Matrix methods in Skorokhod problems*

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ABSTRACT

The *Skorokhod problem* is a determined mathematical model used for the construction and analysis of constrained processes, both determined and stochastic, such as queing networks, processor sharing in communication networks, stochastic approximation schemes for problems with constraints, etc. The model we deal with consists of a convex polyhedral set Z in a finite-dimensional space $\mathbb{R}^n$ and a family of "reflection vectors" d_i associated with $(n-1)$-dimensional faces F_i of Z. According to certain rules of reflection, an output $x(t) \subseteq Z$ is generated for each continuous input $u(t)$, $u(0) \in Z$. The corresponding input-output operator, if it exists, is called the *Skorokhod map*. The properties of the Skorokhod problem such as existence and uniqueness of an output for any admissible input, and different continuity properties of the associated Skorokhod map can be studied in terms of different types of stability of finite families of special $n \times n$-matrices, namely, the projection matrices onto the hyperplanes L_i parallel to faces F_i along the vectors d_i. We present both necessary and sufficient conditions of some of the above properties and also establish new relations between such notions as absolute stability, BV-stability, Perron stability, etc., of finite sets of projections and, more generally, of arbitrary $n \times n$-matrices.

INTRODUCTION

Skorokhod problem

The *Skorokhod problem* (SP) is a mathematical model which is often used for the construction and analysis of constrained processes, both determined and stochastic. A given set $Z \subseteq \mathbb{R}^n$ is the *domain* or the *characteristic set* of the SP. A set-valued vector-field $D(x) \subseteq \mathbb{R}^n$ called the *reflection map* is defined on the boundary ∂Z. The vectors of $D(x)$ are generalizations of inward normals to Z. Usually, sets $D(x)$ are closed convex cones. A process is studied on a time interval $[0,T]$, where T is either finite or $+\infty$. For a given input $u : [0,T] \to \mathbb{R}^n$, $u(0) \in Z$, an output $x(t)$, $x(0) = u(0)$, is sought according to the following *reflection conditions*: If $x(t) \in \mathrm{int}(Z)$, the derivative $\dot{x}(t)$ of the output coincides with $\dot{u}(t)$; otherwise

$$\dot{x}(t) = \dot{u}(t) + p,$$

where p is some *compensation vector* belonging to $D(x(t))$. This informal definition works only for inputs smooth enough. Often, however, SP's are supposed to handle broader classes of inputs such as continuous ones, which necessitates more sofisticated definitions. Here we give two alternative definitions of an SP and compare their properties.

In this talk we will mainly study *polyhedral SP's*: the set Z is a polyhedral set in $\mathbb{R}^n$ represented as

$$Z = \{x \in \mathbb{R}^n : \langle n_i, x \rangle \geq c_i, i = 1, \dots, N\},$$

and $D(x) = \{\alpha d_i : \alpha \geq 0\}$ in the relative interior of the i-th $(n-1)$-dimensional face F_i of Z. At faces of lower dimension, the set $D(x)$ is defined as the convex cone spanned by the vectors d_i corresponding to "active" faces F_i at x. An SP of this kind can be written in the form $\{n_i, d_i, c_i\}$, $i = 1, \dots, N$.

Historically the term SP was introduced for the description of stochastic processes with boundary conditions [1, 4, 25, 28]. In recent years this model also found applications in queing networks [14–17], processor sharing in communication networks [23], Leontief models in mathematical economics [5, 11], transport processes [7], stochastic approximation schemes for problems with constraints [9, 12].

The first problems one faces with a given SP is whether it has solutions for any input of a given class (existence property) and whether a solution is unique (uniqueness property). If a given SP is uniquely solvable then the input-output correspondence $\Gamma : u(\cdot) \to x(\cdot)$ is defined; it is called the *Skorokhod map*. When Γ is regular enough (Lipschitz continuous, for instance) the study of many problems in applications is greatly simplified.

The first case of an SP has been studied by A.V. Skorokhod [25]. He considered the simplest one-dimensional model in order to construct stochastic differential equations on $\mathbb{R}_+ = \{x \in \mathbb{R} : x \geq 0\}$. This is one of rare cases when the operator Γ can be found explicitely:

$$x(t) = u(t) - \left(0 \wedge \inf_{0 \leq s \leq t} u(s)\right).$$

Here $a \wedge b$ is the smaller of a and b.

