Linear Covering Codes of Radius 2 and 3

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Abstract. Infinite families of linear codes with covering radius $R = 2$ and $R = 3$ and codimension $r = 2t + 1$ and $r = 3t + 1, 3t + 2$, respectively, are constructed on the base of q^m-concatenating constructions with starting codes of codimension $r = 3$ and $r = 4, 5$. Parity check matrices of the starting codes are treated as saturating sets in the projective spaces $PG(v, q)$, $v = 2, 3, 4$. The sets are obtained by a theoretical way and by computer search using geometrical properties of objects. A new concept of N-fold strong blocking set in the projective spaces is introduced. New upper bounds on the length function and on the smallest sizes of saturating sets in the projective spaces are given. The asymptotic covering densities of the new code families are bounded from above by constants independent of q.

1 Introduction

Let F_q be the Galois field of q elements. Let F_q^n be the n-dimensional vector space over F_q. Denote by $[n, n-r]_q$ a q-ary *linear code* of length n and codimension r. The *covering radius* of an $[n, n-r]_q$ code is the least integer R such that F_q^n is covered by spheres of radius R centered on codewords [1]. Equivalently, an $[n, n-r]_q$ code has covering radius R if every column of F_q^r is equal to a linear combination of R columns of a parity check matrix of the code, and R is the smallest value with such property. An $[n, n-r]_qR$ code is an $[n, n-r]_q$ code of covering radius R. An $[n, n-r, d]_qR$ code is an $[n, n-r]_qR$ code of minimum distance d.

Linear covering codes are deeply connected with *saturating sets* in the *projective spaces* over finite fields. Let $PG(v, q)$ be the v-dimensional projective space over F_q. We say that a set of points $S \subseteq PG(v, q)$ is *ϱ-saturating* if for any point $x \in PG(v, q)$ there exist $\varrho + 1$ points in S generating a subspace of $PG(v, q)$ in which x lies and ϱ is the smallest value with such property, cf. [2]-[5].

Points of an $(R-1)$-saturating n-set in the projective space $PG(r-1, q)$, given by an r-positional vector $(x_0, x_1, \ldots, x_{r-1})$ with $x_i \in F_q$, can be treated as columns of a *parity-check matrix* of an $[n, n-r]_qR$ *related* covering code [3]-[6]. In the literature the *saturating sets* are called also "*saturated sets*" [3],[6], "*spanning sets*", and "*dense sets*" [7]-[9].

The covering quality of an $[n, n-r(\mathcal{C})]_qR$ code $\mathcal{C}$ can be measured by its *covering density*

$$\mu_q(n, R, \mathcal{C}) = q^{-r(\mathcal{C})} \sum_{i=0}^{R} (q-1)^i \binom{n}{i} \geq 1.$$

The *length function* $\ell_q(r,R)$ is the smallest length of a q-ary linear code with codimension r and covering radius R. The *smallest known* length of such code is denoted by $\overline{\ell}_q(r,R)$. Evidently, $\ell_q(r,R) \leq \overline{\ell}_q(r,R)$. Existence of an $[n,n-r]_qR$ code or an $(R-1)$-saturating n-set in $PG(r-1,q)$ implies upper bounds $\ell_q(r,R) \leq \overline{\ell}_q(r,R) \leq n$.

For given integers R,η with $R>0$, $0 \leq \eta \leq R-1$, and for a fixed prime power q, let $\mathcal{A}_{R,q}^{(\eta)} = (\mathcal{C}_{n_i})_i$ denote an infinite sequence of q-ary linear $[n_i, n_i - r_i]_qR$ codes $\mathcal{C}_{n_i}$ with covering radius R and such that the following two properties hold:

(a) the codimension $r_i = Rt_i + \eta$, where $(t_i)_i$ is an increasing sequence of integers;

(b) the length n_i of $\mathcal{C}_i$ coincides with $f_q^{(\eta)}(r_i)$, where $f_q^{(\eta)}$ is an increasing function.

