] H. Gropp, "Configurations", in *The CRC Handbook of Combinatorial Designs*, 2-nd edition, C. J. Colbourn and J. Dinitz, Editors, Chapter VI.7, 2006, pp. 353–355.

TABLE II
THE UPDATED TABLE OF THE KNOWN PARAMETERS OF CYCLIC SYMMETRIC v_k-CONFIGURATIONS

k	$P(k)$	$P(k) \leq v < G(k)$	$G(k)$	$E_c(k) \leq$
7	43	$\overline{43}, \overline{44}, \overline{45^c}$-$\overline{47^c}$, 48-50	51	48
8	57	57, $\overline{58}, \overline{59^c}$-$\overline{62^c}$, 63-68	69	63
9	73	73, $\overline{74}, \overline{75^c}$-$\overline{79^c}$, 80, $\overline{81^c}$-$\overline{84^c}$, 85-88	89	85
10	91	91, $\overline{92}$, 107-110	111	107
11	111	$\overline{111}, \overline{112}$, 120, 133, 137, 139, 142-144	145	142
12	133	133, $\overline{134}$, 156, 158, 162-165, 167-170	171	167
13	157	$\overline{158}$, 168, 183, 197, 201, 203-212	213	203
14	183	183, $\overline{184}$, 226, 227, 231, 233-254	255	233
15	211	$\overline{211}, \overline{212}$, 255, 272, 273, 278, 282, 284, 286-288, 290-302	303	290
16	241	255, 272, 273, 288, 307, 318, 320-329, 331-354	355	331
17	273	273, $\overline{274}$, 288, 307, 342, 343, 353, 357-363, 365-398	399	365
18	307	307, 342, 360, 381, 401, 403, 405-407, 410, 412-418, 420-432	433	420
19	343	$\overline{344}$, 360, 381, 455, 457, 464, 467, 468, 470-477, 479, 481-492	493	481
20	381	381, $\overline{382}$, 503, 506, 508, 513, 516, 519, 520, 525, 527-530, 532, 534-566	567	534
21	421	$\overline{422}$, 506, 528, 553, 592, 597, 601, 602, 606-609, 611, 614-666	667	614
22	463	$\overline{463}, \overline{464}$, 506, 528, 553, 624, 640, 644, 645, 649-712	713	649
23	507	$\overline{507}, \overline{508}$, 528, 553, 624, 651, 683, 696, 698, 699, 702, 707, 709-711, 713-744	745	713
24	553	553, $\overline{554}$, 624, 651, 728, 739, 757, 759, 761, 763, 765-770, 772, 775-850	851	775
25	601	624, 651, 728, 757, 812, 837, 840, 842-844, 846-854, 856-863, 865-960	961	865
26	651	651, $\overline{652}$, 728, 757, 812, 840, 871, 900, 905-907, 910, 912, 913, 916, 917, 919-921, 924, 925, 929, 930, 932-941, 943-984	985	943
27	703	728, 757, 812, 840, 871, 930, 960, 971, 975, 977, 978, 987, 991, 993, 994, 997, 1000, 1001, 1003-1006, 1008, 1010-1015, 1017, 1019, 1021-1106	1107	1021
28	757	757, $\overline{758}$, 812, 840, 871, 930, 960, 993, 1023, 1045, 1057, 1063, 1067, 1070, 1074, 1075, 1077, 1079-1082, 1085-1170	1171	1085

[12] M. J. Lipman, “The Existence of small tactical configurations,” in: *Graphs and Combinatorics,* Springer Lecture Notes in Mathematics, vol. 406, Springer-Verlag, 1974, pp. 319–324.

[13] N. S. Mendelsohn, R. Padmanabhan, B. Wolk, “Planar projective configurations I,” *Note di Matematica,* vol. 7, pp. 91–112, 1987, [Online]. Available: http://siba2.unile.it/ese/issues/1/13/Notematv7n1p91.pdf

[14] N. Miladinović and M. P. C. Fossorier, “Generalized LDPC codes and generalized stopping sets,” *IEEE Trans. Commun.*, vol. 56, pp. 201–212, 2008.

[15] J. Shearer, “Difference Triangle Sets”, in *The CRC Handbook of Combinatorial Designs,* 2-nd edition, C. J. Colbourn and J. Dinitz, Editors, Chapter VI.19, 2006, pp. 436–440.

[16] J. Shearer, Golomb ruler tables. [Online]. Available: http://www.research.ibm.com/people/s/shearer/grtab.html