An interesting particular case of SP arises in so called fluid apploximations of network traffic. It is a polyhedral one, its domain is the positive orthant $\mathbb{R}^n_+$. This problem has been studied by Harrison and Reiman [15] and proved to have a Lipschitz continuous Skorokhod map under mild assumptions.

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Later a problem on the positive orthant $\mathbb{R}^n_+$ with constant reflection vectors on each one of its $(n-1)$-dimensional faces was addresed in context of so called dynamic complementarity problem [6, 19, 20, 22]. In case when the solutions to an SP are understood as pairs $(u(\cdot), x(\cdot))$ with the compensation $y(t) = x(t) - u(t)$ of bounded variation, the problem of uniqueness has been completely reduced to that of the absence of non-trivial solutions of bounded variation of a certain differential inclusion [22], and then, to that of absolute stability of a system of projections plus some non-singularity conditions on the system of reflection vectors [19, 20].

Continuity properties of the Skorokhod map Γ for polyhedral SP's have been addressed in [10,18,27]. In particular, Dupuis and Ishii derived sufficient conditions for *Lipschitz continuity* of Γ. Earlier results for the case of normal reflection have been obtained by Vladimirov and Kleptsyn [27], see also [21].

Still, in many cases there is no complete characterization of polyhedral SP's with important properties. Though these cases are "marginal" (they constitute a residual set in the parametric space of polyhedral SP's), they might be very important because the SP's arising in applications often have additional symmetries, etc., which makes them not generic. This consideration justifies the attempts to derive as complete algebraic and geometric conditions of regularity of SP's as possible.

The aim of this talk is to derive both necessary and sufficient conditions of regularity of SP's in terms of stability-like properties of finite families of special matrices associated with polyhedral sets Z, namely, the projection matrices onto the hyperplanes parallel to the $(n-1)$-dimensional faces F_i of Z along the corresponding reflection vectors d_i. In parallel, we will state and prove results concerning finite systems of general $n \times n$-matrices, see [3, 8].

1. SKOROKHOD PROBLEM AND SKOROKHOD MAP

We will agree that any formal definition of an SP has to determine an input-output operator $\Gamma : u(\cdot) \to x(\cdot)$ possessing the following four characteristic properties:

- **Causality**. This is a standard requirement to non-anticipating operators meaning that the behavior of the output $x(t)$ after a moment t_1 depends only on the state $x(t_1)$ and the corresponding "tail" of the input $u(t)$, $t \geq t_1$.

- **Rate independence**. This property means solutions $(u(\cdot), x(\cdot))$ of any SP are invariant to monotone changes of time $\tau(t)$ such that both $\tilde{u}(\tau) = u(t(\tau))$ and $\tilde{x}(\tau) = x(t(\tau))$ are continuous functions again. This property is sometimes referred to as the characteristic property of *hysteresis* [21, 26] which can justify treating SP's as particular cases of *hysteresis nonlinearities*.

- **Laziness**. The trajectory of the output follows that of the input as long as the output can stay within Z. Formally, if $u(t) \in \text{int}(Z)$ for all $t \in [0, T]$ then $x(t) \equiv u(t)$ is the only solution of the SP on $[0, T]$. The "lazy" part of the model is the compensation $y(t) = x(t) - u(t)$ which does not change without need, see also analogous constructions in viability theory [2] and mathematical hysteresis [21, 26].

- **Reflection**. The following *complementarity condition* has to hold whenever the output $x(t)$ is at the boundary of Z: the infinitesimal increment of the compensation $y(t) = x(t) - u(t)$ should belong to the cone $D(x(t))$. This condition can be understood in different ways, see the alternative definitions of an SP below. More generally, this condition can be interpreted as existence of a mechanism ensuring that the process stays within the characteristic set. This mechanism need not be realized as a compensation vector from the reflection set; for example, one can also consider different kinds of limit procedures such as penalty functions, discrete iterations with projections onto Z, lexicographic schemes of compensation, etc.