We call $\mathcal{A}_{R,q}^{(\eta)}$ an *infinite family of covering codes.* Considering families of type $\mathcal{A}_{R,q}^{(\eta)}$ is a standard method of investigation of *linear* covering codes [1],[3]-[6],[10]-[12]. Through the paper, distinct families $\mathcal{A}_{R,q}^{(\eta)}$ with the same values of R,η will be denoted as $\mathcal{A}_{R,q,i}^{(\eta)}$, $i=1,2,\ldots$

For an infinite code family $\mathcal{A}_{R,q}^{(\eta)}$, its *asymptotic covering density* is defined as follows:

$$\overline{\mu}_q(R, \mathcal{A}_{R,q}^{(\eta)}) = \liminf_{i\to\infty} \mu_q(n_i, R, \mathcal{C}_{n_i}).$$

We use the notations $\overline{\mu}_q(R)$ if the family $\mathcal{A}_{R,q}^{(\eta)}$ is clear by context.

In this work we concentrate on infinite code families $\mathcal{A}_{2,q}^{(1)}, \mathcal{A}_{3,q}^{(1)}$, and $\mathcal{A}_{3,q}^{(2)}$.

We use two basic tools: the q^m-concatenating constructions of covering codes and the connection of covering codes and saturating sets in projective spaces.

The q^m-concatenating constructions, see [1],[3]-[6],[10]-[12], and the references therein, are the fundamental instrument for obtaining infinite families of covering codes with a fixed radius. Using a starting code as a "seed", the q^m-concatenating constructions yield an infinite family of new codes with the same covering radius and with almost the same covering density.

Small saturating sets in the spaces of little dimension are highly convenient to take them as the starting codes for the q^m-concatenating constructions [3]-[6],[12].

In this work new constructions of small 1-saturating sets in planes $PG(2,q)$ and 2-saturating sets in spaces $PG(3,q), PG(4,q)$ are proposed. For it a new concept of N-fold strong blocking set in projective spaces is introduced. Also, many small saturating sets are obtained by computer. Using the saturating sets as the starting points of the q^m-concatenating constructions, we obtained new infinite code families $\mathcal{A}_{2,q}^{(1)}, \mathcal{A}_{3,q}^{(1)}$, and $\mathcal{A}_{3,q}^{(2)}$. Their asymptotic covering density is smaller than that of the known families. Moreover, the densities $\overline{\mu}_q(R, \mathcal{A}_{R,q}^{(\eta)})$ of the new families are bounded from above by constants independent of q. As results, many new upper bounds on the length function are obtained. They are also new upper bounds on the smallest possible sizes of saturating sets in projective spaces.

2 Small Saturating Sets in Spaces $PG(v,q)$, $v=2,3,4$

Definition 2.1. A subset B of a projective space $PG(N,q)$ is an *N-fold strong blocking set* if every hyperplane of $PG(N,q)$ is spanned by N points in B.

Theorem 2.2. *Let $q = (q')^{\rho+1}$. Any $(\rho+1)$-fold strong blocking set in a subspace $PG(\rho+1, q') \subset PG(\rho+1, q)$ is a ρ-saturating set in the space $PG(\rho+1, q)$.*

Theorem 2.3. *Let $q = (q')^4$. In $PG(2, q)$ there is a 1-saturating set of size $2\sqrt{q} + 2\sqrt[4]{q} + 2$.*

Theorem 2.4. *Let $q = (q')^6$, q' prime, $q' \leq 73$. In $PG(2, q)$ there is a 1-saturating set of size $2\sqrt{q} + 2\sqrt[3]{q} + 2\sqrt[6]{q} + 2$.*

Theorem 2.5. *Let $q = (q')^3$. In $PG(3, q)$ there is a 2-saturating set of size $4\sqrt[3]{q} + 4$.*

Theorem 2.6. *Let $q = (q')^3$. In $PG(4, q)$ there is a 2-saturating set of size $9\sqrt[3]{q^2} - 8\sqrt[3]{q} + 4$.*

We give tables of *smallest known* lengths $\overline{\ell}_q(r, R)$ of an $[n, n-r]_q R$ codes. A subscript indicates the minimum distance d of the corresponding $[\,\overline{\ell}_q(r, 2), \overline{\ell}_q(r, 2) - r, d]_q 2$ code. A double subscript "a, b" means that the value of $\overline{\ell}_q(r, R)$ is provided by related codes with distinct distances a and b. The dot "▪" notes the exact bound $\ell_q(r, R) = \overline{\ell}_q(r, R)$.

