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Minimal 1-saturating sets and complete caps in binary projective spaces

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Abstract

In binary projective spaces $PG(v,2)$, minimal 1-saturating sets, including sets with inner lines and complete caps, are considered. A number of constructions of the minimal 1-saturating sets are described. They give infinite families of sets with inner lines and complete caps in spaces with increasing dimension. Some constructions produce sets with an interesting symmetrical structure connected with inner lines, polygons, and orbits of stabilizer groups. As an example we note an 11-set in $PG(4,2)$ called "Pentagon with center". The complete classification of minimal 1-saturating sets in small geometries is obtained by computer and is connected with the constructions described.

Keywords: Binary minimal saturating sets; Binary complete caps; Projective spaces

1. Introduction

Let $PG(v,q)$ be the v-dimensional projective space over the Galois field of q elements. For an introduction to geometrical objects in such spaces, see [5,16,17].

For an integer ϱ with $0 \leqslant \varrho \leqslant v$ we say that a set of points $S \subseteq PG(v,q)$ is ϱ-*saturating* if for any point $x \in PG(v,q)$ there exist $\varrho+1$ points in S generating a subspace of $PG(v,q)$ in which x lies and ϱ is the smallest value with such property, cf. [8,12,22,23].

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One can say that the points in S are "saturating" and points in $PG(v,q)$ are "saturated" or "covered" [8,12]. The saturating sets are called "R-spanning sets" in [2] and "saturated sets" in [23].

A ϱ-saturating set of k points is called *minimal* if it does not contain a ϱ-saturating set of $k-1$ points [8,23].

A set $S \subset PG(v,q)$ is 1-saturating if any point of $PG(v,q) \setminus S$ lies on a t-secant of S with $t \geqslant 2$ if $q \geqslant 3$ and $t=2$ if $q=2$.

Arcs in $PG(2,q)$ and caps in $PG(v,q)$, $v \geqslant 3$, are sets of points, no three of which are collinear. Further we consider arcs only in $PG(2,q)$. An arc or cap $S \subset PG(v,q)$ is *complete* if its bisecants cover all points of $PG(v,q) \setminus S$ [16,17]. Complete arcs and caps are minimal 1-saturating sets [8,23]. For binary complete arcs and caps, see [3–5,7,9,11,13,14,16,17,19,20,24].

On the other hand, a minimal 1-saturating set may contain three points of the same line. In the binary case this line entirely lies into the set as a binary line consists of three points. We call such line an *inner line*. For binary minimal 1-saturating sets containing inner lines, see [1,2,6,8,14,15,18,22,23].

Similarly to binary complete caps [3,4,7], we say that a minimal 1-saturating k-set in $PG(v,2)$ is "small" if $k \leqslant 2^{v-1}$ and "large" otherwise.

For minimal 1-saturating sets we can use results of the linear covering codes theory, e.g., of [1,2,6,14,15,18,21]. A q-ary linear code with codimension r has *covering radius* R if every r-positional q-ary column is equal to a linear combination of R columns of a parity check matrix of this code and R is the smallest value with such property. For an introduction to the concept of code covering radius, see [1,6,21]. The points of a ϱ-saturating n-set in $PG(r-1,q)$ can be considered as columns of a parity check matrix of a q-ary linear code of length n, codimension r, and covering radius $\varrho+1$ [2,8,12].

For given codimension and covering radius, the linear covering codes theory [1,6] is interested in codes of the smallest length as they have small covering density. In a geometric perspective, saturating sets of the smallest size are also interesting as extremal objects.

At the same time, in projective spaces, *minimal* saturating sets are introduced whereas the corresponding concept for linear covering codes is not considered. Problems connected with minimal saturating sets essentially enhance the theory of saturating sets.

At present binary minimal 1-saturating sets, especially small and large sets with inner lines and small complete caps, seem to be studied insufficiently. In general, their smallest sizes and the spectrum of possible sizes are unknown. Relatively a few corresponding constructions are described in literature.

In this work, in Sections 2–5, for binary projective spaces $PG(v,2)$ we describe a number of constructions of minimal 1-saturating sets including sets with inner lines and complete caps. The constructions give infinite families of small and large minimal 1-saturating sets in spaces with increasing dimension and essentially increase our knowledge on the spectrum of possible sizes of these sets. Construction "Polygon" connected with polygons in projective spaces and orbits of stabilizer groups produces sets with an interesting symmetrical structure, see Section 3. Construction "addition of space lift" (ASL) is convenient for both sets with inner lines and complete caps. It can produce a wide spectrum of sizes for suitable starting sets, see Sections 4 and 5.

