New constructions of small complete caps in $PG(N, q)$, q even

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Abstract

New families of small complete caps in $PG(N, q)$, q even, are described. The problem of the construction of small complete caps in projective spaces of arbitrary dimensions is reduced to the same problem in the plane, by using inductive arguments. Apart from small values of either N or q, the caps constructed in this paper provide an improvement on the currently known upper bounds on the size of the smallest complete cap in $PG(N, q)$.

I. INTRODUCTION

Let $PG(N, q)$ be the projective N-dimensional space over the finite field $\mathbb{F}_q$ with q elements. A k-cap is a set of k points no three of which are collinear and in the planar case it is also called a k-arc.

A cap is called complete if it is not properly contained in a larger cap. The most important problem on caps in $PG(N, q)$ is to determine the spectrum of possible values of k for which there exists a complete k-cap. In this context the smallest and the largest sizes are of particular interest. This work is mainly devoted to constructions of small complete caps that provide upper bounds on the smallest possible sizes of complete caps. Besides, the constructions proposed give some successions of values of k.

This problem is related to Coding Theory as complete k-caps in $PG(N, q)$ with $k > N + 1$ and linear quasi-perfect $[k, k - N - 1, 4]_q 2$-codes over $\mathbb{F}_q$ with covering radius 2 are equivalent objects (with the exceptions of the complete 5-cap in $PG(3, 2)$ and the complete 11-cap in $PG(4, 3)$), see e.g. [5], [6].

The best known results on the size $t_2(N, q)$ of the smallest complete cap in $PG(N, q)$ (or, equivalently, the minimal length k for which there exists a $[k, k - N - 1, 4]_q 2$-code) are given in [3]–[5], [7], [9]. As a consequence of our constructions, essential improvements on the known results are obtained for $q \geq 8$. Our main achievement is the following result, which directly follows from Theorems 11 and 17.

Theorem 1: Let $q > 8$, q even. Assume that there exists a complete k-arc in $PG(2, q)$ with $k < q - 5$. Then there exists a complete n-cap in $PG(N, q)$ with

$$n = \begin{cases} (k+3) \cdot q^{\frac{N-2}{2}} + 3(q^{\frac{N-4}{2}} + q^{\frac{N-6}{2}} + \ldots + q) - N + 3, & N \geq 4 \text{ even} \\ 2q^{\frac{N-1}{2}} + (k+3) \cdot q^{\frac{N-3}{2}} + 3(q^{\frac{N-5}{2}} + q^{\frac{N-7}{2}} + \ldots + q) - N + 4, & N \geq 5 \text{ odd} \end{cases}. \tag{1}$$

Moreover, it is shown that when the k-arc has some special properties, smaller complete caps can be obtained with sizes approximately $(k+1)q^{\frac{N-2}{2}}$ and $kq^{\frac{N-2}{2}}$ for N even and $2q^{\frac{N-1}{2}} + kq^{\frac{N-3}{2}}$ for N odd. These special properties are connected with a new concept of "sum-points" introduced in this work.

All the known upper bounds on $t_2(N, q)$ are improved in this paper. By Theorems 14 and 19, we have.

Theorem 2: For $q > 8$, q even,

$$t_2(N, q) \leq \begin{cases} t_2(2, q) \cdot q^{\frac{N-2}{2}} + s_{N,q} - N + 1, & N \text{ even} \\ 2q^{\frac{N-1}{2}} + t_2(2, q) q^{\frac{N-3}{2}} + s_{N,q} - N + 2, & N \text{ odd} \end{cases}, \tag{2}$$

where $s_{N,q} = 3(q^{\lfloor \frac{N-2}{2} \rfloor} + q^{\lfloor \frac{N-2}{2} \rfloor - 1} + \ldots + q) + 2$ and $\lfloor \frac{N-2}{2} \rfloor$ denotes the integer part of $\frac{N-2}{2}$.