In Table 1 we use [2, Tab. 1],[5, Tabs III,IV] for $q \leq 7$ and the computer search made in this work for 2-saturating sets in the spaces $PG(4, q)$, $q \geq 8$.

Table 1. Upper Bounds $\overline{\ell}_q = \overline{\ell}_q(5, 3)$ on the Length Function $\ell_q(5, 3) < 4.5\sqrt[3]{q^2}$

q	$\overline{\ell}_q$	q	$\overline{\ell}_q$	q	$\overline{\ell}_q$	q	$\overline{\ell}_q$	q	$\overline{\ell}_q$	q	$\overline{\ell}_q$	q	$\overline{\ell}_q$	q	$\overline{\ell}_q$	q	$\overline{\ell}_q$
2	$6_{5,6}$▪	5	10_4▪	7	$13_{3,4}$	9	$16_{3,4}$	13	$21_{3,4}$	17	$25_{3,4}$	23	$32_{3,4}$	27	$36_{3,4}$	31	$40_{3,4}$
3	$8_{3,4}$▪	4	$9_{3,4}$▪	8	$14_{3,4}$	11	$18_{3,4}$	16	$24_{3,4}$	19	27_4	25	$34_{3,4}$	29	$38_{3,4}$	32	$41_{3,4}$

Theorem 2.7. *For the length function $\ell_q(5, 3)$ it holds that*

$$\ell_q(5, 3) \leq c_q\sqrt[3]{q^2}, \;\; c_q < 4 \text{ if } q \leq 27, \;\; c_q < 4.5 \text{ if } q \leq 32.$$

In Table 2 we use [4, Tab. I], see also the references therein, [3, Th. 5.2], Theorem 2.3, and computer search made in this work for 1-saturating sets in the planes $PG(2, q)$.

Theorem 2.8. *For the length function $\ell_q(3, 2)$ it holds that*

$$\ell_q(3, 2) \leq a_q\sqrt{q}, \quad a_q < 3 \text{ if } q \leq 109, \quad a_q < 3.5 \text{ if } q \leq 349, \quad a_q < 4 \text{ if } q \leq 1217.$$

For large q the existence of 1-saturating sets in $PG(2, q)$ of size at most $5\sqrt{q \log q}$ was shown by means of *probabilistic methods* in [8].

The following results are given by explicit constructions. In $PG(2, q)$, $q = (q')^2$, a 1-saturating set of size $3\sqrt{q} - 1$ is obtained in [3, Th. 5.2]. In the plane $PG(2, q)$, $q = (q')^m$, $m \geq 2$, projectively nonequivalent to each other 1-saturating sets of size $2q^{\frac{m-1}{m}} + \sqrt[m]{q}$ are obtained in [2, Th. 2],[9, Th. 3.2]. In a few papers, see [7],[8] and the reference therein, 1-saturating sets in $PG(2, q)$ of size approximately $cq^{\frac{3}{4}}$ with a constant c independent of q are constructed. In [9] constructions of 1-saturating n-sets in $PG(2, q)$ of size n about $3q^{\frac{2}{3}}$ are proposed, in particular, numerous 1-saturating n-sets with $n < 5\sqrt{q \log q}$ are obtained.

