Math. Nachr. **191** (1998), 59 – 79

Boundedness and Dissipativity of Truncated Rotations on Uniform Planar Lattices

By Phil Diamond of Brisbane, Peter Kloeden of Geelong, Victor Kozyakin of Moscow, Alexei Pokrovskii of Geelong

(Received October 20, 1995)

Abstract. The finiteness of computer arithmetic can lead to some dramatic differences between the behaviour of a continuous dynamical system and a computer simulation. A thorough rigorous theoretical analysis of what may or what does happen is usually extremely difficult and to date little has been done even in relatively simple contexts. The comparative behaviour of a rotation mapping in the plane and on a uniform lattice in the plane is one such example. Simulations show that the rounding operator applied to a planar rotation mapping more or less preserves the qualitative behaviour of the original mapping, whereas the application of the truncation operator to a planar rotation can lead to quite different dynamical features. In the paper a theoretical justification of the properties of the planar rotation mappings under truncation to a uniform integer lattice is provided, in particular properties of boundedness and dissipativity are investigated.

1. Introduction

The finiteness of computer arithmetic can lead to some dramatic differences between the behaviour of a continuous dynamical system and a computer simulation, such as the collapsing of chaotic behaviour onto simple cyclic behaviour and the occurrence of spurious equilibria and cycles, see for example [B, KK]. This is, in fact, characteristic of the spatial discretization of dynamical systems in general, that is to arbitrary discretions of a state space continuum, and is not just restricted to computer arithmetic fields. A thorough rigorous theoretical analysis of what may or what does happen is usually extremely difficult and to date little has been done even in relatively simple contexts. The comparative behaviour of a rotation mapping in the plane and on a uniform lattice in the plane is one such example. Simulations presented in [KK] show that the rounding operator applied to a planar rotation mapping more or less preserves the qualitative behaviour of the original mapping, whereas the application of the truncation operator to a planar rotation can lead to quite different dynamical features (for example, see Figure 3 below).

The purpose of this paper is to provide a theoretical justification of the properties of the planar rotation mappings under truncation to a uniform integer lattice,

1991 *Mathematics Subject Classification.* Primary 65C20; Secondary 52C05, 68U20
Keywords and phrases. Rotation mapping, truncation, dissipativity, boundedness, lattices.

in particular properties of boundedness and dissipativity. The paper consists of five Sections. Section 2 gives the main definitions and states conditions for the boundedness of a general class of planar mappings under truncation (Theorem 2.1), which are then specialized to planar rotation mappings (Theorem 2.2). Properties of the truncation operator in the plane $\mathbf{R}^2$ are then investigated in Section 3, while the proofs of Theorems 2.1 and 2.2 are given in Section 4 The properties of periodic trajectories of truncated rotation mappings needed for the proof of Theorem 6.2 are investigated in Section 5 and, finally, Theorems 6.1 and 6.2 on dissipativity of truncated rotation mapping are proved in Section 6

2. Boundedness

Let $\mathbf{Z} = \{m : \ m = 0, \pm1, \pm2, \ldots\}$ and $\mathbf{Z}^2 = \{(m,n) : \ m,n = 0, \pm1, \pm2, \ldots\}$ be integer valued lattices in $\mathbf{R}^1$ and $\mathbf{R}^2$, respectively. For a real number x let $[\![x]\!]$ denote the *truncation* of x, that is the integer closest to x for which the absolute value does not exceed that of x:

$$[\![x]\!] = i \in \mathbf{Z} : \quad |i - x| < 1, \ |i| \le |x|.$$

Analogously, define the *truncation* of the point $x = (x_1, x_2) \in \mathbf{R}^2$ component-wise as

$$[\![x]\!] = ([\![x_1]\!], [\![x_2]\!]) \in \mathbf{Z}^2.$$

Given $\gamma > 0$, define an *elliptical norm* $\|\cdot\|_\gamma$ on $\mathbf{R}^2$ by $\|x\|_\gamma = \sqrt{x_1^2 + \gamma^2 x_2^2}$ for $x = (x_1, x_2) \in \mathbf{R}^2$ and write

$$\mathbf{B}_\gamma(z^*, r) = \{x \in \mathbf{R}^2 : \ \|x - z^*\|_\gamma \le r\},$$

$$\partial\mathbf{B}_\gamma(z^*, r) = \{x \in \mathbf{R}^2 : \ \|x - z^*\|_\gamma = r\}.$$

The subscript will be omitted when $\gamma = 1$, that is $\|\cdot\|_1$ and $\mathbf{B}_1$ will be written simple as $\|\cdot\|$ and $\mathbf{B}$, respectively.

Let $W : \mathbf{R}^2 \mapsto \mathbf{R}^2$ be a mapping and denote its truncated counterpart by $W_{tr}(x) = [\![W(x)]\!]$ for $x \in \mathbf{R}^2$; clearly, $W_{tr} : \mathbf{Z}^2 \mapsto \mathbf{Z}^2$. The following Theorem, which will be proved in Section 4, shows the relationship between properties of such a mapping W and its truncation counterpart W_{tr}.

Theorem 2.1. *Let $z^* \in \mathbf{Z}^2$ and suppose that the mapping $W : \mathbf{R}^2 \mapsto \mathbf{R}^2$ satisfies*

$$\|W(x) - z^*\|_\gamma \le \|x - z^*\|_\gamma, \qquad x \in \mathbf{R}^2 \tag{2.1}$$

for some $\gamma > 0$. Then for any integer $i \ge 2\max\left\{(1+\gamma^{-1})(z_1^*)^2, 2\gamma^2(z_2^*)^2\right\}$ there exists a number $r \in [i, i+1)$ such that

$$W_{tr}(\mathbf{B}_\gamma(z^*, r) \bigcap \mathbf{Z}^2) \subseteq \mathbf{B}_\gamma(z^*, r) \bigcap \mathbf{Z}^2.$$

Moreover, if $z^* = 0$, then

$$W_{tr}(\mathbf{B}_\gamma(0, r)) \subseteq \mathbf{B}_\gamma(0, r) \bigcap \mathbf{Z}^2$$

for any $r \geq 0$.

This Theorem says that any trajectory

$$z(n+1) = W_{tr}(z(n)), \qquad n = 0, 1, 2, \dots, z(n) \in \mathbf{Z}^2,$$

of the mapping W_{tr} is bounded, that is the dynamical system generated by this mapping is *Lagrange stable* (cf. [S]).

This paper will focus on a specific class of mappings satisfying the conditions of Theorem 2.1, the rotation mappings. For a given point $z^* = (z_1^*, z_2^*) \in \mathbf{Z}^2$ and a given real number θ, the mapping $T_\theta(z^*, x) = (T_{\theta,1}(z^*, x), T_{\theta,2}(z^*, x))$ defined by

$$\begin{array}{rcl} T_{\theta,1}(z^*, x) & = & z_1^* + (x_1 - z_1^*)\cos\theta - (x_2 - z_2^*)\sin\theta, \\ T_{\theta,2}(z^*, x) & = & z_2^* + (x_1 - z_1^*)\sin\theta + (x_2 - z_2^*)\cos\theta, \end{array} \tag{2.2}$$

for all $x = (x_1, x_2) \in \mathbf{R}^2$ is the *rotation mapping* by angle θ centered at z^*. Clearly, $T_\theta(z^*, \cdot)$ maps $\mathbf{R}^2$ into itself and satisfies $\|T_\theta(z^*, x) - z^*\| = \|x - z^*\|$ for all $x \in \mathbf{R}^2$, while its truncated counterpart $U_\theta(z^*, \cdot) = [\![T_\theta(z^*, \cdot)]\!]$ with components

$$U_{\theta,1}(z^*, x) = [\![T_{\theta,1}(z^*, x)]\!], \qquad U_{\theta,2}(z^*, x) = [\![T_{\theta,2}(z^*, x)]\!] \tag{2.3}$$

for $x \in \mathbf{R}^2$ maps $\mathbf{Z}^2$ into itself.

From Theorem 2.1 it follows that there exists an $r \in [i, i+1)$ for any integer $i \geq 4\max\{(z_1^*)^2, (z_2^*)^2\}$ such that $U_\theta(z^*, \mathbf{B}(z^*, r)) \subseteq \mathbf{B}(z^*, r) \bigcap \mathbf{Z}^2$. However, more accurate estimates for the radii of invariant balls of the truncated rotation mapping $U_\theta(z^*, \cdot)$ can also be determined. Let $\mathbf{X}(\sigma)$ denote the cross set centered at the point z^*, defined by $\mathbf{X}(\sigma) = \mathbf{X}_1(\sigma) \bigcup \mathbf{X}_2(\sigma)$ where

$$\mathbf{X}_i(\sigma) = \left\{x = (x_1, x_2) \in \mathbf{R}^2 : \ |x_i - z_i^*| \leq \sigma\right\}, \qquad i = 1, 2.$$

Theorem 2.2. *Let* $\sigma \geq \max\{|z_1^*|, |z_2^*|\}$ *and let* $r > \frac{1}{2}\sigma^2 + 1$ *be such that* $r = \|z - z^*\|$ *for some* $z \in \mathbf{X}(\sigma) \bigcap \mathbf{Z}^2$. *Then*

$$U_\theta(z^*, \mathbf{B}(z^*, r)) \subseteq \mathbf{B}(z^*, r) \bigcap \mathbf{Z}^2.$$

If $z^* = 0$ *this inclusion holds for any* $r > 0$.