Directly from (2), using plane $(\frac{q+8}{3})$-arcs of [1], the upper bound on $t_2(2,q)$ of [10, Remark 2], and the relation $t_2(2,q) < \sqrt{q} \log^c q$ given in [8], we obtained the following upper bounds on $t_2(N,q)$:

- $t_2(N,q) \leq \frac{7}{3} q^{\frac{N-1}{2}} + \frac{8}{3} q^{\frac{N-3}{2}} + s_{N,q} - N + 2, \ N \text{ odd}, \quad q = 2^{2m}, m \geq 3.$
- $t_2(N,q) \leq \begin{cases} 2q^{-\frac{1}{2C}} q^{\frac{N}{2}} + s_{N,q} - N + 1, & N \text{ even} \\ 2q^{\frac{N-1}{2}} + 2q^{-\frac{1}{2C}} q^{\frac{N-1}{2}} + s_{N,q} - N + 2, & N \text{ odd} \end{cases}, \ q = 2^{2Cm} \geq 2^{30}, C \geq 5, m \geq 2.$
- $t_2(N,q) \leq \begin{cases} q^{\frac{N-1}{2}} \log^c q + s_{N,q} - N + 1, & N \text{ even} \\ 2q^{\frac{N-1}{2}} + q^{\frac{N-2}{2}} \log^c q + s_{N,q} - N + 2, & N \text{ odd} \end{cases}, c > 0 \text{ constant}, q \text{ large sufficiently}.$

Note that in [8] it is proved that $c \leq 300$ and the authors of [8] claim that $c \leq 10$ can be assumed.

For specific values of q, further improvements are provided by Theorems 15, 16, and 20.

Theorem 3: Let $q \leq 2^{15}$, q even. Then,

$$t_2(N,q) \leq \begin{cases} t_q(q^{\frac{N-2}{2}} + q^{\frac{N-4}{2}} + q^{\frac{N-6}{2}} + \ldots + q + 1), & N \geq 4 \text{ even} \\ 2q^{\frac{N-1}{2}} + t_q(q^{\frac{N-3}{2}} + q^{\frac{N-5}{2}} + q^{\frac{N-7}{2}} + \ldots + q + 1), & N \geq 5 \text{ odd} \end{cases},$$

where t_q is as in the following table.

$\log_2 q$	3	4	5	6	7	8	9	10	11	12	13	14	15
t_q	6	9	14	22	34	56	86	130	258	514	514	1026	1026

II. A CONCEPT OF SUM-POINTS. SOME PRELIMINARIES ON PLANE ARCS

Throughout the paper, q is a power of 2. Let $\mathbb{F}_q$ be the finite field with q elements, and $\mathbb{F}_q^* = \mathbb{F}_q \setminus \{0\}$. Let X_0, X_1, X_2 be homogeneous coordinates of points of $PG(2,q)$. Denote by l_∞ the line of $PG(2,q)$ of equation $X_0 = 0$. Points of an arc K not on l_∞ are the *affine points* of K, and the subset of affine points of K is the *affine part* of K. An arc is said to be affine if it coincides with it affine part. An *affinely complete arc* is an affine arc whose secants cover all the points in $PG(2,q) \setminus l_\infty$. As usual, we say that a point is written in his normalized form if the first nonzero coordinate is equal to 1.

Let K be a complete arc in $PG(2,q)$, and let Q be a point in $PG(2,q) \setminus K$ written in its normalized form. For every secant l of K through Q, let $c_1^{(l)}, c_2^{(l)}$ be the elements in $\mathbb{F}_q^*$ such that $Q = c_1^{(l)} P_1 + c_2^{(l)} P_2$ where P_1 and P_2 are the points on $l \cap K$ written in their normalized form.

Definition 4: The point Q is said to be a *sum-point* for K if $c_1^{(l)} = c_2^{(l)}$ for every secant l of K through Q.

Remark 5: In general, collineations do not preserve the number of sum-points for an arc. In this sense the concept of "sum-points" is "not geometrical".

We denote by $\beta(K)$ the number of sum-points for a complete arc K. When $\beta(K) = 1$, we denote by $p(K)$ the number of secants of K passing through the only sum-point.