In Section 6, for small geometries $PG(v,2)$, using computer, we obtain the complete classification of minimal 1-saturating k-sets for all k if $v \leqslant 5$ and for $k \leqslant 20$ if $v = 6$. As an element of the classification we give the order of the stabilizer group for the objects found. We give also the known sizes of minimal 1-saturating sets in $PG(v,2)$, $6 \leqslant v \leqslant 9$. The computer results are connected with the constructions described.

Some results of this work were represented without proofs in [10].

2. Some constructions of minimal 1-saturating sets

We consider constructions of minimal 1-saturating sets in $PG(v,2)$. We denote

$$\Sigma_r = PG(r-1,2). \tag{1}$$

Assume that Σ_{r_1} and Σ_{r_2} are *disjoint* subspaces of the space Σ_r with $r_1 + r_2 = r$. Every point of Σ_r is represented by an r-positional binary column vector $(i_1 i_2)$ where i_1 (resp., i_2) is the value of the binary representation written in the first r_1 (resp., in the last r_2) positions. We represent points of the subspaces Σ_{r_1} and Σ_{r_2} by vectors of the form

$$\begin{aligned} &(i_1 0) \in \Sigma_{r_1} \subset \Sigma_r,\ (0 i_2) \in \Sigma_{r_2} \subset \Sigma_r,\ r_1 + r_2 = r,\\ &\quad i_j = 1, 2, \ldots, 2^{r_j} - 1, \quad j = 1, 2. \end{aligned} \tag{2}$$

We denote by $\Sigma^*_{r,f}$ the subset of Σ_r consisting of all points represented by a vector the first f positions of which do not contain the zero vector, $f < r$. By (2), $\Sigma^*_{r,r_1} = \Sigma_r \setminus \Sigma_{r_2}$.

A 1-saturating set $S \subset \Sigma_r$ has an *f-peculiarity* if for every point $P \in S$ at least one point of $\Sigma^*_{r,f}$ is unsaturated by $S \setminus \{P\}$.

Note that the formulas of (2) have been taken only for *definiteness* of a matrix representation of geometrical objects, in particular for studying the f-peculiarity important for Construction ASL.

Let A, B, C be points of Σ_r. The relation $A + B = C$ means that a vector representing the point C is the sum of vectors representing the points A, B, i.e., $\{A, B, C\}$ is a line.

If S is a 1-saturating set in Σ_r and if for some point $C \in \Sigma_r \setminus S$ there exists only one pair of points $\{A, B\} \subset S$ with $A + B = C$, then $\{A, B\}$ is called a *critical pair of points*, A, B are called *critical points* and (A, B, C) is called a *critical bisecant*. In this case, if $C \in \Sigma^*_{r,f}$ the points A, B are also called *f-critical*. The following lemma is evident.

Lemma 1. *If every point of a* 1-*saturating set is critical then the set is minimal. If every point of a* 1-*saturating set is f-critical then the set has the f-peculiarity.*

The conditions of Lemma 1 are not necessary. For example, if S is the complete cap of the maximal size then no its point is critical but nevertheless S is a minimal 1-saturating set. If one removes one point from S, this point becomes unsaturated.

We describe Constructions A, B, C connected with the direct sum and its modifications. They give codes with covering radius two [1, Chapter 8, 6, Chapters 3, 5, 15]. Here we note that these constructions produce *minimal* 1-saturating sets having an f-peculiarity.

Construction A. We consider the direct sum construction for representation (2). We form a k-set S with

$$S = \Sigma_{r_1} \cup \Sigma_{r_2} \subset \Sigma_r, \quad r_1, r_2 \geqslant 1, \quad k = 2^{r_1} + 2^{r_2} - 2. \tag{3}$$

By (2) and (3), the following theorem is evident.

Theorem 1. *The set S of* (3) *is a minimal* 1*-saturating set with the r_1-peculiarity.*

Construction A was considered in [23, Lemma 10] without studying the *f*-peculiarity. For $r_1 = 1$ Construction A forms the greatest minimal 1-saturating set [8, Constrction A, Corollary 1].