The proof will be given in Section 4

3. Properties of the Truncation Operator in $\mathbf{R}^2$

To prove Theorem 2.1 we need first to investigate some properties of the truncation operator $[\![\cdot]\!]$ and we shall assume without loss of generality that $z_1^*, z_2^* \geq 0$. For this we define some auxiliary sets associated with the lattice $\mathbf{Z}^2$, namely for

$$\begin{array}{rcl} \mathbf{R}_i^+(x) & = & \{x = (x_1, x_2) \in \mathbf{R}^2 : \ x_i \geq x\}, \\ \mathbf{R}_i^-(x) & = & \{x = (x_1, x_2) \in \mathbf{R}^2 : \ x_i \leq x\}, \\ \mathbf{L}_i(x) & = & \{x = (x_1, x_2) \in \mathbf{R}^2 : \ x_i = x\}, \\ \mathbf{C}_1^+ & = & \{x = (x_1, x_2) \in \mathbf{R}^2 : \ 0 \leq x_1 \leq z_1^*, \ z_2^* < x_2\}, \\ \mathbf{C}_1^- & = & \{x = (x_1, x_2) \in \mathbf{R}^2 : \ 0 \leq x_1 \leq z_1^*, \ x_2 < 0\}, \\ \mathbf{C}_2^+ & = & \{x = (x_1, x_2) \in \mathbf{R}^2 : \ z_1^* < x_1, \ 0 \leq x_2 \leq z_2^*\}, \\ \mathbf{C}_2^- & = & \{x = (x_1, x_2) \in \mathbf{R}^2 : \ x_1 < 0, \ 0 \leq x_2 \leq z_2^*\}, \\ \mathbf{C}_0 & = & \{x = (x_1, x_2) \in \mathbf{R}^2 : \ 0 \leq x_1 \leq z_1^*, \ 0 \leq x_2 \leq z_2^*\}, \end{array} \tag{3.1}$$

for $i = 1$ and 2 (see Figure 1) and also

$$\mathbf{C}_1 = \mathbf{C}_1^+ \bigcup \mathbf{C}_1^- \bigcup \mathbf{C}_0, \qquad \mathbf{C}_2 = \mathbf{C}_2^+ \bigcup \mathbf{C}_2^- \bigcup \mathbf{C}_0, \qquad \mathbf{C} = \mathbf{C}_1 \bigcup \mathbf{C}_2.$$

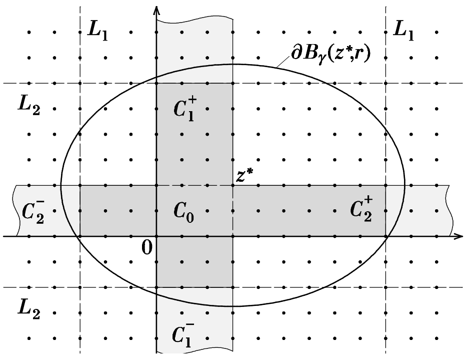

Figure 1: Location of the set $\partial \mathbf{B}_\gamma(z^*, r)$, the strips $\mathbf{C}_1^\pm, \mathbf{C}_2^\pm$ and the lines $\mathbf{L}_1(x), \mathbf{L}_2(x)$. The dark grey cross is the set $\mathbf{C}(r)$.

Lemma 3.1. *For $i = 1, 2$ and $j \in \mathbf{Z}$ the following inclusions are valid:*

$$[\![\mathbf{L}_i(j)]\!] \subseteq \mathbf{L}_i(j) \bigcap \mathbf{Z}^2, \qquad [\![\mathbf{C}_i]\!] \subseteq \mathbf{C}_i \bigcap \mathbf{Z}^2,$$

$$[\![\mathbf{R}_i^+(j)]\!] \subseteq \mathbf{R}_i^+(j) \bigcap \mathbf{Z}^2 \text{ for } j \le -1, \qquad [\![\mathbf{R}_i^-(j)]\!] \subseteq \mathbf{R}_i^-(j) \bigcap \mathbf{Z}^2 \text{ for } j \ge 1.$$

Proof. By direct calculation. □

Denote the interior of the set $\mathbf{C}$ by $\mathrm{int}(\mathbf{C})$.

Lemma 3.2. *If $x \in \mathbf{R}^2 \backslash \mathrm{int}(\mathbf{C})$, then*

$$\|[\![x]\!] - z^*\|_\gamma \le \|x - z^*\|_\gamma \tag{3.2}$$

for any $\gamma > 0$. If, in addition, $[\![x]\!] \ne x$ and $\|x - z^*\|^2$ is an integer, then

$$\|[\![x]\!] - z^*\|^2 \le \|x - z^*\|^2 - 1.$$

Proof. If $x \in \mathbf{R}^2 \backslash \mathrm{int}(\mathbf{C})$, then the inequalities $|[\![x_i]\!] - z_i^*| \le |x_i - z_i^*|$ for $i = 1$ and 2 hold, from which (3.2) follows immediately .

If $[\![x]\!] \ne x$ and $\|x - z^*\|^2$ is an integer, then $\|[\![x]\!] - z^*\|^2$ is also integer valued and by the inequality (3.2) just proved these values are different. Hence they differ at least by 1, that is $\|[\![x]\!] - z^*\|^2 \le \|x - z^*\|^2 - 1$. □

Lemma 3.3. *If $T_\theta(z^*, z) \notin \mathrm{int}(\mathbf{C})$ and $U_\theta(z^*, z) \ne T_\theta(z^*, z)$ for given $z, z^* \in \mathbf{Z}^2$, then $\|U_\theta(z^*, z) - z^*\|^2 \le \|z - z^*\|^2 - 1$.*

Proof. This follows immediately from Lemma 3.2 and the identity $U_\theta(z^*, z) \equiv [\![T_\theta(z^*, z)]\!]$. □

Lemma 3.2 shows that the truncation operator $[\![\cdot]\!]$ attracts vectors from $R^2\backslash\mathrm{int}(\mathbf{C})$ to the point z^* in any norm $\|\cdot\|_\gamma$. Clearly, the truncation operator can increase the distance of vectors from the point z^* if these vectors belong to $\mathrm{int}(\mathbf{C})$, though, as is seen from the next lemma, this increase is rather mild.

Lemma 3.4. *If* $x \in \mathrm{int}(\mathbf{C})\backslash\mathbf{C}_0$, *then*

$$\|[\![x]\!] - z^*\|_\gamma^2 \le \|x - z^*\|_\gamma^2 + 2\max\{|z_1^*|, \gamma|z_2^*|\}.$$

Proof. For any $z \in \mathbf{Z}^2$ let $\mathbf{H}(z)$ denote the set of all points in $\mathbf{R}^2$ which truncate to z denote, that is

$$\mathbf{H}(z) = \{w \in \mathbf{R}^2 : \; z = [\![w]\!]\}. \tag{3.3}$$

Consider first the case when $x = (x_1, x_2) \in \mathrm{int}(\mathbf{C}_2^+)$ and let $z = (z_1, z_2) = [\![x]\!]$. Then by Lemma 3.1 $z_1 \ge z_1^*$ and $0 \le z_2 \le z_2^*$, so

$$\mathbf{H}(z) = \{y = (y_1, y_2) \in \mathbf{R}^2 : \; z_1 \le y_1 < z_1 + 1, \; z_2 \le y_2 < z_2 + 1 \le z_2^*\}.$$

Finally denote

$$\beta = \inf_{y \in \mathbf{H}(z)} \|y - z^*\|_\gamma^2.$$

This infimum is attained for $y = (z_1, z_2+1)$, so $\beta = (z_1 - z_1^*)^2 + \gamma(z_2 + 1 - z_2^*)^2$. Since $z = [\![x]\!] \in \mathbf{H}(z)$, then

$$\begin{aligned} \|x - z^*\|_\gamma^2 &\ge \beta = ((z_1 - z_1^*)^2 + \gamma(z_2 - z_2^*)^2) + \\ &+ (z_1 - z_1^*)^2 + \gamma(z_2 + 1 - z_2^*)^2 - (z_1 - z_1^*)^2 + \gamma(z_2 - z_2^*)^2 = \\ &= \|[\![x]\!] - z^*\|_\gamma^2 + 2\gamma(z_2 - z_2^*) + \gamma \end{aligned}$$

and hence

$$\|[\![x]\!] - z^*\|_\gamma^2 \le \|x - z^*\|_\gamma^2 + 2\gamma|z_2^*| \le \|x - z^*\|^2 + 2\max\{|z_1^*|, \gamma|z_2^*|\}. \tag{3.4}$$

As is easy to see, the same inequality is also valid in the case when $x = (x_1, x_2) \in \mathrm{int}(\mathbf{C}_2^-)$.

Analogously, in the cases when $x \in \mathrm{int}(\mathbf{C}_1^\pm)$ the following inequality can be proved:

$$\|[\![x]\!] - z^*\|_\gamma^2 \le \|x - z^*\|_\gamma^2 + 2|z_1^*| \le \|x - z^*\|^2 + 2\max\{|z_1^*|, \gamma|z_2^*|\}. \tag{3.5}$$

The assertion of Lemma then follows from inequalities (3.4), (3.5). □

A real number $r \ge 0$ will be called *γ–proper* for a given $\gamma > 0$ if neither of the sets $\partial\mathbf{B}_\gamma(z^*, r) \bigcap \mathbf{C}_1^+$ and $\partial\mathbf{B}_\gamma(z^*, r) \bigcap \mathbf{C}_1^-$ intersects with any lines $\mathbf{L}_2(i)$ and if neither of the sets $\partial\mathbf{B}_\gamma(z^*, r) \bigcap \mathbf{C}_2^+$ and $\partial\mathbf{B}_\gamma(z^*, r) \bigcap \mathbf{C}_2^-$ intersects with any lines $\mathbf{L}_1(i)$ (see Figure 1).

Lemma 3.5. *For any $\gamma > 0$ and integer i satisfying*

$$i \geq 2 \max \left\{ (1+\gamma^{-1})(z_1^*)^2, 2\gamma^2 (z_2^*)^2 \right\} \tag{3.6}$$

there exists at least one γ–proper value $r \in [i, i+1)$.