For an arc K in $PG(2,q)$, and for an element $m \in \mathbb{F}_q$, let

$$Cov_\infty(K) = \{m \mid (0,1,m) \text{ is covered by the secants of } K\},$$

$$S_m(K) = \left\{ X_1 + Y_1 \mid (X_0, X_1, X_2), (Y_0, Y_1, Y_2) \in K, \ X_0 = Y_0, X_1 \neq Y_1, \frac{X_2 + Y_2}{X_1 + Y_1} = m \right\},$$

$$S_\infty(K) = \{X_2 + Y_2 \mid (X_0, X_1, X_2), (Y_0, Y_1, Y_2) \in K, \ X_0 = Y_0, X_1 = Y_1\}.$$

Lemma 6: For every complete arc K in $PG(2,q)$ with $\beta(K) = 1$ there are projectivities ψ_i such that:

$\beta(\psi_1(K)) = 1$, $p(\psi_1(K)) = p(K)$, and the only sum-point for $\psi_1(K)$ is $(0,0,1)$;

$\beta(\psi_2(K)) = 1$, $p(\psi_2(K)) = p(K)$, the only sum-point for $\psi_2(K)$ is $(0,0,1)$, and $1 \notin S_\infty(\psi_2(K))$;

$\beta(\psi_3(K)) = 1$, $\psi_3(K) \cap l_\infty = \{(0,0,1),(0,1,0)\}$, $p(\psi_3(K)) = p(K)$, and the only sum-point for $\psi_3(K)$ is $(0,1,1)$.

Lemma 7: In $PG(2,q)$ for every complete k-arc K with $\beta(K) = 1$ and $(k-2)p(K) < q - 1$ there is a collineation ψ such that $\psi(K) \cap l_\infty = \{(0,0,1),(0,1,0)\}$, with $\beta(\psi(K)) = 1$, $p(\psi(K)) = p(K)$, the only sum-point for $\psi(K)$ is $(0,1,1)$, and with the property $\psi(K) \cap \{(1,a,Aa^2) | A \in S_1(\psi(K)), a \in \mathbb{F}_q\} = \emptyset$.

Lemma 8: Let K be an affinely complete k-arc K in $PG(2,q)$ such that $(0,0,1)$ is covered by the secants of K. Then it can be assumed that $1 \notin S_\infty(\psi_2(K))$. Moreover, if $k < q - 5$ then there exist $m_1, m_2 \in \mathbb{F}_q^*$ with $m_1 \neq m_2$, $(m_1 + m_2)^3 \neq 1$, $m_i \neq 1$, such that $1 \notin S_{m_1}(K) \cup S_{m_2}(K)$.

Conjecture 9: Every complete arc in $PG(2,q)$ is projectively equivalent to an arc with only one sum-point.

III. Caps in projective spaces of even dimension

A. New inductive constructions of complete caps

Let s be a positive integer. Let $X_0, X_1, \ldots, X_{2s+2}$ be homogeneous coordinates of points of $PG(2s+2, q)$. For $i = 0, \ldots, 2s+1$, let H_i be the subspace of $PG(2s+2)$ of equations $X_0 = \ldots = X_i = 0$. Let $AG(N,q)$ be the N-dimensional affine space over $\mathbb{F}_q$. As usual, a point in $AG(N,q)$ is identified with a vector in $\mathbb{F}_q^N$. For any integer $j \geq 1$, let

$$\mathcal{P}^j = \{(a_1, a_1^2, \ldots, a_j, a_j^2) \mid a_1, \ldots, a_j \in \mathbb{F}_q\} \subset AG(2j, q).$$

The so called *product construction*, see e.g. [2], is the starting point for our constructions of small complete caps in $PG(2s+2, q)$. Let $C_1 \subset \mathbb{F}_q^{N_1+1}$ be a set of representatives of a cap $C = \langle C_1 \rangle \subset PG(N_1, q)$, and let $C_2 \subset AG(N_2, q)$ be a cap. Then, by [2], the product

$$(C : C_2) := \{(P, Q) \mid P \in C_1, Q \in C_2\} \subset PG(N_1 + N_2, q) \text{ is a cap.}$$

