Linear Covering Codes of Radius 2 and 3

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Abstract. Infinite families of linear codes with covering radius \( R = 2 \) and \( R = 3 \) and codimension \( r = 2t + 1 \) and \( r = 3t + 1, 3t + 2 \), respectively, are constructed on the base of \( q^m \)-concatenating constructions with starting codes of codimension \( r = 3 \) and \( r = 4, 5 \). Parity check matrices of the starting codes are treated as saturating sets in the projective spaces \( PG(v, q) \), \( v = 2, 3, 4 \). The sets are obtained by a theoretical way and by computer search using geometrical properties of objects. A new concept of \( N \)-fold strong blocking set in the projective spaces is introduced. New upper bounds on the length function and on the smallest sizes of saturating sets in the projective spaces are given. The asymptotic covering densities of the new code families are bounded from above by constants independent of \( q \).

1 Introduction

Let \( F_q \) be the Galois field of \( q \) elements. Let \( F_q^n \) be the \( n \)-dimensional vector space over \( F_q \). Denote by \( [n, n-r]_q \) a \( q \)-ary linear code of length \( n \) and codimension \( r \). The covering radius of an \( [n, n-r]_q \) code is the least integer \( R \) such that \( F_q^n \) is covered by spheres of radius \( R \) centered on codewords [1]. Equivalently, an \( [n, n-r]_q \) code has covering radius \( R \) if every column of \( F_q^n \) is equal to a linear combination of \( R \) columns of a parity check matrix of the code, and \( R \) is the smallest value with such property. An \( [n, n-r]_q R \) code is an \( [n, n-r]_q \) code of covering radius \( R \). An \( [n, n-r, d]_q R \) code is an \( [n, n-r]_q R \) code of minimum distance \( d \).

Linear covering codes are deeply connected with saturating sets in the projective spaces over finite fields. Let \( PG(v, q) \) be the \( v \)-dimensional projective space over \( F_q \). We say that a set of points \( S \subseteq PG(v, q) \) is \( \rho \)-saturating if for any point \( x \in PG(v, q) \) there exist \( \rho + 1 \) points in \( S \) generating a subspace of \( PG(v, q) \) in which \( x \) lies and \( \rho \) is the smallest value with such property, cf. [2]-[5].

Points of an \( (R-1) \)-saturating \( n \)-set in the projective space \( PG(r-1, q) \), given by an \( r \)-positional vector \( (x_0, x_1, \ldots, x_{r-1}) \) with \( x_i \in F_q \), can be treated as columns of a parity-check matrix of an \( [n, n-r]_q R \) related covering code [3]-[6]. In the literature the saturating sets are called also “saturated sets” [3],[6], “spanning sets”, and “dense sets” [7]-[9].

The covering quality of an \( [n, n-r(C)]_q R \) code \( C \) can be measured by its covering density

\[
\mu_q(n, R, C) = q^{-r(C)} \sum_{i=0}^{R} (q-1)^i \binom{n}{i} \geq 1.
\]
The length function \( \ell_q(r, R) \) is the smallest length of a \( q \)-ary linear code with codimension \( r \) and covering radius \( R \). The smallest known length of such code is denoted by \( \tilde{\ell}_q(r, R) \). Evidently, \( \ell_q(r, R) \leq \tilde{\ell}_q(r, R) \). Existence of an \([n, n-r]_qR\) code or an \((R-1)\)-saturating \( n \)-set in \( PG(r-1, q) \) implies upper bounds \( \ell_q(r, R) \leq \tilde{\ell}_q(r, R) \leq n \).

For given integers \( R, \eta \) with \( R > 0 \), \( 0 \leq \eta \leq R - 1 \), and for a fixed prime power \( q \), let \( A_{R,q}^{(\eta)} = (C_n) \) denote an infinite sequence of \( q \)-ary linear \([n_i, n_i-r_i]_qR\) codes \( C_n \) with covering radius \( R \) and such that the following two properties hold:

(a) the codimension \( r_i = Rt_i + \eta \), where \( (t_i) \) is an increasing sequence of integers;

(b) the length \( n_i \) of \( C_i \) coincides with \( f_q^{(\eta)}(r_i) \), where \( f_q^{(\eta)} \) is an increasing function.