Additionally, some kind of continuity of the Skorokhod map is often useful in applications. We will use two formal definitions complying with the four principles above; our main goal will be to find "computable" conditions of regularity of the corresponding Skorokhod maps.

Definition 1 The Skorokhod map $\Gamma : u(\cdot) \to x(\cdot)$ associated with an *SP* is defined on a class of continuous inputs $u(t)$, $u(0) \in Z$, its output $x(t)$ coincides with $u(t)$ as long as $u(t)$ remains within Z, in the general case it satisfies the *integral reflection conditions*, see, for example, [13]:

1. $u(t) + y(t) = x(t)$, $t \in [0, T]$,
2. $|y|(T) < \infty$,
3. $|y(t)| = \int_{(0,t]} I_{x(t) \in \partial Z} d|y(t)|$,
4. $y(t) = \int_{(0,t]} \gamma(s) d|y|(s)$, where $\gamma(s) \in D(x(s))$ for almost all $s \in [0, t]$ with respect to $d|y|$.

Here by $|y|(t)$ we denote the full variation of $y(\cdot)$ on $[0, t]$.

It is often required that the class of feasible compensations was not restricted to functions of bounded variation; this situation is typical, for instance, if the characteristic set Z has empty interior. The following definition does not require boundedness of variation of the compensation $y(t)$ and is the least restrictive definition of a solution to an SP.

Definition 2 Suppose $u(t)$, $0 \leq t \leq T$, is a continuous input to the SP and $u(0) \in Z$. The solution of SP is a pair of continuous functions $x(t), y(t) = x(t) - u(t)$, $0 \leq t \leq T$, such that $x(0) = u(0)$ and, for any, $0 \leq t' < t'' \leq T$, the following inclusion holds:

$$y(t'') - y(t') \in \overline{\text{co}\{D(x(t)) : t \in [t', t'']\}}.$$

The above means that the total compensation $y(t'') - y(t')$ on an arbitrary segment $[t', t'']$ belongs to the closed convex hull of the union of reflection cones $D(x(t))$, $t \in [t', t'']$.

Definition 3 We say that an SP $\{Z, D(\cdot)\}$ is *closed* if the map $D(x)$ is upper semicontinuous on $\mathbb{R}^n$ (that is its graph is closed in $\mathbb{R}^n \times \mathbb{R}^n$) and its values $D(x)$ are convex closed cones for all $x \in \partial Z$.

Theorem 4 *Suppose the pair $(u(t), x(t))$ is a solution of a closed SP according to Definition 2 and the difference $y(t) = x(t) - u(t)$ is a function of bounded variation. Then it is also a solution according to Definition 1.*

Any polyhedral SP is, obviously, closed. Moreover, if the interior of Z is not empty and the cones $D(x)$ do not contain whole lines, that is are *pointed*, the total variation of any compensation $y(t)$ is necessarily bounded on a finite time interval. Thus, for the above class of polyhedral SP's (we will call them *pointed SP's*) the two definitions of solutions are equivalent.

2. ASSOCIATED PROJECTION SYSTEMS

For the analysis of different kinds of regularity of polyhedral SP's such as existence, uniqueness or Lipschitz continuous dependence of the output on the input, we will need several additional notions.

Definition 5 For a pair $\{p, d\}$, $p, d \in \mathbb{R}^n$, $\|p\| = 1$, $\langle d, p\rangle > 0$, the linear operator

$$P_{n,d}x = x - \frac{d\langle p, x\rangle}{\langle d, p\rangle}$$

is called a projection onto the plane $L = \{x : \langle p, x\rangle = 0\}$ along the direction d.

Definition 6 For any $x \in Z$, the finite set of projections

$$\mathcal{P}(x) = \{P_{n_i,d_i} : i \in I(x)\},$$

where $I(x) = \{i : \langle n_i, x\rangle = c_i\}$, is called an associated projection system for the SP $\{n_i, d_i, c_i\}$ at the point x.

In particular, the set $\mathcal{P}(x)$ is empty whenever $x \in \text{int}(Z)$. Clearly, there exists only a finite number of APS's for a given polyhedral SP, namely, one for each face of Z of any dimension. The union $\mathcal{P}$ of all possible APS's will be called the *maximal APS* of the SP.