TABLE 2. Upper Bounds $\overline{\ell}_q = \overline{\ell}_q(3,2)$ on the Length Function $\ell_q(3,2) < 4\sqrt{q}$

q	$\overline{\ell}_q$	q	$\overline{\ell}_q$	q	$\overline{\ell}_q$	q	$\overline{\ell}_q$	q	$\overline{\ell}_q$	q	$\overline{\ell}_q$	q	$\overline{\ell}_q$	q	$\overline{\ell}_q$	q	$\overline{\ell}_q$
3	4_4 ■	64	19_3	167	$42_{3,4}$	283	58_3	431	75_3	577	89_3	729	80_3	887	116_3	1051	128_3
4	5_3 ■	67	$23_{3,4}$	169	38_3	289	50_3	433	75_3	587	90_3	733	102_3	907	117_3	1061	129_3
5	$6_{3,4}$ ■	71	22_4	173	42_3	293	59_3	439	75_3	593	90_3	739	103_3	911	118_3	1063	129_3
7	$6_{3,4}$ ■	73	24_4	179	43_3	307	60_3	443	76_3	599	91_3	743	104_3	919	118_3	1069	129_3
8	6_4 ■	79	$26_{3,4}$	181	43_3	311	61_3	449	76_3	601	91_3	751	105_3	929	119_3	1087	130_3
9	6_4 ■	81	$26_{3,4}$	191	45_3	313	61_3	457	77_3	607	91_3	757	105_3	937	120_3	1091	131_3
11	7_4 ■	83	26_3	193	45_3	317	62_3	461	77_3	613	92_3	761	105_3	941	120_3	1093	131_3
13	8_4 ■	89	$28_{3,4}$	197	46_3	331	63_3	463	77_3	617	92_3	769	106_3	947	121_3	1097	131_3
16	$9_{3,4}$ ■	97	29_3	199	46_3	337	64_3	467	78_3	619	92_3	773	106_3	953	121_3	1103	131_3
17	$10_{3,4}$	101	$30_{3,4}$	211	48_3	343	64_3	479	79_3	625	62_3	787	107_3	961	92_3	1109	132_3
19	$10_{3,4}$	103	30_3	223	49_3	347	65_3	487	80_3	631	94_3	797	108_3	967	123_3	1117	132_3
23	10_4 ■	107	31_3	227	50_3	349	65_3	491	81_3	641	95_3	809	109_3	971	123_3	1123	133_3
25	$12_{3,4}$	109	31_3	229	50_3	353	66_3	499	81_3	643	95_3	811	110_3	977	124_3	1129	133_3
27	$12_{3,4}$	113	32_3	233	51_3	359	66_3	503	82_3	647	95_3	821	110_3	983	124_3	1151	135_3
29	$13_{3,4}$	121	32_3	239	51_3	361	56_3	509	82_3	653	96_3	823	110_3	991	124_3	1153	135_3
31	$14_{3,4}$	125	34_3	241	52_3	367	67_3	512	82_3	659	96_3	827	110_3	997	125_3	1163	135_3
32	13_3	127	$35_{3,4}$	243	52_3	373	68_3	521	84_3	661	96_3	829	110_3	1009	124_3	1171	136_3
37	15_4	128	$34_{3,4}$	251	53_3	379	69_3	523	83_3	673	98_3	839	111_3	1013	125_3	1181	137_3
41	16_4	131	35_3	256	42_3	383	69_3	529	68_3	677	98_3	853	113_3	1019	126_3	1187	137_3
43	16_4	137	36_3	257	54_3	389	70_3	541	85_3	683	99_3	857	113_3	1021	126_3	1193	138_3
47	$18_{3,4}$	139	$37_{3,4}$	263	55_3	397	71_3	547	86_3	691	99_3	859	113_3	1024	95_3	1201	138_3
49	18_4	149	$39_{3,4}$	269	56_3	401	71_3	557	87_3	701	100_3	863	114_3	1031	127_3	1213	139_3
53	18_4	151	$39_{3,4}$	271	56_3	409	72_3	563	87_3	709	101_3	877	115_3	1033	127_3	1217	139_3
59	20_4	157	$40_{3,4}$	277	57_3	419	73_3	569	88_3	719	102_3	881	115_3	1039	127_3		
61	20_4	163	$41_{3,4}$	281	57_3	421	73_3	571	88_3	727	102_3	883	115_3	1049	128_3		

3 Infinite Families of Codes with Covering Radius 2

Combining the results of [10, Ths 4,9] we obtain an infinite family $\mathcal{A}^{(1)}_{2,3,1}$ of $[n,n-r]_3 2$ codes.