Construction B. We consider the amalgamated direct sum construction for representation (2). Let $p_1, p_2 \neq 0$, $P_1 = (p_1 0) \in \Sigma_{r_1}$, $P_2 = (0 p_2) \in \Sigma_{r_2}$, $P = P_1 + P_2 = (p_1 p_2) \in \Sigma_r \setminus (\Sigma_{r_1} \cup \Sigma_{r_2})$. We form a k-set S with

$$S = (\Sigma_{r_1} \setminus \{P_1\}) \cup \{P\} \cup (\Sigma_{r_2} \setminus \{P_2\}) \subset \Sigma_r, \quad r_1, r_2 \geqslant 2, \quad k = 2^{r_1} + 2^{r_2} - 3. \tag{4}$$

By (2) and (4), the following theorem can be easily proved.

Theorem 2. *The set S of* (4) *is a minimal* 1*-saturating set with the r_1-peculiarity.*

Construction C. We consider a modification of the direct sum construction for representation (2). We put $p_j = 2^{r_j} - 1$, $g_j = 2^{r_j - 1}$, $t_j = 2^{r_j - 1} - 1$, $j = 1, 2$; $v_2 = 1$; $P_1 = (p_1 0)$, $G_1 = (g_1 0)$, $T_1 = (t_1 0)$; $P_2 = (0 p_2)$, $G_2 = (0 g_2)$, $T_2 = (0 t_2)$, $V_2 = (0 v_2)$; $G_{12} = (g_1 p_2)$, $T_{12} = (t_1 p_2)$, $G_{21} = (p_1 g_2)$, $T_{21} = (p_1 t_2)$, $V_{21} = (p_1 v_2)$. We denote $\mathbf{M}_1 = \Sigma_{r_1} \setminus \{P_1, G_1, T_1\}$, $\mathbf{M}_2 = \Sigma_{r_2} \setminus \{P_2, G_2, T_2, V_2\}$, and form a k-set S with

$$\begin{aligned} S = \mathbf{M}_1 \cup \{G_{12}, T_{12}\} \cup \mathbf{M}_2 \cup \{G_{21}, T_{21}, V_{21}\} \subset \Sigma_r, \quad r_1 \geqslant 3,\ r_2 \geqslant 4, \\ k = 2^{r_1} + 2^{r_2} - 4. \end{aligned} \tag{5}$$

Example 1. Let $r_1 = 3,\ r_2 = 4,\ r = 7$. Construction C gives the 20-set

$$S = \begin{bmatrix} 0011\ 10\ 00000000000\ 111 \\ 0101\ 01\ 00000000000\ 111 \\ 1010\ 01\ 00000000000\ 111 \\ 0000\ 11\ 00000111111\ 100 \\ 0000\ 11\ 00111000111\ 010 \\ 0000\ 11\ 11001011001\ 010 \\ 0000\ 11\ 01010101010\ 011 \end{bmatrix} \subset \Sigma_7.$$

Theorem 3. *The set S of* (5) *is a minimal* 1*-saturating set with the r_1-peculiarity.*

Proof. We take a point $A = (i_1 i_2) \in \Sigma_r \setminus S$ and consider the following cases:

(1) The point $A \in \{P_1, G_1, T_1, P_2, G_2, T_2, V_2\}$ is covered as $r_1 \geqslant 3,\ r_2 \geqslant 4$.

(2) $i_1, i_2 \neq 0, i_1 \notin \{p_1, g_1, t_1\}, i_2 \notin \{p_2, g_2, t_2, v_2\}$.

There is *only one* pair $\{A_1, A_2\} \subset S$ with $A_1 + A_2 = A$. Here $A_1 \in \mathbf{M}_1$, $A_2 \in \mathbf{M}_2$. Every point of $S \setminus \{G_{12}, T_{12}, G_{21}, T_{21}, V_{21}\}$ is included to at least one such pair and it is r_1-critical.

(3) $i_1 = p_1, i_2 \in \{p_2, g_2 + 1, t_2 - 1\}$.

For every i_2 there is *only one* variant: $(p_1 p_2) = E_2 + V_{21}$, $(p_1, g_2 + 1) = E_2 + T_{21}$, $(p_1, t_2 - 1) = E_2 + G_{21}$, where $E_2 = (0e_2)$, $e_2 = 2^{r_2} - 2$. So, the points G_{21}, T_{21}, V_{21} are r_1-critical.

(4) $i_1 \in \{g_1, t_1\}, i_2 \neq 0, i_2 \notin \{p_2, g_2, t_2, e_2\}$.