We call \( A_{R,q}^{(\eta)} \) an infinite family of covering codes. Considering families of type \( A_{R,q}^{(\eta)} \) is a standard method of investigation of linear covering codes \([1],[3]-[6],[10]-[12]\). Through the paper, distinct families \( A_{R,q}^{(\eta)} \) with the same values of \( R, \eta \) will be denoted as \( A_{R,q,i}^{(\eta)} \), \( i = 1, 2, \ldots \).

For an infinite code family \( A_{R,q}^{(\eta)} \), its asymptotic covering density is defined as follows:

\[ \overline{\mu}_q(R, A_{R,q}^{(\eta)}) = \liminf_{i \to \infty} \mu_q(n_i, R, C_n). \]

We use the notations \( \overline{\mu}_q(R) \) if the family \( A_{R,q}^{(\eta)} \) is clear by context.

In this work we concentrate on infinite code families \( A_{2,q}^{(1)}, A_{3,q}^{(1)}, \) and \( A_{3,q}^{(2)} \).

We use two basic tools: the \( q^m \)-concatenating constructions of covering codes and the connection of covering codes and saturating sets in projective spaces.

The \( q^m \)-concatenating constructions, see \([1],[3]-[6],[10]-[12]\), and the references therein, are the fundamental instrument for obtaining infinite families of covering codes with a fixed radius. Using a starting code as a “seed”, the \( q^m \)-concatenating constructions yield an infinite family of new codes with the same covering radius and with almost the same covering density.

Small saturating sets in the spaces of little dimension are highly convenient to take them as the starting codes for the \( q^m \)-concatenating constructions \([3]-[6],[12]\).

In this work new constructions of small \( 1 \)-saturating sets in planes \( PG(2, q) \) and \( 2 \)-saturating sets in spaces \( PG(3, q) \), \( PG(4, q) \) are proposed. For it a new concept of \( N \)-fold strong blocking set in projective spaces is introduced. Also, many small saturating sets are obtained by computer. Using the saturating sets as the starting points of the \( q^m \)-concatenating constructions, we obtained new infinite code families \( A_{2,q}^{(1)}, A_{3,q}^{(1)}, \) and \( A_{3,q}^{(2)} \). Their asymptotic covering density is smaller than that of the known families. Moreover, the densities \( \overline{\mu}_q(R, A_{R,q}^{(\eta)}) \) of the new families are bounded from above by constants independent of \( q \). As results, many new upper bounds on the length function are obtained. They are also new upper bounds on the smallest possible sizes of saturating sets in projective spaces.

2 Small Saturating Sets in Spaces \( PG(v, q) \), \( v = 2, 3, 4 \)

Definition 2.1. A subset \( B \) of a projective space \( PG(N, q) \) is an \( N \)-fold strong blocking set if every hyperplane of \( PG(N, q) \) is spanned by \( N \) points in \( B \).
Theorem 2.2. Let \( q = (q')^{\rho+1} \). Any \( (\rho+1) \)-fold strong blocking set in a subspace \( PG(\rho+1, q') \subset PG(\rho+1, q) \) is a \( \rho \)-saturating set in the space \( PG(\rho+1, q) \).

Theorem 2.3. Let \( q = (q')^4 \). In \( PG(2, q) \) there is a 1-saturating set of size \( 2\sqrt{q} + 2\sqrt{q} + 2 \).

Theorem 2.4. Let \( q = (q')^6 \), \( q' \) prime, \( q' \leq 73 \). In \( PG(2, q) \) there is a 1-saturating set of size \( 2\sqrt{q} + 2\sqrt{q} + 2\sqrt{q} + 2 \).

Theorem 2.5. Let \( q = (q')^3 \). In \( PG(3, q) \) there is a 2-saturating set of size \( 4\sqrt{q} + 4 \).

Theorem 2.6. Let \( q = (q')^3 \). In \( PG(4, q) \) there is a 2-saturating set of size \( 9\sqrt{q} - 8\sqrt{q} + 4 \).

We give tables of smallest known lengths \( \ell_q(r, R) \) of an \([n, n-r]_q R \) codes. A subscript indicates the minimum distance \( d \) of the corresponding \([\ell_q(r, 2), \ell_q(r, 2) - r, d]_q \) code. A double subscript “a,b” means that the value of \( \ell_q(r, R) \) is provided by related codes with distinct distances \( a \) and \( b \). The dot “.” notes the exact bound \( \ell_q(r, R) = \ell_q(r, R) \).