$$\mathcal{A}^{(1)}_{2,3,1} \quad : \quad R=2,\ r=2t+1\geq 5,\ q=3,\ r\neq 7,\ \overline{\mu}_3(2)\approx\frac{25}{24},$$
$$n \quad = \quad \begin{cases} \frac{5}{4}\cdot 3^{\frac{r-1}{2}}-\frac{1}{4} & \text{if } r=4k+1\geq 5\\ \frac{5}{4}\cdot 3^{\frac{r-1}{2}}+\frac{3}{4}\cdot 3^{\frac{r+1}{4}+\psi(\frac{r-3}{4})}-1 & \text{if } r=4k+3\geq 11\end{cases},$$

where $\psi(x)\equiv x \pmod 2$, $\psi(x)\in\{0,1\}$.

Theorem 3.1. *For $q=(q')^4$ there is an infinite family $\mathcal{A}^{(1)}_{2,q,2}$ of $[n,n-r]_q 2$ codes such that*

$$\mathcal{A}^{(1)}_{2,q,2} \quad : \quad R=2,\ r=2t+1\geq 3,\ q=(q')^4,\ n=2(\sqrt{q}+\sqrt[4]{q}+1)q^{\frac{r-3}{2}}+\left\lfloor q^{\frac{r-5}{2}}\right\rfloor,$$
$$\overline{\mu}_q(2) \quad \approx \quad 2+\frac{4}{\sqrt[4]{q}}+\frac{6}{\sqrt{q}}+\frac{4}{\sqrt[4]{q^3}}-\frac{2}{q}+O\left(\frac{1}{q\sqrt[4]{q}}\right).$$

Theorem 3.2. *Let $q \geq 7$. Assume that there is an $[n_q, n_q - 3]_q 2$ code V_0 with $n_q < q$. Define a 2-partition as in [6, Def. 1],[5, Def. 2]. Let a parity-check matrix of V_0 admit a 2-partition to $p(V_0)$ subsets. Then there is the following infinite family $\mathcal{A}^{(1)}_{2,q,3}$ of $[n, n-r]_q 2$ codes.*

$$\mathcal{A}^{(1)}_{2,q,3} \quad : \quad R = 2,\ r = 2t+1 \geq 3,\ q \geq 7,\ r \neq 9, 13,$$
$$n \quad = \quad \begin{cases} n_q q^{\frac{r-3}{2}} + 2\lfloor q^{\frac{r-5}{2}} \rfloor & \text{if } 2p(V_0) \leq q+1 \\ n_q q^{\frac{r-3}{2}} + 2\lfloor q^{\frac{r-5}{2}} \rfloor + \lfloor q^{\frac{r-7}{2}} \rfloor & \text{if } 2p(V_0) > q+1 \end{cases}.$$

For $r = 9, 13$ it holds that $n = n_q q^{\frac{r-3}{2}} + 2q^{\frac{r-5}{2}} + q^{\frac{r-7}{2}} + q^{\frac{r-9}{2}}$.

Theorem 3.2 is the main tool to obtain infinite code families $\mathcal{A}^{(1)}_{2,q}$.

Theorem 3.3. *For $q = (q')^6$ there is an infinite family $\mathcal{A}^{(1)}_{2,q,4}$ of $[n, n-r]_q 2$ codes such that*

$$\mathcal{A}^{(1)}_{2,q,4}\ :\ R = 2,\ r = 2t+1 \geq 3,\ q = (q')^6,\ q' \text{ prime},\ q' \leq 73,\ r \neq 9, 13,$$
$$n = 2(\sqrt{q} + \sqrt[3]{q} + \sqrt[6]{q} + 1) q^{\frac{r-3}{2}} + 2\lfloor q^{\frac{r-5}{2}} \rfloor,\ \overline{\mu}_q(2) \approx 2 + \frac{4}{\sqrt[6]{q}} + \frac{6}{\sqrt[3]{q}} + \frac{8}{\sqrt{q}} + O\left(\frac{1}{\sqrt[3]{q^2}}\right).$$