For every pair $\{i_1, i_2\}$ there is *only one* pair of points $\{A_1, A_2\} \subset S$ with $A_1 + A_2 = A$. Here $A_1 \in \{G_{12}, T_{12}\}$, $A_2 \in \mathbf{M}_2$. So, the points G_{12}, T_{12} are r_1-critical.

(5) $i_1 = p_1, i_2 \neq 0, i_2 \notin \{p_2, g_2 + 1, t_2 - 1, g_2, t_2, v_2\}$.

There are three pairs $\{A_2, D\} \subset S$ with $A_2 + D = A$. Here $A_2 \in \mathbf{M}_2$, $D \in \{G_{21}, T_{21}, V_{21}\}$.

(6) $i_1 \in \{g_1, t_1\}, i_2 \in \{g_2, t_2, e_2\}$.

If $i_1 = g_1$ for every i_2 there is *only one* variant: $(g_1 g_2) = T_{12} + T_{21}$, $(g_1 t_2) = T_{12} + G_{21}$, $(g_1 e_2) = T_{12} + V_{21}$. Similarly for $i_1 = t_1$.

(7) $i_1 \neq 0, i_1 \notin \{p_1, g_1, t_1\}, i_2 = p_2$.

We use bisecants $\{A_1, A_2, A\}$ with $A_1 \in \mathbf{M}_1$, $A_2 \in \{G_{12}, T_{12}\}$.

(8) $i_1 \neq 0, i_1 \notin \{p_1, g_1, t_1\}, i_2 \in \{g_2, t_2, v_2\}$.

There is *only one* pair $\{A_1, A_2\} \subset S$ with $A_1 + A_2 = A$. Here $A_1 \in \mathbf{M}_1$, $A_2 \in \{G_{21}, T_{21}, V_{21}\}$.

So, S is a 1-saturating set and all its points are r_1-critical. □

Construction D. We consider the doubling construction or Plotkin construction, see [3,4,7,11]. Let Σ_r be a hyperplane of Σ_{r+1} and let Q be a point of $\Sigma_{r+1} \setminus \Sigma_r$. We take a minimal 1-saturating k_0-set $S_0 \subset \Sigma_r$ and form the $2k_0$-set $S = S_0 \cup (S_0 + \{Q\}) \subset \Sigma_{r+1}$. Evidently, S is 1-saturating. If S_0 is a complete arc or cap then S is a complete cap, i.e., a minimal 1-saturating set. But if S_0 contains inner lines it is possible that S is no minimal. In this case we form a $(k'_0 + k_0)$-set $S = S'_0 \cup (S_0 + \{Q\})$ where S'_0 is a k'_0-subset of S_0.

Construction $\mathbf{D_A}$. We take the $(2^{r_1} + 2^{r_2} - 2)$-set $S_0 = \Sigma_{r_1} \cup \Sigma_{r_2}$ of Construction A with $r_1, r_2 \geqslant 2$ and put $S'_0 = \Sigma_{r_1}$. Then $S = \Sigma_{r_1} \cup ((\Sigma_{r_1} \cup \Sigma_{r_2}) + \{Q\})$, $k'_0 + k_0 = 2^{r_1+1} + 2^{r_2} - 3$. By (2) and (3), it can be shown that S is a minimal 1-saturating set with $(r_1 + 1)$-peculiarity. Its parameters are the same as in (4) but often these sets are projectively distinct.

Construction E. Using a *starting* minimal 1-saturating k_0-set S_0 in a space Σ_r the construction designs an infinite family of minimal 1-saturating sets in spaces Σ_{r+t} with increasing dimension $r+t$. We partition S_0 into two nonempty subsets $S_0^{(1)}$, $S_0^{(2)}$ so that

(a) Every point $C \in \Sigma_r \setminus S_0$ lies on a bisecant $\{A, B, C\}$ with $A \in S_0,\ B \in S_0^{(2)}$.

(b) Every point E of $S_0^{(2)}$ belongs to a critical pair of points $\{D, E\}$ with $D \in S_0^{(1)}$.

(c) $S_0^{(1)} \not\subseteq S_0 + S_0^{(2)}$.