In Table 1 we use \([2, Tab. 1],[5, Tabs III,IV]\) for \( q \leq 7 \) and the computer search made in this work for 2-saturating sets in the spaces \( PG(4, q), q \geq 8 \).

Table 1. Upper Bounds \( \ell_q = \ell_q(5, 3) \) on the Length Function \( \ell_q(5, 3) < 4.5\sqrt{q}^2 \)

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Theorem 2.7. For the length function \( \ell_q(5, 3) \) it holds that

\[
\ell_q(5, 3) \leq c_q \sqrt{q^2}, \quad c_q < 4 \text{ if } q \leq 27, \quad c_q < 4.5 \text{ if } q \leq 32.
\]

In Table 2 we use \([4, Tab. 1]\), see also the references therein, \([3, Th. 5.2]\), Theorem 2.3, and computer search made in this work for 1-saturating sets in the planes \( PG(2, q) \).

Theorem 2.8. For the length function \( \ell_q(3, 2) \) it holds that

\[
\ell_q(3, 2) \leq a_q \sqrt{q}, \quad a_q < 3 \text{ if } q \leq 109, \quad a_q < 3.5 \text{ if } q \leq 349, \quad a_q < 4 \text{ if } q \leq 1217.
\]

For large \( q \) the existence of 1-saturating sets in \( PG(2, q) \) of size at most \( 5\sqrt{q} \log \sqrt{q} \) was shown by means of probabilistic methods in \([8]\).

The following results are given by explicit constructions. In \( PG(2, q), q = (q')^2 \), a 1-saturating set of size \( 3\sqrt{q} - 1 \) is obtained in \([3, Th. 5.2]\). In the plane \( PG(2, q), q = (q')^m \), \( m \geq 2 \), projectively nonequivalent to each other 1-saturating sets of size \( 2q^{m-1} + \sqrt{q} \) are obtained in \([2, Th. 2],[9, Th. 3.2]\). In a few papers, see \([7],[8]\) and the reference therein, 1-saturating sets in \( PG(2, q) \) of size approximately \( q^3 \) with a constant \( c \) independent of \( q \) are constructed. In \([9]\) constructions of 1-saturating \( n \)-sets in \( PG(2, q) \) of size \( n \) about \( 3q^2 \) are proposed, in particular, numerous 1-saturating \( n \)-sets with \( n < 5\sqrt{q} \log \sqrt{q} \) are obtained.
TABLE 2. Upper Bounds $\ell_q = \ell_q(3, 2)$ on the Length Function $\ell_q(3, 2) < 4\sqrt{q}$

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3 Infinite Families of Codes with Covering Radius 2

Combining the results of [10, Ths 4,9] we obtain an infinite family $A_{2,3,1}^{(1)}$ of $[n, n-r]_3^2$ codes.

$$n = \left\{ \begin{array}{ll}
\frac{5}{4} \cdot 3^{-\frac{r-1}{2}} - \frac{1}{4} & \text{if } r = 4k + 1 \geq 5 \\
\frac{5}{4} \cdot 3^{-\frac{r-3}{2}} + \frac{3}{4} \cdot 3^{-\frac{r+1}{2} + \psi(\frac{r-3}{2}) - 1} & \text{if } r = 4k + 3 \geq 11
\end{array} \right.$$

where $\psi(x) \equiv x (\mod 2)$, $\psi(x) \in \{0, 1\}$.

**Theorem 3.1.** For $q = (q')^4$ there is an infinite family $A_{2,q,2}^{(1)}$ of $[n, n-r]_q^2$ codes such that

$$A_{2,q,2}^{(1)} : R = 2, r = 2t + 1 \geq 3, q = (q')^4, n = 2(\sqrt{q} + \sqrt[q^2]{q} + 1)q^{-\frac{r-3}{2}} + \left[ q^{-\frac{r-5}{2}} \right],$$

$$\bar{\mu}_q(2) \approx 2 + \frac{4}{\sqrt[q^2]{q}} + \frac{6}{\sqrt{q}} + \frac{4}{\sqrt[q^2]{q}} - \frac{2}{q} + O \left( \frac{1}{q^{\frac{1}{2}}} \right).$$