Another important tool is the following inductive construction. Let $m_1, m_2 \in \mathbb{F}_q^*$ with $m_1 \neq m_2$, $(m_1 + m_2)^3 \neq 1$, $m_i \neq 1$. Let $K^{(1)}_{m_1,m_2}$ be the subset of $PG(1,q)$ consisting of points $\{(1,0),(0,1)\}$. For $i \geq 1$, let

$$K^{(2i+1)}_{m_1,m_2} = A_1^{(2i+1)} \cup A_2^{(2i+1)} \cup A_3^{(2i+1)} \cup \{(1,0,\ldots,0)\} \subset PG(2i+1, q)$$

where

$$\begin{aligned} A_1^{(2i+1)} = \;& \{(1, m_1, a_1, a_1^2, a_2, a_2^2, \ldots, a_i, a_i^2), (1, m_2, a_1, a_1^2, a_2, a_2^2, \ldots, a_i, a_i^2) \mid \\ & a_1, \ldots, a_i \in \mathbb{F}_q, (a_1, \ldots, a_i) \neq (0, \ldots, 0)\}, \end{aligned}$$

$$A_2^{(2i+1)} = \{(0, 1, a_1, a_1^2, a_2, a_2^2, \ldots, a_i, a_i^2) \mid a_1, \ldots, a_i \in \mathbb{F}_q, (a_1, \ldots, a_i) \neq (0, \ldots, 0)\},$$

$$A_3^{(2i+1)} = \{(0, 0, b_0, b_1, \ldots, b_{2i-1}) \mid (b_0, b_1, \ldots, b_{2i-1}) \in K^{(2i-1)}_{m_1,m_2}\}.$$

Let $K_2^*(m_1, m_2) = K^{(2s+1)}_{m_1,m_2} \setminus \{(1,0,\ldots,0)\}$.

Proposition 10: If $q > 4$, then the set $K_2^*(m_1,m_2)$ is a cap in $PG(2s+1,q)$ which covers all the points in $PG(2s+1,q)$ with the exception of points $(1, m, 0, 0 \ldots, 0)$, $m \in \mathbb{F}_q$.

Let m_1, m_2 be as in Lemma 8. Let $K_2(m_1,m_2)$ be the natural embedding of $K_2^*(m_1,m_2)$ in $H_0 \subset PG(2s+2, q)$.

Theorem 11: Let $M = 2s+2$, $s \geq 1$, $q > 8$. Assume that K is an affinely complete k-arc in $PG(2,q)$ and $k < q-5$. Let $\mathcal{X}$ be the set

$$\mathcal{X} = (K : \mathcal{P}^s) \cup K_2(m_1, m_2) \subset PG(M, q).$$

Then $\mathcal{X}$ is a cap of size

$$(k+3) \cdot q^{\frac{M-2}{2}} + 3(q^{\frac{M-4}{2}} + q^{\frac{M-6}{2}} + \ldots + q) - M + 3.$$

Moreover,

- if $Cov_\infty(K) = \mathbb{F}_q$, then $\mathcal{X}$ is a complete cap;
- if $\mathbb{F}_q \setminus Cov_\infty(K) = \{m_0\}$, then

$$\mathcal{X}' = \mathcal{X} \cup \{(0, 1, m_0, 0, \ldots, 0)\}$$

  is a complete cap;
- if $\mathbb{F}_q \setminus Cov_\infty(K) \supseteq \{m_0, m_0'\}$, then

$$\mathcal{X}' = \mathcal{X} \cup \{(0, 1, m_0, 0, \ldots, 0), (0, 1, m_0', 0, \ldots, 0)\}$$

  is a complete cap.

Let m_1 and m_2 be as in Lemma 8. Let $\bar{K}_2^*(m_1, m_2) = K_{m_1,m_2}^{(2s-1)} \setminus \{(1, 0, \ldots, 0)\}$, and let $\bar{K}_2(m_1, m_2)$ be the natural embedding of $\bar{K}_2^*(m_1, m_2)$ in the subspace H_2 of $PG(2s + 2, q)$.

Theorem 12: Let $M = 2s + 2$, $s \geq 1$, $q > 8$. Assume that K is a complete k-arc in $PG(2, q)$ with $\beta(K) = 1$ and $k < q - 5$. Let $\mathcal{X}$ be the set

$$\mathcal{X} := (K : \mathcal{P}^s) \cup \bar{K}_2(m_1, m_2) \cup \{(0, 0, 1, a_1, a_1^2, \ldots, a_s, a_s^2) \mid a_i \in \mathbb{F}_q\} \subset PG(M, q).$$

Then,

- the size of $\mathcal{X}$ is
$$(k + 1) \cdot q^{\frac{M-2}{2}} + 3(q^{\frac{M-4}{2}} + q^{\frac{M-6}{2}} + \ldots + q) - M + 5;$$
- $\mathcal{X}$ is a complete cap.