We put $S_0 \subset \Sigma_r \subset \Sigma_{r+t},\ \Sigma_t \subset \Sigma_{r+t},\ \Sigma_r \cap \Sigma_t = \emptyset$, and form a k-set S as follows:

$$S = \Sigma_t \cup S_0 \cup (S_0^{(2)} + \Sigma_t) \subset \Sigma_{r+t},$$
$$k = k_0 + (k_0^{(2)} + 1)(2^t - 1),\quad k_0^{(2)} = |S_0^{(2)}|,\quad t \geqslant 1. \tag{6}$$

Theorem 4. *The set S of* (6) *is a minimal* 1*-saturating set.*

Proof. As $S_0 = S_0^{(1)} \cup S_0^{(2)}$, we have by (6), $S \cup (S+S) \supseteq \Sigma_t \cup S_0 \cup (S_0 + S_0) \cup (S_0^{(1)} + \Sigma_t) \cup (S_0^{(2)} + \Sigma_t) \cup (S_0 + (S_0^{(2)} + \Sigma_t)) \cup ((S_0^{(2)} + \Sigma_t) + (S_0^{(2)} + \Sigma_t))$. By condition (c), we need all points of Σ_t to get $S_0^{(1)} + \Sigma_t$. As S_0 is a minimal 1-saturating set, $S_0 \cup (S_0 + S_0) = \Sigma_r$ and all points of S_0 are necessary. By condition (a), $S_0 + (S_0^{(2)} + \Sigma_t) \supseteq (\Sigma_r \setminus S_0) + \Sigma_t$. For the last relation we need all points of $S_0^{(2)} + \Sigma_t$, see condition (b). So, $S \cup (S+S) \supseteq \Sigma_t \cup \Sigma_r \cup (\Sigma_r + \Sigma_t) = \Sigma_{r+t}$ and all the points of S are needed. □

Construction $\mathbf{E_B}$. We take the $(2^{r_1}+2^{r_2}-3)$-set $S_0 = (\Sigma_{r_1} \setminus \{P_1\}) \cup \{P\} \cup (\Sigma_{r_2} \setminus \{P_2\}) \subset \Sigma_r$ of Construction B and put $S_0^{(2)} = S_0 \setminus \{P\}$. Conditions (a)–(c) hold. We have

$$S = \Sigma_t \cup S_0 \cup (S_0^{(2)} + \Sigma_t) \subset \Sigma_{r+t},\ k = 2^t(2^{r_1} + 2^{r_2} - 3),$$
$$r = r_1 + r_2,\ r_1, r_2 \geqslant 2,\ t \geqslant 1. \tag{7}$$

In the space Σ_{r+t} for S_0 we keep representation (2) adding t zeroes to the end of every vector. Every point of Σ_t is represented in Σ_{r+t} by an $(r+t)$-positional vector with r zeroes in the beginning. For the such representation one can easily prove the following.

Corollary 1. *The set S of* (7) *is a minimal* 1*-saturating set with the r_1-peculiarity.*

3. Construction "Polygon". Pentagon-C

We use representation (2) putting $r_2 = r-1$. So, $\Sigma_{r_2} = \Sigma_{r-1}$ is a hyperplane of Σ_r. We consider a $2r$-set $\mathbf{P}_{2r} = \mathbf{A} \cup \mathbf{B}$ where $\mathbf{A} = \{A_1, \ldots, A_r\}$, $\mathbf{B} = \{B_1, \ldots, B_r\}$, A_i and B_j are points of Σ_r, $A_i = (0a_i) \in \Sigma_{r-1}$, $a_i = 2^{i-1} - \lfloor i/r \rfloor$, $B_j = (1b_j) \in \Sigma_r \setminus \Sigma_{r-1}$, $b_j = 2^{j-1} - 1$. The set $\mathbf{P}_{2r}$ contains exactly r inner lines forming a *polygon* with r vertexes B_j. We obtain a minimal 1-saturating $(2r+s)$-set $\mathbf{P}_{2r}^{(s)} = \mathbf{P}_{2r} \cup \mathbf{M}$ where $\mathbf{M} \subset \Sigma_r$ is an s-set. The stabilizer group for the polygon $\mathbf{P}_{2r}$ is the dihedral group D_r [16, Section 2.13] with $|D_r| = 2r$. The dihedral D_r partitions the space Σ_r into l-orbits where the size l is a divisor of $2r$. Always there are one 1-orbit and two r-orbits coinciding with the subsets $\mathbf{A}$ and $\mathbf{B}$. To find $\mathbf{M}$ we add to $\mathbf{P}_{2r}$ (using computer) some *whole* orbits until the set obtained becomes 1-saturating. Then we check the minimality and find its stabilizer group G. As we added whole orbits, either $G = D_r$ or $|G|$ is multiple of $2r$. Usually if $G = D_r$ the structure of $\mathbf{P}_{2r}^{(s)}$ is interesting.