Proof. Given an integer $j \geq 0$, denote by $\mathbf{J}_1(j)$ the set of real numbers r for which the set $\partial \mathbf{B}_\gamma(z^*, r) \bigcap \mathbf{C}_2$ intersects the lines $\mathbf{L}_1(z_1^* \pm j)$ (because z_1^* is integer, both these lines then intersect the set $\partial \mathbf{B}_\gamma(z^*, r) \bigcap \mathbf{C}_2$ simultaneously). Analogously, denote by $\mathbf{J}_2(j)$ the set of r for which the set $\partial \mathbf{B}_\gamma(z^*, r) \bigcap \mathbf{C}_1$ intersects the lines $\mathbf{L}_2(z_2^* \pm i)$. Clearly, a number $r \geq \|z^*\|_\gamma$ is γ–proper if and only if

$$r \notin \left(\bigcup_{j \geq 0} \mathbf{J}_1(j) \right) \bigcup \left(\bigcup_{j \geq 0} \mathbf{J}_2(j) \right). \tag{3.7}$$

In order to prove that γ–proper values r can be found in any interval $[i, i+1)$ for i sufficiently large, we estimate the Lebesgue measure of the sets $\mathbf{J}_1(j)$ and $\mathbf{J}_2(j)$. The set $\mathbf{J}_1(j)$ is the interval $\mathbf{J}_1(j) = \left[j,\ \sqrt{j^2 + \gamma^2 (z_2^*)^2} \right]$, so its Lebesgue measure satisfies

$$\operatorname{mes} \mathbf{J}_1(j) = \sqrt{j^2 + \gamma^2 (z_2^*)^2} - j < \frac{\gamma^2 (z_2^*)^2}{\sqrt{j^2 + \gamma^2 (z_2^*)^2}}. \tag{3.8}$$

If the interval $\mathbf{J}_1(j)$ has nonempty intersection with the interval $[i, i+1)$, then $i \leq \sqrt{j^2 + \gamma^2 (z_2^*)^2}$. By (3.6), $4\gamma^2 (z_2^*)^2 \leq \sqrt{j^2 + \gamma^2 (z_2^*)^2}$ and hence by (3.8)

$$\operatorname{mes} \mathbf{J}_1(i) < \frac{1}{4}. \tag{3.9}$$

Analogously, the set $\mathbf{J}_2(j)$ $\left[\gamma j,\ \sqrt{(z_1^*)^2 + (\gamma j)^2} \right]$ and its Lebesgue measure can be estimated by

$$\operatorname{mes} \mathbf{J}_2(i) < \frac{(z_1^*)^2}{\sqrt{(z_1^*)^2 + (\gamma j)^2}}. \tag{3.10}$$

If the interval $\mathbf{J}_2(j)$ has nonempty intersection with the interval $[i, i+1)$, then $i \leq \sqrt{(z_1^*)^2 + (\gamma j)^2}$ and by (3.6)

$$2(1+\gamma)^{-1})(z_1^*)^2 \leq \sqrt{(z_1^*)^2 + (\gamma j)^2},$$

so by (3.10)

$$\operatorname{mes} \mathbf{J}_2(j) < \frac{\gamma}{2(1+\gamma)}. \tag{3.11}$$

Since the left-hand endpoints of the intervals $\mathbf{J}_1(j)$ are distributed periodically with period 1 over the real numbers for $j \geq 0$, an interval $[i, i+1)$ with i satisfying (3.6) can intersect no more than one interval $\mathbf{J}_1(j)$. Then by (3.9),

$$\operatorname{mes} \left([i, i+1) \bigcap \left(\bigcup_{j \geq 0} \mathbf{J}_1(j) \right) \right) < \frac{1}{4}. \tag{3.12}$$

Similarly, the left-hand endpoints of the intervals $\mathbf{J}_2(j)$ are distributed periodically with period $\gamma > 0$ over the real numbers for $j \geq 0$, so an interval $[i, i+1)$ with i satisfying (3.6) can intersect no more than $1 + \gamma^{-1}$ intervals $\mathbf{J}_2(j)$. Therefore, by (3.11),

$$\text{mes}\left([i, i+1) \bigcap \left(\bigcup_{j \geq 0} \mathbf{J}_2(j)\right)\right) < \frac{1}{2}. \tag{3.13}$$

From (3.12) and (3.13) we conclude that

$$\text{mes}\left([i, i+1) \bigcap \left(\left(\bigcup_{j \geq 0} \mathbf{J}_1(j)\right) \bigcup \left(\bigcup_{j \geq 0} \mathbf{J}_2(j)\right)\right)\right) < 1,$$

and thus there must exist an $r \in [i, i+1)$ satisfying (3.7). □

The following Lemma gives an explicit description of some invariant balls in the norm $\|\cdot\|$ for the truncation operator. Associate with a point $z = (z_1, z_2) \in \mathbf{Z}^2$ the set

$$\mathbf{D}(z) = \left\{(z_1^* \pm \widetilde{z}_1, z_2^* \pm \widetilde{z}_2)), \ (z_1^* \pm \widetilde{z}_2, z_2^* \pm \widetilde{z}_1)\right\}, \tag{3.14}$$

where $\widetilde{z}_1 = z_1 - z_1^*$, $\widetilde{z}_2 = z_2 - z_2^*$ (see Figure 2).

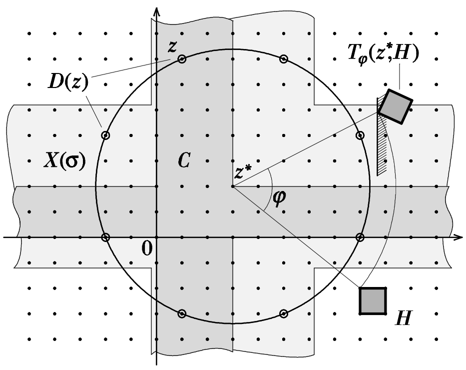

Figure 2: Location of the cross $\mathbf{X}(\sigma)$ (light grey), the cross $\mathbf{C}$ (grey), the elements of the set $\mathbf{D}(z)$ (encircled points) and the square $\mathbf{H}$ with its image $T_\varphi(z^*, \mathbf{H})$ (dark grey).

Lemma 3.6. *Let $r > \frac{1}{2}\sigma^2 + 1$ where $\sigma \geq \max\{|z_1^*|, |z_2^*|\}$. Then*
(i) the set $\partial\mathbf{B}(z^*, r) \bigcap \mathbf{X}(\sigma) \bigcap \mathbf{Z}^2$ contains no more than eight points.
If, in addition, the value of r is such that $r = \|z - z^*\|$ for some $z = (z_1, z_2) \in \mathbf{X}(\sigma) \bigcap \mathbf{Z}^2$, then
(ii) the set $\partial\mathbf{B}(z^*, r) \bigcap \mathbf{X}(\sigma) \bigcap \mathbf{Z}^2$ coincides with $\mathbf{D}(z)$ and so consists of exactly eight points;

(iii) the set $\partial\mathbf{B}(z^*, r)\bigcap\mathbf{C}\bigcap\mathbf{Z}^2$ *contains no more than four points from the set* $\mathbf{D}(z)$*;*
(iv) $[\![\mathbf{B}(z^*, r)]\!] \subseteq \mathbf{B}(z^*, r)\bigcap\mathbf{Z}^2$.

Proof. Direct calculations show that the circle $\partial\mathbf{B}(z^*, r)$ with $r > \frac{1}{2}\sigma^2 + 1$ can intersect only four different intervals of the form $\mathbf{L}_1(i)\bigcap\mathbf{X}_2(\sigma)$ or $\mathbf{L}_2(i)\bigcap\mathbf{X}_1(\sigma)$ in no more than 2 points each. From the statement (i) of Lemma follows.

If $r = \|z - z^*\|$ for some $z = (z_1, z_2) \in \mathbf{X}(\sigma)\bigcap\mathbf{Z}^2$, then each point from $\mathbf{D}(z)$ belongs to the set $\partial\mathbf{B}(z^*, r)\bigcap\mathbf{X}(\sigma)\bigcap\mathbf{Z}^2$ and simultaneously to some interval $\mathbf{L}_1(i)\bigcap\mathbf{X}_2(\sigma)$ or $\mathbf{L}_2(i)\bigcap\mathbf{X}_1(\sigma)$ with integer i. Statement (ii) follows from this and statement (i).

Statement (iii) follows directly from the statement (ii) since no more than half of the points from $\mathbf{D}(z)$ can belong to the set $\mathbf{C} \subset \mathbf{X}(\sigma)$ (see Figure 2).

To prove statement (iv), note that $\frac{1}{2}\sigma^2 + 1 \geq \sqrt{(z_1^*)^2 + (z_2^*)^2}$ for $\sigma \geq \max\{|z_1^*|, |z_2^*|\}$, so

$$\mathbf{C}_0 \subseteq \mathbf{B}(z^*, r). \tag{3.15}$$

Let $x \in \mathbf{B}(z^*, r)$ and consider the following cases: $x \in \mathbf{C}_0$; $x \notin \operatorname{int}(\mathbf{C})$; and $x \in \operatorname{int}(\mathbf{C})\backslash\mathbf{C}_0$. When $x \in \mathbf{C}_0$, statement (iv) follows from Lemma 3.1 and the inclusion (3.15) since

$$[\![\mathbf{C}_0]\!] \subseteq [\![\mathbf{C}_1]\!]\bigcap[\![\mathbf{C}_2]\!] \subseteq \mathbf{C}_1\bigcap\mathbf{C}_2\bigcap\mathbf{Z}^2 \subseteq \mathbf{B}(z^*, r)\bigcap\mathbf{Z}^2$$

and when $x \notin \operatorname{int}(\mathbf{C})$ it follows from Lemma 3.2. In the final case, $x \in \operatorname{int}(\mathbf{C})\backslash\mathbf{C}_0$, but for definiteness let $x \in \mathbf{C}_1^+$, that is $0 \leq x_1 \leq z_1^*$ and $x_2 \geq z_2^* \geq 0$. Suppose that the statement of Lemma is not valid, that is $\|x - z^*\| \leq r$ and at the same time $\|[\![x]\!] - z^*\| > r$. Then the circle $\partial\mathbf{B}(z^*, r)$ intersects the interval $\mathbf{L}_2([\![x_2]\!])\bigcap\mathbf{C}_1$ at a point with non-integer first coordinate. But according to the already proved statement (ii) of the Lemma, the circle $\partial\mathbf{B}(z^*, r)$ can intersect the intervals $\mathbf{L}_1(i)\bigcap\mathbf{C}_2$ and $\mathbf{L}_2(i)\bigcap\mathbf{C}_1$ only at the points in $\mathbf{D}(z)$. Since all of these have integer valued coordinates, we have obtained a contradiction and so statement (iv) must be valid. □