Together with the projection systems we shall also consider finite systems of general $n \times n$-matrices $\mathcal{A} = \{A_1, \ldots, A_m\}$.

Definition 7 Let $\mathcal{N}$ be a segment of integers. A sequence $\{x_k\}$, $k \in \mathcal{N}$ is called a path of the system $\mathcal{A}$ if, for any $k \in \mathcal{N}$, $k+1 \in \mathcal{N}$, there exists an $A \in \mathcal{A}$ such that $x_{k+1} = Ax_k$.

For projection systems we shall also consider continuous paths. Let us denote by $V(x)$ the following finite set of vectors: $d_i \in V(x)$ iff $\langle n_i, d_i\rangle \geq 0$ and $-d_i \in V(x)$ iff $\langle n_i, d_i\rangle \leq 0$.

Definition 8 A continuous function $u(t) : [0, T] \to \mathbb{R}^n$ is called a continuous path of the system $\mathcal{P}$ if, for any t_1, t_2, $0 \leq t_1 \leq t_2 \leq T$, the difference $u(t_2) - u(t_1)$ belongs to the set $D(u(\cdot), t_1, t_2)$ which is the minimal closed convex cone containing all the sets $V(u(t))$, $t \in [t_1, t_2]$.

3. STABLE SYSTEMS OF MATRICES

Here we introduce important notions of stability type for finite collections of $n \times n$-matrices.

Definition 9 A family $\mathcal{A} = \{A_1, \ldots, A_k\}$ is called *absolutely stable* or just *stable* if there exists $M > 0$ such that

$$\|A_{i_1} \ldots A_{i_m}\| \leq M$$

for all $m > 0$ and $1 \leq i_j \leq k$, $j = 1, \ldots, m$. Here $\|\cdot\|$ is a matrix norm in $\mathbb{R}^{n\times n}$ (they are all equivalent to each other). In this case we will also say that the difference equation

$$x_{m+1} = A_{i_m} x_m \tag{1}$$

is absolutely stable in the class of matrices

$$\mathcal{A} = \{A_1, A_2, \ldots, A_k\}.$$

The set of absolutely stable systems of $n \times n$-matrices will be denoted AS.

Theorem 10 *A finite system of matrices is not absolutely stable iff it has an unbounded path.*

PROOF. One half of the assertion is obvious. Let now $\mathcal{A}$ be an unstable system of matrices. Let us define a function

$$F(x) = \sup_{\{x, x_1, \ldots, x_m\}} \|x_m\|.$$

Here the supremum is taken over all the finite trajectories $\{x, x_1, \ldots, x_m\}$ of the system $\mathcal{A}$ that start at x. The function F is convex and it is defined on a linear subspace $L \in \mathbb{R}^n$. Being a convex function, it is bounded on the intersection of L with any compact set. Let us choose an arbitrary point $x \notin L$ and demonstrate the existence of a finite path of the system that starts at x and terminates within $y \notin L$, $\|y\| > 2\|x\|$. Indeed, otherwise all the trajectories from x would enter the subspace L without leaving the ball $\{z : \|z\| \leq 2\|x\| + 2\sup_i \|A_i\|\|x\|\}$. However, the function F is equibounded on the intersection of L with this ball which means that the variations of all the paths from x are uniformly bounded.

Iterating this construction we will get the required trajectory. □

Definition 11 The system $\mathcal{A}$ is called *asymptotically absolutely stable* (AAS) if, for any sequence of indices $\{i_m\}$, any solution x_m of the corresponding equation (1) vanishes as $m \to \infty$. The set of all AAS systems will be denoted AAS.

System of projections are never asymptotically absolutely stable, this is why we introduce the following:

Definition 12 A system $\mathcal{A}$ will be called *r-asymptotically stable* (RS) if, for any regular sequence of indices $\{i_m\}$, any solution x_m of the associated equation (1) vanishes as $m \to \infty$. The set of all RS-systems will be denoted RS.

Let us introduce some additional properties of families of matrices.