Lemma 3.4. *For an $[n_q, n_q - 3, 3]_q 2$ code V_0 we have $p(V_0) \leq n_q - 1$.*

Theorem 3.5. *For $q \leq 1217$ there is an infinite family $\mathcal{A}^{(1)}_{2,q,5}$ of $[n, n-r]_q 2$ codes such that*

$$\mathcal{A}^{(1)}_{2,q,5} \quad : \quad R = 2,\ r = 2t+1 \geq 3,\ q \leq 1217,\ r \neq 9, 13,$$
$$n \quad = \quad \begin{cases} \overline{\ell}_q(3,2) q^{\frac{r-3}{2}} + 2\lfloor q^{\frac{r-5}{2}} \rfloor & \text{if } 16 \leq q \leq 1217 \\ \overline{\ell}_q(3,2) q^{\frac{r-3}{2}} + 2\lfloor q^{\frac{r-5}{2}} \rfloor + \lfloor q^{\frac{r-7}{2}} \rfloor & \text{if } 7 \leq q \leq 13 \end{cases},$$
$$\overline{\mu}_q(2) \quad < \quad 4.5 \text{ if } q \leq 109,\ \overline{\mu}_q(2) < 6.125 \text{ if } q \leq 349,\ \overline{\mu}_q(2) < 8 \text{ if } q \leq 1217.$$

where values of $\overline{\ell}_q(3,2)$ can be found in Table 2.

Table 3 uses the families $\mathcal{A}^{(1)}_{2,q,i}$ written above, Table 1, and the papers [3, Tab. I],[6, Tab. I],[10, Tab. 1],[11, Tab. I],[12, Tab. I],[13, Tab. II].

Table 3. Upper Bounds $\overline{\ell}_q(r,2)$ on the Length Function $\ell_q(r,2)$, $q = 3,4,5,7$, $r \leq 24$

r	$\overline{\ell}_3(r,2)$	$\overline{\ell}_4(r,2)$	$\overline{\ell}_5(r,2)$	$\overline{\ell}_7(r,2)$	r	$\overline{\ell}_3(r,2)$	$\overline{\ell}_4(r,2)$	$\overline{\ell}_5(r,2)$	$\overline{\ell}_7(r,2)$
3	**4.**	**5.**	**6.**	**6.**	14	1822	9522	35000	252105
4	**8.**	9	11	15	15	2915	19456	78256	741909
5	**11.**	19	28	44	16	5588	37888	175000	1764735
6	22	37	56	105	17	8201	77824	410937	5193363
7	40	85	131	309	18	16402	151552	875000	12353145
8	76	154	281	743	19	24785	316672	1953828	36353541
9	101	304	703	2164	20	49328	611328	4375000	86472015
10	202	592	1400	5145	21	73811	1245184	9853906	254474787
11	323	1237	3153	15141	22	147622	2424832	21875000	605304105
12	620	2389	7031	36407	23	223073	4980736	48831278	1781323509
13	911	4948	16406	106036	24	443960	9699328	109375000	4237128735

4 Infinite Families of Codes with Covering Radius 3

Let $\theta_{t,q} = \frac{q^t-1}{q-1}$. The direct sum (DS) [1] of the codes of the family $\mathcal{A}_{2,3,1}^{(1)}$ and the $[\theta_{t,3}, \theta_{t,3}-t]_3 1$ Hamming codes forms an infinite family $\mathcal{A}_{3,3,1}^{(1)}$ of $[n, n-r]_3 3$ codes such that

$$\begin{aligned} \mathcal{A}_{3,3,1}^{(1)} \quad &: \quad R = 3,\ r = 3t+1 \geq 7,\ q = 3,\ r \neq 10,\ \overline{\mu}_3(3) \approx 2.38, \\ n \quad &= \quad \begin{cases} \frac{7}{4} \cdot 3^{\frac{r-1}{3}} - \frac{3}{4} & \text{if } r = 6k+1 \geq 7 \\ \frac{7}{4} \cdot 3^{\frac{r-1}{3}} + \frac{3}{4} \cdot 3^{\frac{r+2}{6} + \psi(\frac{r-4}{6})} - \frac{3}{2} & \text{if } r = 6k+4 \geq 16 \end{cases}. \end{aligned}$$