Partition the set $\mathbf{X}(\sigma)$ into disjoint subsets

$$\mathbf{X}(\sigma) = \mathbf{X}_1^+(\sigma)\bigcup\mathbf{X}_1^-(\sigma)\bigcup\mathbf{X}_2^+(\sigma)\bigcup\mathbf{X}_2^-(\sigma)\bigcup\mathbf{X}_0(\sigma)$$

defined by

$$\begin{aligned}
\mathbf{X}_1^+(\sigma) &= \{x = (x_1, x_2) \in \mathbf{R}^2 :\ |x_1 - z_1^*| \leq \sigma,\ z_2^* + \sigma < x_2\},\\
\mathbf{X}_1^+(\sigma) &= \{x = (x_1, x_2) \in \mathbf{R}^2 :\ |x_1 - z_1^*| \leq \sigma,\ x_2 < z_2^* - \sigma\},\\
\mathbf{X}_2^+(\sigma) &= \{x = (x_1, x_2) \in \mathbf{R}^2 :\ z_1^* + \sigma < x_1,\ |x_2 - z_2^*| \leq \sigma\},\\
\mathbf{X}_2^+(\sigma) &= \{x = (x_1, x_2) \in \mathbf{R}^2 :\ x_1 < z_1^* + \sigma,\ |x_2 - z_2^*| \leq \sigma\},\\
\mathbf{X}_0(\sigma) &= \{x = (x_1, x_2) \in \mathbf{R}^2 :\ |x_1 - z_1^*| \leq \sigma,\ |x_2 - z_2^*| \leq \sigma\}.
\end{aligned}$$

Lemma 3.7. *Let* $\sigma > 0$ *be such that* $\mathbf{C} \subseteq \mathbf{X}(\sigma)$*. If* $z \in \mathbf{Z}^2\backslash\mathbf{X}(\sigma)$ *and* $u = T_\varphi(z^*, z) \in \mathbf{X}(\sigma)$ *for some* φ*, then*

$$u \in \mathbf{X}(\sigma)\backslash\mathbf{X}_0(\sigma) = \mathbf{X}_1^+(\sigma)\bigcup\mathbf{X}_1^-(\sigma)\bigcup\mathbf{X}_2^+(\sigma)\bigcup\mathbf{X}_2^-(\sigma)$$

and (see Figure 2)

$$T_\varphi(z^*, \mathbf{H}(z))\backslash\{u\} \subseteq \operatorname{int}(\mathbf{R}_1^\pm(u_1)) \quad \text{if} \quad u \in \mathbf{X}_2^\pm(\sigma),$$

$$T_\varphi(z^*, \mathbf{H}(z))\backslash\{u\} \subseteq \mathrm{int}(\mathbf{R}_2^\pm(u_2)) \quad \text{if} \quad u \in \mathbf{X}_1^\pm(\sigma).$$

Proof. By direct calculation. □

4. Proofs of Theorems 2.1 and 2.2

4.1. Proof of Theorem 2.1

Given $i > 0$ satisfying the condition of the Theorem, cf. (3.6), choose a γ–proper number $r \in [i, i+1)$ in accordance with Lemma 3.5. If the point $x \in \mathbf{B}_\gamma(z^*, r)$ does not belong to $\mathrm{int}(\mathbf{C})$, then by Lemma 3.1 $\|[\![x]\!] - z^*\|_\gamma \le \|x - z^*\|_\gamma$ and so $[\![x]\!] \in \mathbf{B}_\gamma(z^*, r) \bigcap \mathbf{Z}^2$.

If the point x belongs to the finite cross

$$\mathbf{C}(r) = \mathbf{C} \bigcap \mathbf{R}_1^+(\mu_1^-) \bigcap \mathbf{R}_1^-(\mu_1^+) \bigcap \mathbf{R}_2^+(\mu_2^-) \bigcap \mathbf{R}_2^-(\mu_2^+)$$

where

$$\begin{aligned} \mu_1^+ &= \max\{i \in \mathbf{Z}:\ i > z_1^*,\ (i, z_2^*) \in \mathbf{B}_\gamma(z^*, r)\},\\ \mu_1^- &= \min\{i \in \mathbf{Z}:\ i < z_1^*,\ (i, z_2^*) \in \mathbf{B}_\gamma(z^*, r)\},\\ \mu_2^+ &= \max\{i \in \mathbf{Z}:\ i > z_2^*,\ (z_1^*, i) \in \mathbf{B}_\gamma(z^*, r)\},\\ \mu_2^- &= \min\{i \in \mathbf{Z}:\ i < z_2^*,\ (z_1^*, i) \in \mathbf{B}_\gamma(z^*, r)\}. \end{aligned}$$

(see Figure 1), then $[\![x]\!] \in \mathbf{C}(r) \bigcap \mathbf{Z}^2 \subseteq \mathbf{B}_\gamma(z^*, r) \bigcap \mathbf{Z}^2$.

Finally, if the point $x = (x_1, x_2)$ belongs to the set $(\mathbf{B}_\gamma(z^*, r) \bigcup \mathbf{C}) \setminus \mathbf{C}(r)$, say $x \in \mathbf{B}_\gamma(z^*, r) \bigcup \mathbf{C}_2^+$, then by Lemma 3.5 $[\![x_1]\!] = \mu_1^+$ and $0 \le [\![x_2]\!] \le z_2^*$, so $[\![x]\!] \in \mathbf{B}_\gamma(z^*, r) \bigcap \mathbf{Z}^2$. The Theorem is thus proved in the case $z^* \ne 0$.

For $z^* = 0$ it suffices to repeat the reasoning above for the case that $x \in \mathbf{B}_\gamma(z^*, r)$ does not belong to $\mathrm{int}(\mathbf{C})$. This completes the proof of Theorem 2.1.

4.2. Proof of Theorem 2.2

Let real r satisfies conditions of Theorem 2.2. By definition of the rotation mapping $T_\theta(z^*, \cdot)$, $\|T_\theta(z^*, x) - z^*\| = \|x - z^*\| \le r$ for any $x \in \mathbf{B}(z^*, r)$. Then

$$\|U_\theta(z^*, x) - z^*\| = \|[\![T_\theta(z^*, x)]\!] - z^*\| \le \|T_\theta(z^*, x) - z^*\| \le r$$

by statement (iv) of Lemma 3.6, and thus $U_\theta(z^*, x) \in \mathbf{B}(z^*, r)$.

The assertion of Theorem for the case $z^* = 0$ follows immediately from the corresponding statement of Theorem 2.1.

5. Periodic Trajectories of Truncated Rotation

Given an arbitrary $z \in \mathbf{Z}^2$, consider the sequence $\{z(n)\}_{n=0}^\infty$ with $z(0) = z$ and

$$z(n+1) = U_\theta(z^*, z(n)), \qquad n = 0, 1, 2, \dots \ . \tag{5.1}$$

By Theorem 2.2 any sequence $\{z(n)\}_{n=0}^{\infty}$ satisfying (5.1) is bounded. Since there are only finitely many $\mathbf{Z}^2$ lattice points in any bounded region, any sequence $\{z(n)\}_{n=0}^{\infty}$ satisfying (5.1) is eventually periodic. Thus, we shall investigate some properties of periodic motions of the mapping $U_\theta(z^*,\cdot)$, which are required in the proofs that follow.

Throughout this Section let $z^* \in \mathbf{Z}^2$ and let$\{z(n)\}_{n=0}^{\infty}$ be a periodic sequence satisfying (5.1) with minimal period p, that is the smallest positive integer p such that $z(n+p) \equiv z(n)$ for all n. Associate with $\{z(n)\}_{n=0}^{\infty}$ the periodic sequence $\{w(n)\}_{n=0}^{\infty}$ defined by $w(0) = T_\theta(z^*, z(p-1))$ and

$$w(n) = T_\theta(z^*, z(n-1)), \qquad n = 1, 2, \ldots \ . \tag{5.2}$$

Then, by definition of the mapping $U_\theta(z^*,\cdot)$,

$$z(n) = [\![w(n)]\!], \qquad n = 0, 1, \ldots \ . \tag{5.3}$$

Since rationality and irrationality of the numbers θ/π, $\cos\theta$ and $\sin\theta$ play an important role in what follows, we state some relevant facts.

Lemma 5.1.[NZM, Theorem 6.16, p. 308] *Let θ be a rational multiple of π. Then $\cos\theta$ and $\sin\theta$ are irrational numbers except when $\cos\theta = 0$, $\pm 1/2$, ± 1 and $\sin\theta = 0$, $\pm 1/2$, ± 1.*

Lemma 5.2. *Given z^*, $z \in \mathbf{Z}^2$. If at least one of the values $\cos\theta$ and $\sin\theta$ is irrational, then $T_\theta(z^*, z) \notin \mathbf{Z}^2$.*

Proof. Suppose the contrary, that is that $w = T_\theta(z^*, z) \in \mathbf{Z}^2$. Then $m = z_1 - z_1^*$, $n = z_1 - z_1^*$, $p = w_1 - z_1^*$, $q = z_1 - z_1^*$ are all integers. By definition of the mapping $T_\theta(z^*,\cdot)$,

$$p = m\cos\theta - n\sin\theta, \qquad q = m\sin\theta + n\cos\theta\ .$$

Hence $\cos\theta = \frac{mp+nq}{m^2+n^2}$ and $\sin\theta = \frac{mq-np}{m^2+n^2}$, which is a contradiction. □

We now establish three important localization properties for trajectories of the mapping $T_\theta(z^*,\cdot)$.