Definition 13 A system $\mathcal{A}$ will be called *variationally stable* or *BV-stable* if all its left-finite paths $\{x_0, x_1, \ldots\}$ have bounded variation, that is $\sum_{i=0,1,\ldots} |x_{i+1} - x_i| < \infty$. The set of all variationally stable systems will be denoted BV.

Definition 14 A sequence $\{x_0, x_1, \ldots\}$ is called an *ε-path* of the system $\mathcal{A}$ if there exists a sequence of vectors $h_i \in \mathbb{R}^n$, $i = 0, 1, \ldots$, such that $\|h_i\| \leq \varepsilon$ for any i and

$$x_{i+1} = h_i + A_{j_i}(x_i - h_i), \qquad i = 0, 1, \ldots$$

Definition 15 A system $\mathcal{A}$ is *Perron stable* if all its 1-paths starting at the origin are bounded. The set of all Perron stable systems will be denoted PS.

The class AAS belongs to all other four classes, the class AS contains all other four classes. The class AAS is open with respect to the finite-dimensional topology on the set of its parameters. The closures of all the five classes are the same and coincide with the set of systems with spectral radius less or equal to one.

Now, let us study the relations between the classes BV, PS, and RS. The complements to these classe will be denoted $\overline{BV}$, $\overline{PS}$, and $\overline{RS}$ correspondingly.

Theorem 16 *Any one of eight different combinations of these three classes is non-empty; moreover, there exist systems within each one of these combinations that are absolutely continuous but not asymptotically absolutely continuous.*

PROOF. Let us construct eight examples. In all these examples $n = 2$ and the matrices of the system $\mathcal{A}$ are oblique projection matrices. Any projection matrix can be completely defined by two vectors: a normal vector p to the hyperplane of projection and a vector d of the direction of projection. For instance, the projection $((0,1),(0,1))$ is the orthogonal projection onto the x-axis.

1) A system in (BV, PS, RS) consists of the two projections $((1,0),(1,0))$ and $((0,1),(0,1))$, that is orthogonal projections onto coordinate axes.

2) A system in (BV, PS, $\overline{RS}$) consists of one projection (anyone).

3) A system in (BV, $\overline{PS}$, RS) consists of the three projections $((1,0),(1,0))$, $((0,1),(0,1))$ and $((0,1),(1,1))$, that is, it can be constructed as System 1) plus an oblique projection onto the x-axis.

4) A system in (BV, $\overline{PS}$, $\overline{RS}$) consists of the last two projections of the previous system.

5) A system in ($\overline{BV}$, PS, RS) consists of the three projections $((1,0),(1,0))$, $((1,1),(0,1))$, and $((-1,1),(0,1))$.

6) A system in ($\overline{BV}$, PS, $\overline{RS}$) consists of the last two projections of the previous system.

7) A system in ($\overline{BV}$, $\overline{PS}$, RS) consists of the three projections of System 5 plus the projection $((0,1),(1,1))$.

8) A system in ($\overline{BV}$, $\overline{PS}$, $\overline{RS}$) consists of the two projections $((0,1),(1,1))$ and $((1,0),(-1,1))$. $\square$

We will show that the class BV coincides with the class of so called LCP-systems (see [3, 8]). An LCP-system is a set of matrices $\mathcal{A}$ such that any sequence $\{M_1, M_2, \ldots\}$ of matrices $M_k = A_k A_{k-1} \ldots A_1$, $A_i \in S$, $i = 1, 2, \ldots, k$, has a limit. For absolutely stable systems this means that any path of the system converges (not necessarily to the origin).

Theorem 17 *The system $\mathcal{A} = \{A_1, \ldots, A_k\}$ possesses the BV-property iff it is an LCP-system.*

4. UNIQUE SOLVABILITY AND CONTINUITY OF THE SKOROKHOD MAP

Let us note that the question of unique solvability of a polyhedral SP is local, that is we can assume that Z is a cone, namely, all the possible tangent cones of the original polyhedral set has to qualify. If Z is a convex polyhedral cone, the APS at its vertex includes any other one of its APS's.

If we accept Definition 2 of solution to an SP, the following assertion holds.