DS of the codes of [10, Ths 5,10]) with codimension $2t+1$ and the $[\theta_{t,5}, \theta_{t,5}-t]_5 1$ Hamming codes forms an infinite family $\mathcal{A}_{3,3,2}^{(1)}$ of $[n, n-r]_5 3$ codes with the following parameters

$$\begin{aligned} \mathcal{A}_{3,3,2}^{(1)} \quad &: \quad R = 3,\ r = 3t+1 \geq 10,\ q = 5,\ r \neq 13,\ b = \psi(\frac{r-1}{6}),\ \overline{\mu}_5(3) \approx 4.16, \\ n \quad &= \quad \begin{cases} \frac{1}{4}(5^{\frac{r+2}{3}} - 1) + \overline{\ell}_5(\frac{r-1}{3}, 2) & \text{if } r = 6k+4 \geq 10 \\ \frac{1}{4}(5^{\frac{r+2}{3}} - 1) + \overline{\ell}_5(\frac{r-7}{6} + b, 2) \cdot 5^{\frac{r+5}{6} - b} + \frac{2-b}{4}(5^{\frac{r+5}{6} - b} - 1) & \text{if } r = 6k+1 \geq 19 \end{cases}. \end{aligned}$$

Theorem 4.1. *For $q = (q')^3 \geq 64$ there is an infinite family $\mathcal{A}_{3,q,3}^{(1)}$ of $[n, n-r]_q 3$ codes with*

$$\begin{aligned} \mathcal{A}_{3,q,3}^{(1)} \quad &: \quad R = 3,\ r = 3t+1 \geq 7,\ q = (q')^3 \geq 64,\ n = 4(\sqrt[3]{q} + 1) q^{\frac{r-4}{3}}, \\ \overline{\mu}_q(3) \quad &\approx \quad \frac{32}{3} + \frac{32}{\sqrt[3]{q}} + \frac{32}{\sqrt[3]{q^2}} - \frac{64}{3q} + O\left(\frac{1}{q\sqrt[3]{q}}\right). \end{aligned}$$

Theorem 4.2. *For $q = (q')^3 \geq 27$ there is an infinite family $\mathcal{A}_{3,q,1}^{(2)}$ of $[n, n-r]_q 3$ codes with*

$$\begin{aligned} \mathcal{A}_{3,q,1}^{(2)} \quad &: \quad R = 3,\ r = 3t+2 \geq 8,\ n = (9\sqrt[3]{q^2} - 8\sqrt[3]{q} + 4) q^{\frac{r-5}{3}}, \\ \overline{\mu}_q(3) \quad &\approx \quad \frac{243}{2} - \frac{324}{\sqrt[3]{q}} + \frac{450}{\sqrt[3]{q^2}} - \frac{2699}{6q} + O\left(\frac{1}{q\sqrt[3]{q}}\right). \end{aligned}$$

Theorem 4.3. *For $q \leq 32$ there is an infinite family $\mathcal{A}_{3,q,2}^{(2)}$ of $[n, n-r]_q 3$ codes such that*

$$\begin{aligned} \mathcal{A}_{3,q,2}^{(2)} \quad &: \quad R = 3,\ r = 3t+2 \geq 11,\ q \leq 32,\ m = \frac{r-5}{3}, \\ n \quad &= \quad \begin{cases} \overline{\ell}_q(5,3) q^m + 2\frac{q^m-1}{q-1} & \textit{if } q \leq 32, q \neq 2, 5, 19 \\ \overline{\ell}_q(5,3) q^m + 3\frac{q^m-1}{q-1} & \textit{if } q = 2, 5, 19 \end{cases}, \\ \overline{\mu}_q(3) \quad &< \quad \frac{32}{3} \textit{ if } q \leq 27,\ \overline{\mu}_q(3) < \frac{243}{16} \textit{ if } q \leq 32. \end{aligned}$$

where values of $\overline{\ell}_q(5,3)$ can be found in Table 1.

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