Lemma 5.3. *If the sequence $\{z(n)\}_{n=0}^{\infty}$ satisfies (5.1), then*

$$\|z(n) - T_\theta^n(z^*, z(0))\| \le \sqrt{2}n, \qquad n \ge 0.$$

Proof. This follows immediately by induction from the fact that the map $T_\theta(z^*,\cdot)$ does not change the distance in the norm $\|\cdot\|$ and from the obvious inequality $\|U_\theta(z^*, z) - T_\theta(z^*, z)\| \le \sqrt{2}$ for any $z \in \mathbf{R}^2$. □

Lemma 5.4. *If the sequence $\{z(n)\}_{n=0}^{\infty}$ is periodic and $\theta \ne 0$, $\pm\pi$, $\pm\frac{\pi}{2}$, then either $w(n)$, $z(n) \equiv z^*$ or $w(n) \in \mathrm{int}(\mathbf{C})$, $z(n) \in \mathbf{C}$ for some n.*

Proof. Let $w(n) \not\equiv z^*$ and $w(n) \notin \mathrm{int}(\mathbf{C})$ for $n = 0,1,\ldots,p-1$, where p is the minimal period of the sequence $\{z(n)\}$. Then by (5.3) and Lemma 3.2

$$\|z(n) - z^*\| = \|[\![w(n)]\!] - z^*\| \le \|w(n) - z^*\|, \qquad n = 0, 1, \ldots, p-1,$$

and by the definition of the mapping $T_\theta(z^*,\cdot)$

(5.4) $\|w(n)-z^*\| = \|T_\theta(z^*,z(n-1))-z^*\| = \|z(n-1)-z^*\|, \qquad n=1,\dots,p.$

Hence in view of periodicity of both of the sequences $\{z(n)\}$ and $\{w(n)\}$

$$\|z(n)-z^*\| = \|w(n)-z^*\| = \|z(0)-z^*\|, \qquad n=0,1,\dots,p-1. \tag{5.5}$$

Now $\|z(n)-z^*\|^2$ is integer valued for $n=0,1,\dots,p-1$ and thus by (5.4) so is $\|w(n)-z^*\|^2$ for $n=0,1,\dots,p-1$. But then it follows from Lemma 3.2 that the equalities (5.5) are possible only if $z(n)=w(n)$ for $n=0,1,\dots,p-1$. Hence

$$w(n) = T_\theta(z^*,w(n-1)), \qquad n=1,2\dots\ ,$$

and then in view of periodicity of the sequence $\{w(n)\}$

$$w(0) = T_\theta^p(z^*,w(0)), \qquad w(0)\neq z^*. \tag{5.6}$$

Now, from (5.6) and from Lemma 5.2 it follows that both values $\cos\theta$ and $\sin\theta$ are rational. At the same time the equality (5.6) can be satisfied only if $\theta = 2\pi\frac{q}{p}$ for some integer q. But by Lemma 5.1 the rationality of the numbers θ/π, $\cos\theta$ and $\sin\theta$ is incompatible with the condition $\theta\neq 0,\pm\pi,\pm\frac{\pi}{2}$. This contradiction thus completes the proof of Lemma. □

Denote by $\{\rho_n\}_{n=0}^\infty$ a nondecreasing enumeration of the successive numbers $r > \frac{1}{2}\max\{(z_1^*)^2,(z_2^*)^2\}+1$ with $r=\|z\|$ for some $z\in \mathbf{C}\bigcap\mathbf{Z}^2$. Then

$$\rho_0 = \min\left\{r : r > \frac{1}{2}\max\{(z_1^*)^2,(z_2^*)^2\}+1,\ r=\|z\| \text{ for some } z\in\mathbf{C}\bigcap\mathbf{Z}^2\right\}.$$

Note that for any point $z\in\mathbf{C}\bigcap\mathbf{Z}^2$ there exists a $\widetilde{z}\in\mathbf{C}\bigcap\mathbf{Z}^2$ such that $\|z-\widetilde{z}\|\le 1$ and therefore

$$\rho_{k-1}\le\rho_k\le\rho_{k-1}+1, \qquad k=0,1,\dots\ . \tag{5.7}$$

Lemma 5.5. *Suppose that the sequence $\{z(n)\}_{n=0}^\infty$ is periodic. Then*
(i) either $z(n)\in\mathbf{B}(z^*,\rho_0)$ for all $n\ge 0$ or there exists an integer $k\ge 1$ such that

$$z(n)\in\mathbf{B}(z^*,\rho_k)\backslash\mathbf{B}(z^*,\rho_{k-1}), \qquad \forall n\ge 0; \tag{5.8}$$

(ii) if (5.8) holds, $\rho_{k-1}\ge\frac{1}{2}\sigma^2+1$ and $z(0)\in\mathbf{X}(\sigma)$ where $\sigma\ge\max\{|z_1^*|,|z_2^*|\}$, then

$$\|z(0)-z^*\|\ge\|z(n)-z^*\|>\|z(0)-z^*\|-1, \qquad \forall n\ge 0, \tag{5.9}$$

and the inclusion $z(n)\in\mathbf{X}(\sigma)$ is possible only if $z(n)\in\mathbf{D}(z(0))$;
(iii) if the conditions of the case (ii) are valid and at least one of the values $\cos\theta$ and $\sin\theta$ is irrational, then inclusion $z(n)\in\mathbf{X}(\sigma)$ is possible only if $z(n)\in\mathbf{C}\bigcap\mathbf{D}(z(0))$.

Proof. (i) Since the sequence $\{z(n)\}_{n=0}^\infty$ is periodic, it is thus bounded and we can choose a smallest ρ_k such that $z(0)\in\mathbf{B}(z^*,\rho_k)$. By Theorem 2.2 $z(n)\in\mathbf{B}(z^*,\rho_k)$ for all $n\ge 0$. If $k=0$ here, then $z(n)\in\mathbf{B}(z^*,\rho_0)$ for all $n\ge 0$, so we consider the case that $k\ge 1$ and show that $z(n)\notin\mathbf{B}(z^*,\rho_{k-1})$ for all $n\ge 0$ in this case. Indeed, if this were not so there would exist an $n_0 \ge 0$ that $z(n_0) \in \mathbf{B}(z^*, \rho_{k-1})$. Using Theorem 2.2 again, we obtain $z(n) \in \mathbf{B}(z^*, \rho_{k-1})$ for all $n \ge n_0$ and hence by periodicity of the sequence $\{z(n)\}_{n=0}^{\infty}$ for all $n \ge 0$. This contradicts the definition of ρ_k and thus proves the validity of the inclusion (5.8).

(ii) If $z(0) \in \mathbf{X}(\sigma)$, then the left inequality in (5.9) follows immediately from the inclusion (5.8) and the fact that the point $z(0) \in \mathbf{X}(\sigma)$ has integer coordinates, so $\rho_k = \|z(0) - z^*\|$. The right inequality (5.9) follows from (5.7) and from the already established inclusion (5.8).

To prove that the inclusion $z(n) \in \mathbf{X}(\sigma)$ implies that $z(n) \in \mathbf{D}(z(0))$ note first that

$$\|z(0) - z^*\| \le \|z(n) - z^*\|, \qquad \forall z(n) \in \mathbf{X}(\sigma). \tag{5.10}$$

Indeed, in the opposite case, that is if $\|z(n) - z^*\| < \|z(0) - z^*\|$ for some n, then by Theorem 2.2 the inequalities

$$\|z(k) - z^*\| \le \|z(n) - z^*\| < \|z(0) - z^*\|$$

would be valid for all $k \ge n$ and, in view of the periodicity of the sequence $\{z(n)\}$, the inequality $\|z(0) - z^*\| < \|z(0) - z^*\|$ would also be valid, but this is impossible.

From (5.9) and (5.10) it follows that $\|z(0) - z^*\| = \|z(n) - z^*\|$ as soon as $z(n) \in \mathbf{X}(\sigma)$ or, what is the same, $z(n) \in \partial\mathbf{B}(z^*, \|z(0)\|)$ as soon as $z(n) \in \mathbf{X}(\sigma)$. The inclusion $z(n) \in \mathbf{D}(z(0))$ follows now from the statement (ii) of Lemma 3.6.

(iii) Suppose that the inclusion $z(n) \in \mathbf{C}$ is not valid for some $z(n) \in \mathbf{X}(\sigma)$, that is $z(n) \notin \mathbf{C}$. Then by the already proved statement (ii) $\|z(n) - z^*\| = \|z(0) - z^*\|$ and $\|z(n-1) - z^*\| \le \|z(0) - z^*\|$, so

$$\|w(n) - z^*\| = \|T_\theta(z^*, z(n-1)) - z^*\| = \|z(n-1) - z^*\| \le \|z(n) - z^*\| \tag{5.11}$$

where the periodic sequence $\{w(n)\}$ is defined by (5.2). On the other hand, since at least one of $\cos\theta$ and $\sin\theta$ is irrational, by Lemma 5.2 $z(n) \ne w(n)$ and thus by Lemma 3.3 $\|z(n) - z^*\| < \|w(n) - z^*\|$, which contradicts the inequalities (5.11). This contradiction completes the proof of case (iii) and hence the proof of the Lemma. □

6. Dissipativity

Theorems 2.1 and 2.2 show that the dynamics of the rotation maps remain bounded under truncation. The following theorems show that the dynamics are in fact typically absorbed by a bounded set, that is the truncated rotation system is dissipative.