Theorem 18 *If, for any $x \in Z$, the associated system $\mathcal{P}(x)$ has no nontrivial continuous path $u(t)$, $0 \leq t \leq T$ starting at the origin then, for any continuous $u(t)$, $u(0) \in Z$, there is at most one solution $(x(t), y(t))$ of the SP corresponding to the input $u(t)$.*

PROOF. This follows from the obvious assertion that the difference $y(t) = y_2(t) - y_1(t)$ is a continuous path of the system $\mathcal{P}(y(\cdot), t_1, t_2)$. $\square$

Note that, in case of a conical Z, it is sufficient to consider only the APS at its vertex. Note also that, if we accept Definition 1 of solution to a Skorokhod problem, the sufficient condition of uniqueness will be changed to that of absence of a nontrivial continuous BV-trajectory from the origin. It is clear that BV-stability of the APS is sufficient for this.

Thus we have established sufficient conditions of the unique solvability of a polyhedral SP in terms of absence of special continuous trajectories from the origin. It is not, however, clear how to check these conditions. Further we will reduce some of these to more tangible conditions in terms of stability properties of APS's. We will need additional notions:

Definition 19 The *joint spectral radius* of a system $\mathcal{A}$ is defined as

$$\hat{\rho}(\mathcal{A}) = \limsup_{k\to\infty} \hat{\rho}_k(\mathcal{A}, \|\cdot\|)^{1/k},$$

where $\|\cdot\|$ is a matrix norm and

$$\hat{\rho}_k(\mathcal{A}, \|\cdot\|) = \sup\{\|A_{i_1} \ldots A_{i_k}\| : A_{i_j} \in \mathcal{A}\}.$$

The value of $\hat{\rho}(\mathcal{A})$ does not depend on the choice of the norm $\|\cdot\|$. The notion of joint spectral radius have been introduced in [24]. It has been shown lately ([3]) that for any finite system of matrices $\mathcal{A}$ the following is true:

$$\hat{\rho}(\mathcal{A}) = \overline{\rho}(\mathcal{A}),$$

where $\overline{\rho}(\mathcal{A})$ is the *generalized spectral radius* of the system $\mathcal{A}$:

$$\overline{\rho}(\mathcal{A}) = \limsup_{m\to\infty} \overline{\rho}_m(\mathcal{A})^{1/m}, \tag{2}$$

where

$$\overline{\rho}_m(\mathcal{A}) = \sup\{\rho(A_{i_1} \ldots A_{i_m}) : A_{i_j} \in \mathcal{A}\}.$$

Theorem 20 *A system $\mathcal{A}$ has a left-infinite trajectory from the origin iff its spectral radius is bigger than one.*

Let us say that a projection system satisfies the *transversality conditions* if, for any one its subsystems $\mathcal{B}$, the fixed subspace $L(\mathcal{B})$ is transversal to the subspace spanning the vectors d_i for all matrices $A_i \in \mathcal{B}$.

Theorem 21 *If the transversality conditions hold for all APS's of a given polyhedral SP, any continuous trajectory can be approximated by discrete ones with any accuracy.*

It follows from Theorem 21 that a stable transversal system of projections has no nontrivial continuous trajectories from the origin. Indeed, one has only to recall that the stability implies the existence of a norm that does not increase along discrete trajectories of the system. Theorem 21 implies than that it does not increase along continuous trajectories either.

Theorem 22 *If a solution of a Skorokhod problem is unique then it depends continuously on the input with respect to the uniform metric.*

We do not mention existence questions here; note only that in many cases algebraic conditions of existence in terms of the matrix $A = \{a_{ij}\} = \{\langle n_i, d_j \rangle\}$ are known to hold [22].

6. LIPSCHITZ CONTINUITY

It has been shown in [10] that the Skorokhod map Γ is Lipschitz continuous if there exists a special norm in $\mathbb{R}^n$. Its unit ball B has to satisfy the following conditions: If $\nu(z)$ denotes the set of inward normals to B at $z \in \partial B$, then there exists $\delta > 0$ such that for $i = 1, \ldots, N$,

$$\left\{ \begin{array}{l} z \in \partial B \\ |\langle z, n_i \rangle| < \delta \end{array} \right\} \Rightarrow \langle \nu, d_i \rangle = 0$$

for all $\nu \in \nu(z)$.