We consider the product cap $(K : \mathcal{P}^j)$ in $PG(2j + 2, q)$, with $1 \leq j \leq s$. Let $Y_0, \ldots, Y_{2j+2}$ be homogeneous coordinates for points in $PG(2j + 2, q)$. For $j = 0, \ldots, s$, let V_{2j+2} be the $(2j + 2)$-dimensional subspace of $PG(2s + 2, q)$ of equations $X_0 = \ldots = X_{2s-2j-2} = 0, X_{2s-2j-1} = X_{2s-2j}$. Let Φ be the following isomorphism between $PG(2j + 2, q)$ and V_{2j+2}

$$\Phi(Y_0, Y_1, \ldots, Y_{2j+2}) = (0, 0, \ldots, 0, Y_0, Y_0, Y_1, \ldots, Y_{2j+2}).$$

Let $\overline{K}^{(j)} = (K : \mathcal{P}^j) \subset PG(2j + 2, q)$ if $j = 1, \ldots, s$, and let $\overline{K}^{(0)} = K \subset PG(2, q)$. We denote by $K^{(j)} \subset PG(2s + 2, q)$ the image of the cap $\overline{K}^{(j)}$ by Φ.

Theorem 13: Let $M = 2s+2$, $s \geq 1$. Assume that K is a complete k-arc in $PG(2, q)$ with $\beta(K) = 1$, $(k - 2)p(K) < q - 1$. Then the set $\mathcal{X} := \bigcup_{j=0}^{s} K^{(j)} \subset PG(M, q)$ is a complete cap of size

$$k\frac{q^{\frac{M}{2}} - 1}{q - 1} = k(q^{\frac{M-2}{2}} + q^{\frac{M-4}{2}} + q^{\frac{M-6}{2}} + \ldots + q + 1).$$

B. New upper bounds on $t_2(N, q)$

From Theorems 11, 12, and 13 we have Theorems 14, 15, and 16, respectively.

Theorem 14: Let N be even, $N > 2$. If $q > 8$, then

$$t_2(N, q) \leq (t_2(2, q) + 3) \cdot q^{\frac{N-2}{2}} + 3(q^{\frac{N-4}{2}} + q^{\frac{N-6}{2}} + \ldots + q) - N + 3,$$

and

$$t_2(N, q) \leq (t_2^A(2, q) + 3) \cdot q^{\frac{N-2}{2}} + 3(q^{\frac{N-4}{2}} + q^{\frac{N-6}{2}} + \ldots + q) - N + 5,$$

where $t_2^A(2, q) \leq t_2(2, q)$ is the size of the smallest affinely complete arc in $PG(2, q)$.

Theorem 15: Let N be even, $N > 2$. If $q > 8$, then

$$t_2(N, q) \leq (t_2^S(2, q) + 1) \cdot q^{\frac{N-2}{2}} + 3(q^{\frac{N-4}{2}} + q^{\frac{N-6}{2}} + \ldots + q) - N + 5,$$

where $t_2^S(2, q)$ is the size of the smallest complete arc in $PG(2, q)$ with only one sum-point.

Theorem 16: Let N be even, $N > 2$. Then

$$t_2(N, q) \leq t_2^{S^+}(2, q)(q^{\frac{N-2}{2}} + q^{\frac{N-4}{2}} + q^{\frac{N-6}{2}} + \ldots + q + 1),$$

where $t_2^{S^+}(2, q)$ is the size of the smallest complete k-arc K in $PG(2, q)$ with only one sum-point and with the property that $(k - 2)p(K) < q - 1$.

IV. Caps in projective spaces of odd dimension

A. New inductive constructions of complete caps

Let $K_0 = \{(1,1),(1,0)\}$ be the trivial complete cap in $PG(1,q)$. We consider the product cap $(K_0 : \mathcal{P}^{s+1}) \subset PG(2s+3,q)$. Let H_0 be the subspace of $PG(2s+3,q)$ of equation $X_0 = 0$.

Now, let $\mathcal{X} \subset PG(2s+2,q)$ be as in Theorem 11. Let $\bar{\mathcal{X}}$ be the natural embedding of $\mathcal{X}$ in the hyperplane H_0 of $PG(2s+3,q)$.