4
Theorem 3.2. Let $q \geq 7$. Assume that there is an $[n_q, n_q - 3]_q 2$ code $V_0$ with $n_q < q$. Define a 2-partition as in [6, Def. 1],[5, Def. 2]. Let a parity-check matrix of $V_0$ admit a 2-partition to $p(V_0)$ subsets. Then there is the following infinite family $A_{2,q,3}^{(1)}$ of $[n,n - r]_q 2$ codes.

$$A_{2,q,3}^{(1)} : R = 2, \ r = 2t + 1 \geq 3, \ q \geq 7, \ r \neq 9, 13,$$

$$n = \begin{cases} \ n_q \frac{r-3}{r-2} + 2 \lfloor \frac{q}{r-2} \rfloor & \text{if } 2p(V_0) \leq q + 1 \\ \ n_q \frac{r-3}{r-2} + 2 \lfloor \frac{q}{r-2} \rfloor + \lfloor \frac{q}{r-2} \rfloor & \text{if } 2p(V_0) > q + 1 \end{cases}.$$

For $r = 9, 13$ it holds that $n = n_q \frac{r-3}{r-2} + 2q \frac{r-5}{r-2} + q \frac{r-9}{r-2}$.

Theorem 3.2 is the main tool to obtain infinite code families $A_{2,q}^{(1)}$.

Theorem 3.3. For $q = (q')^6$ there is an infinite family $A_{2,q,4}^{(1)}$ of $[n,n - r]_q 2$ codes such that

$$A_{2,q,4}^{(1)} : R = 2, \ r = 2t + 1 \geq 3, \ q = (q')^6, \ q' \text{ prime, } q' \leq 73, \ r \neq 9, 13,$$

$$n = 2(\sqrt{q} + \sqrt[3]{q} + \sqrt[3]{q + 1})q \frac{r-3}{r-2} + 2 \lfloor \frac{q}{r-2} \rfloor, \ p_q(2) \approx 2 + 4 \frac{1}{\sqrt{q}} + \frac{6}{\sqrt{q}} + \frac{8}{\sqrt{q}} + O \left( \frac{1}{\sqrt{q^2}} \right).$$

Lemma 3.4. For an $[n_q, n_q - 3, 3]_q 2$ code $V_0$ we have $p(V_0) \leq n_q - 1$.

Theorem 3.5. For $q \leq 1217$ there is an infinite family $A_{2,q,5}^{(1)}$ of $[n,n - r]_q 2$ codes such that

$$A_{2,q,5}^{(1)} : R = 2, \ r = 2t + 1 \geq 3, \ q \leq 1217, \ r \neq 9, 13,$$

$$n = \begin{cases} \bar{\ell}_q(3,2)q \frac{r-3}{r-2} + 2 \lfloor \frac{q}{r-2} \rfloor & \text{if } 16 \leq q \leq 1217 \\ \bar{\ell}_q(3,2)q \frac{r-3}{r-2} + 2 \lfloor \frac{q}{r-2} \rfloor + \lfloor \frac{q}{r-2} \rfloor & \text{if } 7 \leq q \leq 13 \end{cases},$$

$$p_q(2) < 4.5 \ \text{if } q \leq 109, \ p_q(2) < 6.125 \ \text{if } q \leq 349, \ p_q(2) < 8 \ \text{if } q \leq 1217.$$ 

where values of $\ell_q(3,2)$ can be found in Table 2.

Table 3 uses the families $A_{2,q,i}^{(1)}$, written above, Table 1, and the papers [3, Tab.I],[6, Tab.I],[10, Tab.1],[11, Tab. I],[12, Tab. I],[13, Tab. II].