These conditions are equivalent to the Perron stability of the union of all APS's of the SP. This property holds, for instance, in the normal case. In [27], an upper bound for the corresponding Lipschitz constant has been found, see also [21].

Theorem 23 *A Skorokhod map is Lipschitz continuous whenever its maximal APS is Perron stable. The reverse statement holds if the Skorokhod problem is pointed.*

The sufficiency follows from [10]. The necessity can be proved as follows. Suppose the associated projection system is not Perron stable. Taking any finite 1-trajectory of the projection system with the norm C of its endpoint as a basis, we can construct two piecewise linear inputs $u_1(t)$ and $u_2(t)$ and two corresponding outputs $x_1(t)$, $x_2(t)$ such that

$$\|x_1(t) - x_2(t)\| \geq C\|u_1(t) - u_2(t)\|.$$

7. DUALITY

It is well known that a system $\{A_i\}$ is absolutely stable iff the dual system $\{A_i^*\}$ is absolutely stable. The same is true for AAS-systems and RS-systems. The following theorem shows that the notions of BV-stability and Perron stability are dual to each other.

Theorem 24 *The BV-property of the system $\mathcal{A}^*$ is equivalent to the PS-property of the system $\mathcal{A}$ and vice-versa.*

PROOF. Let us consider a finite 1-trajectory $\{x_0, x_1, \ldots, x_{n+1}\}$ and write the following relations: $x_{m+1} = v_m + A_m(x_m - v_m)$, $\|v_m\| \leq 1$, $m = 0, \ldots, n$. Here, for simplicity, we denote by A_i matrices of the family $\mathcal{A}$. After transformations, we get

$$x_{m+1} = v_m + C_m^m(v_{m-1} - v_m) + C_{m-1}^m(v_{m-2} - v_{m-1}) + \ldots + C_0^m(x_0 - v_0), \quad m = 0, \ldots, n, \qquad (3)$$

where $C_i^m = A_m A_{m-1} \ldots A_i$. Let us rewrite (3) in the form

$$x_{m+1} = (I - C_m^m)v_m + (C_m^m - C_{m-1}^m)v_{m-1} + \ldots + (C_1^m - C_0^m)v_0 + C_0^m x_0.$$

By assumption, norms of different points x_{m+1} have to be uniformly bounded for a fixed x_0 and any choice of matrices A_i from $\mathcal{A}$. This is equivalent to the following. First, the semigroup $\mathcal{L}(\mathcal{A})$ is bounded, and second, all the sums of the form $\|C_m^m - C_{m-1}^m\| + \ldots + \|C_1^m - C_0^m\|$ also have to be unifromly bounded. The latter implies that the total variation of any right-infinite matrix product is unifromly bounded as well. This is, in its turn, equivalent to the unifrom boundedness of variations of left-infinite products of conjugated matrices. Finally, the latter property is equivalent to the BV-property of the system $\mathcal{A}^*$. □

8. EXAMPLE: PROCESSOR SHARING POLICY

The following is an example from [23]. Consider a processor with capacity c providing services to n customers. A processor sharing policy can be identified with a probability vector $\rho = (\rho_1, \rho_2, \ldots, \rho_n)$. The inputs are buffered if immediate processing is not possible. When a positive number of buffers are empty the processing capacity is distributed among remaining buffers in proportion with ρ.

Suppose the inputs $\theta_i(t)$ are non-decreasing absolutely continuous functions. Define ψ by $\dot{\psi}(t) = \dot{\theta}(t) - c\rho$ and $\psi(0) = \theta(0) \in \mathbb{R}_+^n$. The vector function $\psi(t)$ is the input to the polyhedral SP whith $Z = \mathbb{R}_+^n$, $d_i = (e_i - \rho)/(1 - \rho_i)$, where e_i is the i-th unit coordinate vector.

It is demonstrated in [23] that the above SP is well defined and its Skorokhod map is Lipschitz continuous. On this basis, a series of results concerning large deviations is proved.

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