Theorem 6.1. *Let $\{z(n)\}_{n=0}^{\infty}$ be a sequence satisfying (5.1). If $z^* = 0$ and $\theta \ne 0$, $\pm\pi$, $\pm\frac{\pi}{2}$, then $z(n) \to 0$ as $n \to \infty$.*

Proof. By Theorem 2.2 the sequence $\{z(n)\}_{n=0}^{\infty}$ is bounded and thus must be periodic starting after some $n = n_0$. Then the sequence $\{w(n)\}_{n=1}^{\infty}$, where $w(n) = T_\theta(0, z(n-1))$, is also periodic for $n \ge n_0$. By Lemma 5.4 either $w(n) = 0$ for $n \ge n_0$

or $w(n) \in \mathrm{int}(\mathbf{C})$ for some n. But when $z^* = 0$ the set $\mathrm{int}(\mathbf{C})$ is empty. Thus $w(n) = 0$ for $n \geq n_0$ and so $z(n) = [\![w(n)]\!] = 0$ for $n \geq n_0$. □

Theorem 6.2. *If $z^* \neq 0$ and $\theta \neq 0, \pm\pi, \pm\frac{\pi}{2}$, then there exists an $r_0 > 0$ such that*

$$\limsup_{n\to\infty} \|z(n) - z^*\| \leq r_0 \ . \tag{6.1}$$

Theorem 6.2 leaves aside the question of what happens when $\theta = 0, \pm\pi, \pm\frac{\pi}{2}$. The map $U_\theta(z^*, \cdot)$ is clearly not dissipative in such case because it has infinitely many periodic trajectories with initial conditions arbitrarily distant from z^*.

Another interesting question concerns the dynamical behaviour of the map $U_\theta(z^*, \cdot)$ with $\theta \neq 0, \pm\pi, \pm\frac{\pi}{2}$ inside its region of absorption. At present the answer is far from clear and we simply indicate what can happen by presenting Figure 3 in which iterations of the truncated rotation mapping (2.3) are plotted for $\theta = 17°$, $z_1^* = z_2^* = 150$ and the increment of radii of initial points is equal to 3. Trajectories starting well away from z^* tend to a bounded region in accordance with Theorem 6.2, while inside this region the picture is reminiscent of the pattern of the rings of Saturn with typical "condensations" of trajectories and "gaps" between them.

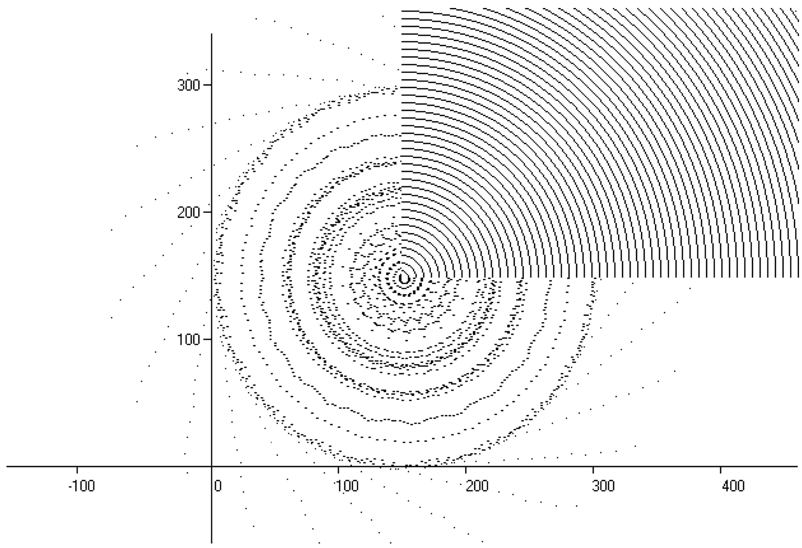

Figure 3: Phase portrait of the rotation mapping on planar integer lattice about the point $(150, 150)$ under truncating of errors of computed coordinates; the angle of rotation $\theta = 17°$ and the increment of radii of initial points is equal to 3. The upper-right quadrant contains a sector of the ideal phase portrait.

6.1. Proof of Theorem 6.2

By Theorem 2.2 the sequence $\{z(n)\}_{n=0}^{\infty}$ is bounded and so, after some $n = n_0$, is periodic. Thus we need only to establish the existence of an r_0 such that any **periodic** sequence $\{z(n)\}_{n=0}^{\infty}$ satisfying (5.1) will also satisfy the inequality

$$\|z(n) - z^*\| \le r_0, \qquad n = 0, 1, \dots \,. \tag{6.2}$$

Hence in what follows the sequence $\{z(n)\}_{n=0}^{\infty}$ will be assumed periodic with a minimal period p.

Consider the following three cases: the value of θ/π is irrational but both the values $\cos\theta$, $\sin\theta$ are rational; the value of θ/π is irrational and at least one of the values $\cos\theta$, $\sin\theta$ is irrational too; and the value of θ/π is rational.

Case 1: the value of θ/π is irrational but both the values $\cos\theta$, $\sin\theta$ are rational. Let

$$\cos\theta = \frac{r}{q}, \qquad \sin\theta = \frac{s}{q}$$

where r,s,$q \neq 0$ and the pairs of integers r, q and s, q are relatively prime. Define for $z = (z_1, z_2) \in \mathbf{C}\backslash\mathbf{C}_0$ the value

$$\delta(z) = \begin{cases} \|z - z^*\| - |z_1 - z_1^*| & \text{if} \quad z \in \mathbf{C}_2\backslash\mathbf{C}_0, \\ \|z - z^*\| - |z_2 - z_2^*| & \text{if} \quad z \in \mathbf{C}_1\backslash\mathbf{C}_0. \end{cases}$$

Then

$$\delta(z) \le \frac{\max\{(z_1^*)^2, (z_2^*)^2\}}{\|z - z^*\|}, \qquad z \in \mathbf{C}\backslash\mathbf{C}_0.$$

Let us choose an integer $k \ge 0$ for which $\rho_k \ge q\max\{(z_1^*)^2, (z_2^*)^2\}$ and show that the inequality (6.2) is valid for $r_0 = \rho_k$. If it were not so, then there would exist at least one value of n for which $\|z(n) - z^*\| > r_0$, and thus by statement (i) of Lemma 5.5

$$\|z(n) - z^*\| > r_0, \qquad \forall n \ge 0. \tag{6.3}$$

To reach a contradiction we need first to establish some estimates. Let $z(m) \notin \mathbf{C}$ for some $m \ge 0$. Then by Lemma 3.3

$$\|z(m) - z^*\| = \|z(m-1) - z^*\| \quad \text{if } z(m) = T_\theta(z^*, z(m-1)), \tag{6.4}$$

$$\|z(m) - z^*\| < \|z(m-1) - z^*\| \quad \text{if } z(m) \neq T_\theta(z^*, z(m-1)). \tag{6.5}$$

In the more complicated case that $z(m) \in \mathbf{C}$, take for definiteness $z(m) \in \mathbf{C}_2^+$ and consider the points $z(m-1)$ and $w(m)$ together with $z(m)$ (see (5.2)). By statement (ii) of Lemma 5.5 (see (5.9) $\|z(n) - z^*\| \le \|z(m) - z^*\|$ for any $n \ge 0$ and thus by periodicity of the sequence $\{z(n)\}$ we also have $\|z(m-1) - z^*\| \le \|z(m) - z^*\|$. Hence

$$\|w(m) - z^*\| = \|T_\theta(z^*, z(m-1)) - z^*\| = \|z(m-1) - z^*\| \le \|z(m) - z^*\|. \tag{6.6}$$

At the same time by definition of the value $\delta(z)$ and by (6.3)

$$\|z(m) - z^*\| > r_0 = \rho_k \ge q\max\{(z_1^*)^2, (z_2^*)^2\}.$$

Then

$$\|z(m) - z^*\| - (z_1(m) - z_1^*) \le \delta(z(m)) < \frac{1}{q}$$

and by (6.6)

$$0 \le w_1(m) - z_1^* \le \|w(m) - z^*\| < z_1(m) - z_1^* + \frac{1}{q}.$$

On the other hand, since $[\![w_1(m)]\!] - z_1^* = z_1(m) - z_1^*$ and $z(m) \in \mathbf{C}_2$, $z_1(m) - z_1^* > 0$, then $w_1(m) - z_1^* \ge z_1(m) - z_1^*$ and so, finally,

$$z_1(m) - z_1^* \le w_1(m) - z_1^* < z_1(m) - z_1^* + \frac{1}{q}. \tag{6.7}$$

By the definition of $\cos\theta$ and $\sin\theta$ we have

$$w_1(m) - z_1^* = \frac{1}{q}\left(r(z_1(m-1) - z_1^*) - s(z_2(m-1) - z_2^*)\right), \tag{6.8}$$

$$w_2(m) - z_2^* = \frac{1}{q}\left(s(z_1(m-1) - z_1^*) + r(z_2(m-1) - z_2^*)\right). \tag{6.9}$$

Since $r(z_1(m-1) - z_1^*) - s(z_2(m-1) - z_2^*)$ in (6.8) is integer, then $w_1(m) - z_1^*$ is a rational number with denominator q and thus (6.7) can be satisfied only if

$$z_1(m) = w_1(m). \tag{6.10}$$

Now $z_2(m) = [\![w_2(m)]\!]$, so $z_2(m)$ is also an integer and by (6.9) $w_2(m)$ is a rational number with denominator q, that is

$$z_2(m) = w_2(m) - \frac{t}{q} \tag{6.11}$$

with some integer $t \in [0, q)$. Without loss of generality we can assume here that the integers t and q are relatively prime or $t = 0$. Since coordinates of vectors $z(m)$, $z(m-1)$ and z^* are integers, then the value

$$u = (z_1(m) - z_1^*)^2 + (z_2(m) - z_2^*)^2 - \big((z_1(m-1) - z_1^*)^2 + (z_2(m-1) - z_2^*)^2\big)$$

is also integer. But

$$(z_1(m-1) - z_1^*)^2 + (z_2(m-1) - z_2^*)^2 = (w_1(m-1) - w_1^*)^2 + (w_2(m-1) - w_2^*)^2.$$