Table 3. Upper Bounds $\bar{\ell}_q(r,2)$ on the Length Function $\ell_q(r,2), q = 3,4,5,7, r \leq 24$

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4 Infinite Families of Codes with Covering Radius 3

Let $\theta_{t,q} = \frac{q^t - 1}{q - 1}$. The direct sum (DS) [1] of the codes of the family $A_{2,3,1}^{(1)}$ and the $[\theta_{t,3}; \theta_{t,3}-t]_{3,1}$ Hamming codes forms an infinite family $A_{3,3,1}^{(1)}$ of $[n, n-r]_{3,3}$ codes such that

$$A_{3,3,1}^{(1)} : R = 3, \ r = 3t + 1 \geq 7, \ q = 3, \ r \neq 10, \ \overline{\mu}_3(3) \approx 2.38,$$

$$n = \begin{cases} \left\lfloor \frac{7}{4} \cdot \frac{3^{t-1}}{3} - \frac{3}{4} \right\rfloor & \text{if } r = 6k + 1 \geq 7 \\ \left\lfloor \frac{7}{4} \cdot \frac{3^{t-1}}{3} + \frac{3}{4} \cdot \frac{3^{t+2} + \psi(\frac{r-2}{6})}{6} - \frac{3}{6} \right\rfloor & \text{if } r = 6k + 4 \geq 16 \end{cases}.$$

DS of the codes of $[10, \text{Ths 5,10}]$ with codimension $2t+1$ and the $[\theta_{t,5}, \theta_{t,5}-t]_{5,1}$ Hamming codes forms an infinite family $A_{3,3,2}^{(1)}$ of $[n, n-r]_{3,3}$ codes with the following parameters

$$A_{3,3,2}^{(1)} : R = 3, \ r = 3t + 1 \geq 10, \ q = 5, \ r \neq 13, \ b = \psi(\frac{r-1}{6}), \ \overline{\mu}_5(3) \approx 4.16,$$

$$n = \begin{cases} \left\lfloor \frac{1}{4} \left(5^{t+2} - 1\right) + \ell_5(\frac{r-1}{3}, 2) \right\rfloor & \text{if } r = 6k + 4 \geq 10 \\ \left\lfloor \frac{1}{4} \left(5^{t+2} - 1\right) + \ell_5\left(\frac{r-7}{6} + b, 2\right) \cdot 5^{\frac{r-b}{6} - b} + \frac{2-b}{4} \left(5^{\frac{r-5}{6} - b} \right) - 1 \right\rfloor & \text{if } r = 6k + 1 \geq 19 \end{cases}.$$

Theorem 4.1. For $q = (q')^3 \geq 64$ there is an infinite family $A_{3,q,3}^{(1)}$ of $[n, n-r]_{3,3}$ codes with

$$A_{3,q,3}^{(1)} : R = 3, \ r = 3t + 1 \geq 7, \ q = (q')^3 \geq 64, \ n = 4(q^2 + 1)q^{r-4},$$

$$\overline{\mu}_q(3) \approx \frac{2}{3} \cdot \frac{3}{\sqrt{q}} + \frac{2}{3} \cdot \frac{3}{\sqrt{q^2}} - \frac{64}{3q} + O\left(\frac{1}{q \sqrt{q}}\right).$$

Theorem 4.2. For $q = (q')^3 \geq 27$ there is an infinite family $A_{3,q,3}^{(2)}$ of $[n, n-r]_{3,3}$ codes with

$$A_{3,q,3}^{(2)} : R = 3, \ r = 3t + 2 \geq 8, \ n = (9\sqrt{q^2 - 8\sqrt{q}} + 4)q^{r-5},$$

$$\overline{\mu}_q(3) \approx \frac{243}{2} - \frac{324}{\sqrt{q}} + \frac{450}{\sqrt{q^2}} - \frac{2699}{6q} + O\left(\frac{1}{q \sqrt{q}}\right).$$

Theorem 4.3. For $q \leq 32$ there is an infinite family $A_{3,q,2}^{(2)}$ of $[n, n-r]_{3,3}$ codes such that

$$A_{3,q,2}^{(2)} : R = 3, \ r = 3t + 2 \geq 11, \ q \leq 32, \ m = \frac{r-5}{3},$$

$$n = \begin{cases} \ell_q(5, 3)q^m + \frac{2^{m-1}}{q-1} & \text{if } q \leq 32, \ q \neq 2, 5, 19 \\ \ell_q(5, 3)q^m + \frac{2^{m-1}}{q-1} & \text{if } q = 2, 5, 19 \end{cases},$$

$$\overline{\mu}_q(3) < \frac{32}{3} \text{ if } q \leq 27, \ \overline{\mu}_q(3) < \frac{243}{16} \text{ if } q \leq 32.$$

where values of $\ell_q(5, 3)$ can be found in Table 1.
References