Theorem 17: Let $M = 2s+3$, $s \geq 1$, $q > 8$. Then the set $(K_0 : \mathcal{P}^{s+1}) \cup \bar{\mathcal{X}}$ is a complete cap in $PG(M,q)$ of size

$$2q^{\frac{M-1}{2}} + (k+3) \cdot q^{\frac{M-3}{2}} + 3(q^{\frac{M-5}{2}} + q^{\frac{M-7}{2}} + \ldots + q) - M + 4.$$

Now we assume that K is a complete k-arc in $PG(2,q)$ with $\beta(K) = 1$, $(k-2)p(K) < q-1$, and that $\mathcal{X} \subset PG(2s+2,q)$ is as in Theorem 13. Let $\bar{\mathcal{X}}'$ be the natural embedding of $\mathcal{X}$ in the hyperplane H_0 of $PG(2s+3,q)$.

Theorem 18: Let $M = 2s+3$, $s \geq 1$. Assume that that K is a complete k-arc in $PG(2,q)$ with $\beta(K) = 1$, $(k-2)p(K) < q-1$. Then the set $(K_0 : \mathcal{P}^{s+1}) \cup \bar{\mathcal{X}}'$ is a complete cap in $PG(2s+3,q)$ of size

$$2q^{\frac{M-1}{2}} + k(q^{\frac{M-3}{2}} + q^{\frac{M-5}{2}} + q^{\frac{M-7}{2}} + \ldots + q + 1).$$

B. New upper bounds on $t_2(N,q)$

From Theorems 17 and 18 we have Theorems 19 and 20, respectively.

Theorem 19: Let N be odd, $N > 3$. If $q > 8$, then

$$t_2(N,q) \leq 2q^{\frac{N-1}{2}} + (t_2(2,q)+3) \cdot q^{\frac{N-3}{2}} + 3(q^{\frac{N-5}{2}} + q^{\frac{N-7}{2}} + \ldots + q) - N + 4.$$

Theorem 20: Let N be odd, $N > 3$. Then

$$t_2(N,q) \leq 2q^{\frac{N-1}{2}} + t_2^{S+}(2,q)(q^{\frac{N-3}{2}} + q^{\frac{N-5}{2}} + q^{\frac{N-7}{2}} + \ldots + q + 1),$$

where $t_2^{S+}(2,q)$ is as in Theorem 16.

References

[1] V. Abatangelo, A class of complete $[(q+8)/3]$-arcs of $PG(2,q)$, with $q = 2^h$ and h (≥ 6) even, *Ars Combin.*, Vol. 16 (1983), pp. 103–111.

[2] Y. Edel, Extensions of generalized product caps, *Des. Codes Cryptogr.*, Vol. 31 (2004), no. 1, pp. 5–14.

[3] M. Giulietti, Small Complete Caps in Galois Affine Spaces, *J. Alg. Comb.*, to appear.

[4] M. Giulietti, Small complete caps in $PG(N,q)$, q even, *J. Comb. Des.*, to appear.

[5] M. Giulietti and F. Pasticci, Quasi-Perfect Linear Codes with Minimum Distance 4, *IEEE Trans. Inform. Theory*, to appear.

[6] J.W.P. Hirschfeld and L. Storme, The packing problem in statistics, coding theory and finite projective spaces, *J. Statist. Planning Infer.*, Vol. 72 (1998), no. 1-2, pp. 355–380.

[7] J.W.P. Hirschfeld and L. Storme, The packing problem in statistics, coding theory and finite projective spaces: update 2001, Blokhuis, A. (ed.) et al., Finite geometries. Proceedings of the fourth Isle of Thorns conference, Brighton, UK, April 2000. Dordrecht: Kluwer Academic Publishers. Dev. Math., Vol. 3 (2001), pp. 201-246.

[8] J.H. Kim and V. Vu, Small Complete Arcs in Projective Planes, *Combinatorica*, Vol. 23 (2003), pp. 311–363.

[9] F. Pambianco and L. Storme, Small complete caps in spaces of even characteristic, *J. Combin. Theory Ser. A*, Vol. 75 (1996), pp. 70–84.

[10] T. Szőnyi, Small complete arcs in Galois planes, *Geom. Dedicata*, Vol. 18 (1985), pp. 161–172.