Hence in view of (6.10) and (6.11)

$$\begin{aligned} u &= (w_1(m-1) - w_1^*)^2 + (w_2(m-1) - w_2^*)^2 \\ &\quad - \big((z_1(m) - z_1^*)^2 + (z_2(m) - z_2^*)^2\big) \\ &= (z_1(m) - z_1^*)^2 + (z_2(m) - z_2^* + \frac{t}{q})^2 \\ &\quad - \big((z_1(m) - z_1^*)^2 + (z_2(m) - z_2^*)^2\big) \\ &= 2(z_2(m) - z_2^*)\frac{t}{q} + \frac{t^2}{q^2}. \end{aligned}$$

From this it follows that $t^2 = uq^2 - 2(z_2(m) - z_2^*)tq$, therefore $t = 0$ since the integers t and q are assumed to be relatively prime. This together with (6.10) and (6.11) means that

$$z(m) = T_\theta(z^*, z(m-1)) \quad \text{if} \quad z(m) \in \mathbf{C}. \tag{6.12}$$

To complete the proof of Case 1, note that from (6.4), (6.5) and (6.12) it follows that

$$\|z(0) - z^*\| \le \|z(1) - z^*\| \le \cdots \le \|z(n) - z^*\| \le \cdots$$

and thus, in view of periodicity of the sequence $\{z(n)\}$,

$$\|z(0) - z^*\| = \|z(1) - z^*\| = \cdots \le \|z(n) - z^*\| = \cdots \ .$$

In view of (6.4), (6.5) and (6.12) the last equalities are valid only when

$$z(n) = T_\theta(z^*, z(n-1)), \qquad n = 1, 2, \ldots$$

which, given the periodicity of the sequence $\{z(n)\}$ with the period p, means that

$$z(0) - z^* = T_\theta^p(0, z(0) - z^*), \qquad \|z(0) - z^*\| > 0.$$

But this last equality is satisfied only for values of θ that are rational multiples of π, which contradicts to the assumption that the value of θ/π is irrational. This contradiction completes the proof of Case 1.

Case 2: the value of θ/π is irrational and at least one of the values $\cos\theta$, $\sin\theta$ is irrational too. Define $\nu = 2\max\{|z_1^*|, |z_2^*|\} + 1$.

Lemma 6.3. *There exists an $r > \rho_0$ such that $z(n) \notin \mathbf{C}$ for $n = 1, \ldots, \nu$ if the sequence $\{z(n)\}_{n=0}^\infty$ is defined by (5.1) and satisfies conditions $z(0) \in \mathbf{C}$ and $\|z(0) - z^*\| \ge r$.*

Proof. Write

$$\xi = \min_{0 \le n \le \nu} \min_{i=0,\pm1,\ldots} \left| n\theta - \frac{\pi}{2} i \right| . \tag{6.13}$$

Since θ/π is irrational, $\xi > 0$ and so we can choose a number $r > 0$ such that

$$\min_{1 \le n \le \nu} \min_{w \in \mathbf{C}} \|U_\theta^n(z^*, z) - w\| > \sqrt{2}\nu \qquad \forall z \in \mathbf{C},\ \|z - z^*\| \ge r.$$

Then by Lemma 5.3 none of the points $z(n)$, $n = 1, \ldots, \nu$, can belong to $\mathbf{C}$. □

We continue the proof of Case 2 by choosing an integer $k \ge 0$ such that $\rho_k > r$ where r is defined by Lemma 6.3 and shall show that the inequality (6.2) is valid for $r_0 = \rho_k$. If this were not so, there would exist at least one value of n for which $\|z(n) - z^*\| > r_0$, and thus, by statement (i) of Lemma 5.5, (6.3) holds.

Let $z(m) \notin \mathbf{C}$ for some $m \ge 1$. By supposition at least one of values $\cos\theta$, $\sin\theta$ is irrational, so by Lemma 5.2 the coordinates of the vector $w(m) = T_\theta(z^*, z(m-1))$ cannot both be integers. Then $z(m) = [\![w(m)]\!] \ne T_\theta(z^*, z(m-1))$ and by Lemma 3.3

$$\|z(m) - z^*\|^2 \le \|z(m-1) - z^*\|^2 - 1, \qquad z(m) \notin \mathbf{C}. \tag{6.14}$$

Now let $z(m) \in \mathbf{C}$. Then by Lemma 3.4

$$\|z(m) - z^*\|^2 \le \|z(m-1) - z^*\|^2 + 2\max\{|z_1^*|, |z_2^*|\}. \tag{6.15}$$

Denote by $0 \le m_1 < m_2 < \ldots < m_s < p-1$ the maximal set of integers for which $z(m_i) \in \mathbf{C}$ (this set is nonempty by Lemma 5.4). From Lemma 6.3 it follows that

$$m_1 + p - m_s \ge \nu + 1 \quad \text{and} \quad m_{i+1} - m_i \ge \nu + 1, \qquad i = 1, 2, \ldots, s.$$

Then, by (6.14), (6.15) and by definition of the number ν,

$$\|z(m_1) - z^*\|^2 \le \|z(m_s) - z^*\|^2 - \nu + 2\max\{|z_1^*|, |z_2^*|\} \le \|z(m_s) - z^*\|^2 - 1$$

and also

$$\|z(m_{i+1}) - z^*\|^2 \le \|z(m_i) - z^*\|^2 - \nu + 2\max\{|z_1^*|, |z_2^*|\} \le \|z(m_i) - z^*\|^2 - 1$$

for $i = 1,2,\ldots,s$ which contradicts to the periodicity of the sequence $\{z(n)\}$. This contradiction completes the proof of Case 2.

Case 3: the value of θ/π is rational. Let $\theta = \pi\frac{s}{q}$ where numbers $s \ne 0$ and $q \ne 0$ are relatively prime and such that $\theta \ne 0$, $\pm\pi$, $\pm\frac{\pi}{2}$. Then $\cos\theta \ne 0$, $\sin\theta \ne 0$ and so, by Lemma 5.1, at least one of the values of $\cos\theta$ and $\sin\theta$ is irrational.

Consider first $q > 4\nu$ where $\nu = 2\max\{|z_1^*|, |z_2^*|\} + 1$. Then the value of ξ defined by (6.13) is strictly positive, so all the reasoning in the proof of Case 2, including that of Lemma 6.3, are valid here. So we need only consider the case $q \le 4\nu$, for which we need first to prove some auxiliary results. Let $\nu^* = (4\sqrt{2} + 1)\nu$.

Lemma 6.4. *There exists an $r \ge \nu^*$ such that for any periodic sequence $\{z(n)\}$ satisfying (5.1) and $\|z(0) - z^*\| \ge r$ has period equal to q. If, in addition, $z(0) \in \mathbf{C}$, then the inclusion $z(n) \in \mathbf{C}$ is valid only when $\theta n = \frac{\pi i}{2}$ for some i.*

Proof. By assumption $\theta \ne 0,\pm\pi,\pm\frac{\pi}{2}$ and $\|z(0) - z^*\| \ge r \ge \rho_0$, so $z(0) \ne z^*$ and then by Lemma 5.4 at least one of elements of the sequence $\{z(n)\}$ belong to the set $\mathbf{C}$. Without loss of generality we can assume that $z(0) \in \mathbf{C}$ and associate with the point $z(0)$ the periodic sequence $\{u(n)\}$ defined by

$$u(n) = T_\theta^n(z^*, z(0)), \qquad n \ge 0.$$

By Lemma 5.3

$$z(n) \in \mathbf{B}(u(n), 4\sqrt{2}\nu), \qquad 0 \le n \le 4\nu. \tag{6.16}$$

Since the width of the strips $\mathbf{C}_1$, $\mathbf{C}_2$ does not exceed $\max\{|z_1^*|,|z_2^*|\} \le \nu$, then any point of the cross $\mathbf{C} = \mathbf{C}_1 \bigcup \mathbf{C}_2$, together with the ball of radius $4\sqrt{2}\nu$ centered at this point, lies in the cross $\mathbf{X}(\nu^*)$. So, by (6.16) with $n = 0$ or q, we obtain

$$z(0), z(q) \in \mathbf{B}(u(q), 4\sqrt{2}\nu) = \mathbf{B}(u(0), 4\sqrt{2}\nu) \subseteq \mathbf{X}(\nu^*).$$

Now, if $r \ge \rho_k \ge \nu^*$, the ball $\mathbf{B}(u(0), 4\sqrt{2}\nu)$ can intersect only one of the half-strips $\mathbf{C}_1^+$, $\mathbf{C}_1^-$, $\mathbf{C}_2^+$ or $\mathbf{C}_2^-$ (see (3.1) for definition). At the same time, by statement (iii) of Lemma 5.5, $z(0)$, $z(q) \in \mathbf{C} \bigcap \mathbf{D}(z(0))$ where the set $\mathbf{D}(\cdot)$ is defined by (3.14). But clearly, the intersection of the set $\mathbf{D}(z(0))$ with any half-strip $\mathbf{C}_1^+$, $\mathbf{C}_1^-$, $\mathbf{C}_2^+$ or $\mathbf{C}_2^-$ contains no more than one point, so $z(0) = z(q)$. This means that period of the sequence $\{z(n)\}$ is equal to q and it is sufficient to prove the remaining part of Lemma for $0 \le n \le q$.

By definition of the sequence $\{u(n)\}$, the inclusion $u(n) \in \mathbf{C}$ is valid for sufficiently large values of r if and only if θn is an integer multiple of $\frac{\pi}{2}$. Note here that the elements of the sequence $\{u(n)\}$, like the cross $\mathbf{X}(\nu^*)$ itself, are symmetric with respect to the rotation on the angle θn around the point z^*. From this and from the already proven inclusion $\mathbf{B}(u(0), 4\sqrt{2}\nu) \subseteq \mathbf{X}(\nu^*)$ it follows that

$$\mathbf{B}(u(n), 4\sqrt{2}\nu) \subset \mathbf{X}(\nu^*) \qquad \text{for} \quad \theta n = \frac{\pi i}{2}. \tag{6.17}$$

Moreover, provided that $r \geq \rho_k \geq \nu^*$ is chosen sufficiently large, the following relations will be also valid

$$\mathbf{B}(u(n), 4\sqrt{2}\nu) \ \bigcap \ \mathbf{X}(\nu^*) = \emptyset \qquad \text{for} \quad \theta n \neq \frac{\pi i}{2}; \tag{6.18}$$

$$\mathbf{B}(u(n), 4\sqrt{2}\nu) \ \bigcap \ \mathbf{B}(u(m), 4\sqrt{2}\nu) = \emptyset \qquad \text{for} \quad 0 \leq m \neq n < q \tag{6.19}$$

Now, it follows immediately from (6.16), (6.17), (6.18), (6.19) that the inclusion $z(n) \in \mathbf{C}$ can hold only when $\theta n = \frac{\pi i}{2}$ for some i. □

Lemma 6.5. *Let $\{z(n)\}$ be a periodic sequence satisfying (*5.1*) and $\|z(0) - z^*\| \geq r$ with r defined by Lemma* 6.4 *and suppose $z(0) \in \mathbf{C}$, $z(m) \in \mathbf{C}_2^+$ with $z(n) \notin \mathbf{C}$ for $0 < n < m$. Then the vectors $u(n) = (u_1(n), u_2(n)) = T_\theta^{m-n}(z^*, z(n))$ satisfy the inequalities (see Figure* 4*)*

$$|u_2(n) - z_2^*| < |z_2(m) - z_2^*|, \quad u_1(n) \geq z_1(m), \qquad n = 0, 1, \ldots m-1. \tag{6.20}$$

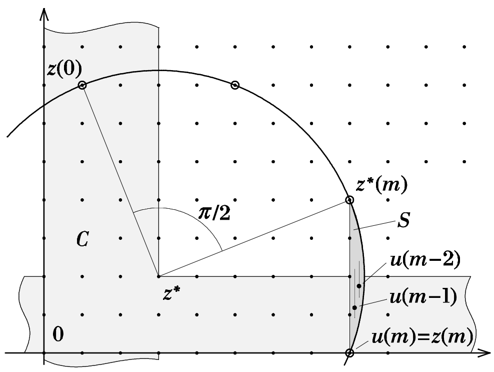

Figure 4: Location of the cross $\mathbf{C}$ (grey), the segment $\mathbf{S}$ (dark grey) and the auxiliary points $\{u(n)\}$.

P r o o f. Consider the segment

$$\mathbf{S} = \{x \in \mathbf{R}^2 : \ \|x\| \leq \|z(m)\|, \ |x_2 - z_2^*| < |z_2(m) - z_2^*|, \ x_1 \geq z_1(m)\}.$$

We shall show by induction for $n = m-1$, $m-2$, …, 0 that

$$u(n) \in \mathbf{S}. \tag{6.21}$$

Clearly, the inequalities (6.20) will follow from (6.21) .

Before proving the inclusion (6.21) some remarks are useful. Since $z(m) \in \mathbf{C} \subset \mathbf{X}(\nu^*)$ and $\mathbf{S} \subset \mathbf{C}$, then

$$\mathbf{S} \subset \mathbf{X}(\nu^*) \tag{6.22}$$

and also by the statement (ii) of Lemma 5.4

$$\|z(n) - z^*\| \leq \|z(m) - z^*\| = \|z(0) - z^*\|, \qquad 0 < n < m. \tag{6.23}$$

Since $z(n) \notin \mathbf{C}$ for $0 < n < m$, by the statement (iii) of Lemma 5.4

$$z(n) \notin \mathbf{X}(\nu^*), \qquad 0 < n < m. \tag{6.24}$$

We first prove the inclusion (6.21) for $n = m-1$. As was mentioned above, at least one of the numbers $\cos\theta$, $\sin\theta$ is irrational, so by Lemma 5.2 the coordinates of the vector $u(m-1) = T_\theta(z^*, z(m-1))$ cannot both be integers. Then $z(m) = [\![u(m-1)]\!] \neq u(m-1)$, from which it follows that either

$$u_1(m-1) \geq z_1(m) > 0, \qquad z_2^* \geq u_2(m-1) > z_2(m) \geq 0 \tag{6.25}$$

or

$$u_1(m-1) > z_1(m) > 0, \qquad z_2^* \geq u_2(m-1) \geq z_2(m) \geq 0. \tag{6.26}$$

At the same time, from $u(m-1) = T_\theta(z^*, z(m-1))$ and from (6.23) we obtain

$$\|u(m-1) - z^*\| = \|z(m-1) - z^*\| \leq \|z(m) - z^*\|$$

which together with (6.25), (6.26) mean that $u(m-1) \in \mathbf{S}$.

Supposing now that inclusion (6.21) holds for some $n > 0$, we shall prove its validity for $n-1$. Set $y = T_\theta(z^*, z(n-1))$. Then, by definition (5.1) of the sequence $\{z(n)\}$,

$$z(n) = U_\theta(z^*, z(n-1)) = [\![y]\!].$$

Since, $z(n) \notin \mathbf{C}$, then $z(n) \notin \mathbf{X}(\sigma)$ by the statement (iii) of Lemma 5.5 and so $y \notin \mathbf{X}(\sigma)$. Moreover, by definition (3.3) of the set $\mathbf{H}(z(n))$,

$$y \in \mathbf{H}(z(n)). \tag{6.27}$$

Again, since at least one of numbers $\cos\theta$, $\sin\theta$ is irrational, then by Lemma 5.2 the coordinates of the vector y cannot both be integers, so

$$y \neq z(n). \tag{6.28}$$

By definition of the sequence $\{u(n)\}$, $u(n-1) = T_\theta^{m-n+1}(z^*, z(n-1)) = T_\theta^{m-n}(z^*, y)$. Thus, by (6.27)

$$u(n-1) \in T_\theta^{m-n}(z^*, \mathbf{H}(z(n)) = T_{\theta(m-n)}(z^*, \mathbf{H}(z(n))$$

and by (6.28)

$$u(n-1) = T_\theta^{m-n}(z^*, y) \neq T_\theta^{m-n}(z^*, z(n)) = u(n) \in T_{\theta(m-n)}(z^*, \mathbf{H}(z(n)).$$

Now, by Lemma 3.7, $u_1(n-1) > u_1(n)$ and by the inductive supposition $u_1(n) \geq z_1(m)$, so

$$u_1(n-1) > z_1(m) \tag{6.29}$$

From (6.23) it then follows that

$$\|u(n-1) - z^*\| = \|T_\theta^{m-n+1}(z^*, z(n-1)) - z^*\| = \|z(n-1) - z^*\| \leq \|z(m) - z^*\|$$

and hence, in view of (6.29), that $|u_2(n-1) - z_2^*| < |z_2(m) - z_2^*|$. Inclusion (6.21) is thus proved and the induction procedure is established. This completes the proof of Lemma 6.5. □

Note that the statement of Lemma 6.5 is valid also with obvious reformulation for the cases $z(m) \in \mathbf{C}_2^-$, $z(m) \in \mathbf{C}_1^+$ or $z(m) \in \mathbf{C}_1^-$.

Returning to the proof of Case 3, we choose an integer $k \geq 0$ such that $\rho_k > r$ where r satisfies both Lemma 6.4 and Lemma 6.5 and shall show that the inequality (6.2) is valid for $r_0 = \rho_k$. If this were not so, there would exist at least one value of n for which $\|z(n) - z^*\| > r_0$ and thus, by statement (i) of Lemma 5.5, (6.3) would hold. By Lemma 5.4 there exists a value of n such that $z(n) \in \mathbf{C}$, so, without loss of generality, we may assume that $z(0) \in \mathbf{C}$. Choose the minimal value of $m > 0$ for which $z(m) \in \mathbf{C}$. By Lemma 6.4 the period of the sequence $\{z(n)\}$ is equal to q, so $m \leq q$.

For definiteness, let $z(m) \in \mathbf{C}_2^+$ and denote (see Figure 4)

$$z^*(m) = (z_1(m), z_2^* + (z_2^* - z_2(m)).$$

By statement (iii) of Lemma 5.5 $z(0)$,$z(m) \in \mathbf{D}(z(m))$ and then in addition $z^*(m) \in \mathbf{D}(z(m))$. Now, by Lemma 6.5, the inequalities (6.20) are valid for the vector $u(0) = T_\theta^m(z^*, z(0))$, and thus

$$u(0) \neq z(m), z^*(m). \tag{6.30}$$

At the same time, $\theta m = \frac{\pi i}{2}$ for some i by Lemma 6.4. As the set $\mathbf{D}(z(0))$ is invariant to the rotation of the plane $\mathbf{R}^2$ by angles $\frac{\pi i}{2}$ around the point z^*, then the point $u(0)$ must coincide either with $z(m)$ or with $z^*(m)$. But neither of these is possible in view of (6.30). The contradiction thus obtained completes the proof of Case 3 and thus Theorem 6.2 is also completely proved.

Acknowledgements

P. Diamond, P. Kloeden and A. Pokrovskii were supported by the Australian Research Council Grant A 8913 2609, V.Kozyakin was partially supported by the Russian Foundation for Fundamental Research Grant 93-01-00884.

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Department of Mathematics
University of Queensland
Brisbane, Qld 4072
Australia.

School of Computing
and Mathematics
Deakin University
Geelong, Victoria 3217
Australia.

Institute of Information
Transmission Problems
Russian Academy of Sciences
19 Bolshoi Karetny lane,
Moscow 101447
Russia.