Example 2. By computer we obtained the following minimal 1-saturating sets $\mathbf{P}_{2r}^{(s)}$: the sets $\mathbf{P}_{10}^{(1)}$, $\mathbf{P}_{12}^{(3)}$, $\mathbf{P}_{14}^{(8)}$ with $G = D_r$; the sets $\mathbf{P}_{12}^{(5)}$ and $\mathbf{P}_{14}^{(15)}$ with $|G| = 120$ and $|G| = 40{,}320$; 75 examples of projectively distinct sets $\mathbf{P}_{16}^{(s)}$ where 64 sets have $G = D_8$, 9 and 2 sets have $|G| = 32$ and $|G| = 256$, respectively, and the sizes $16 + s$ cover completely the region 33–46.

Pentagon-C. In (8) we give an 11-set $\mathbf{P}_{10}^{(1)}$ called "Pentagon with center" (*Pentagon-C*) where a point C is the center. The first 10 points of $\mathbf{P}_{10}^{(1)}$ form the pentagon $\mathbf{P}_{10}$.

$$\mathbf{P}_{10}^{(1)} = \left[\begin{array}{c|c} & C \\ & 1 \\ \mathbf{P}_{10} & 1 \\ & 0 \\ & 1 \\ & 0 \end{array}\right] = \left[\begin{array}{ccccc|ccccc|c} A_1 & A_2 & A_3 & A_4 & A_5 & B_1 & B_2 & B_3 & B_4 & B_5 & C \\ 0 & 0 & 0 & 0 & 0 & 1 & 1 & 1 & 1 & 1 & 1 \\ 0 & 0 & 0 & 1 & 1 & 0 & 0 & 0 & 0 & 1 & 1 \\ 0 & 0 & 1 & 0 & 1 & 0 & 0 & 0 & 1 & 1 & 0 \\ 0 & 1 & 0 & 0 & 1 & 0 & 0 & 1 & 1 & 1 & 1 \\ 1 & 0 & 0 & 0 & 1 & 0 & 1 & 1 & 1 & 1 & 0 \end{array}\right] \subset \Sigma_5. \tag{8}$$

The pentagon is showed by complete lines in Fig. 1. Stroke-lines are bisecants. For $\mathbf{P}_{10}^{(1)}$ we have $G = D_r$. The center C is the 1-orbit. The pentagon $\mathbf{P}_{10}$ consists of two 5-orbits $\mathbf{A} \subset \Sigma_4$, $\mathbf{B} \subset \Sigma_5 \setminus \Sigma_4$, where $\mathbf{A}$ is the complete cap corresponding to the perfect repetition code [21]. For every point $F \in \mathbf{P}_{10}^{(1)}$ *exactly one* point of Σ_5, say N_F, is not covered by the bisecants of $\mathbf{P}_{10}^{(1)} \setminus \{F\}$. There are six points N_F. We have $N_C = C$, while the other points N_F belong to $\Sigma_5 \setminus \mathbf{P}_{10}^{(1)}$ and form a 5-orbit $\{O_1, \ldots, O_5\} \subset \Sigma_5 \setminus \Sigma_4$. The remaining external points form three 5-orbits $\{O_6, \ldots, O_{10}\} \subset \Sigma_5 \setminus \Sigma_4$, $\{H_1, \ldots, H_5\} \subset \Sigma_4$, $\{H_6, \ldots, H_{10}\} \subset \Sigma_4$.

We use $\mathbf{P}_{10}^{(1)}$ as the starting set S_0 in Construction E with $S_0^{(2)} = \mathbf{B}$. By Fig. 1, the conditions (a)–(c) of Construction E hold. By (6), we obtain the infinity family of k-sets S:

$$S \subset \Sigma_{5+t}, \quad k = 5 + 3 \cdot 2^{t+1}, \quad t \geqslant 1. \tag{9}$$

4. Construction "addition of space lift"

We describe a construction ASL based on ideas of [2, Theorem 3.3, 14, Theorems 2, 5], where the minimality of the sets obtained has not been studied. Iterative applying ASL forms infinite families of minimal 1-saturating sets using convenient starting sets.

Definition 1. We denote by c_u^i an u-positional *column* vector which is a binary representation of a nonnegative integer i. A binary matrix has a property U_f if its first f rows contain *all* distinct *nonzero* f-positional binary columns. A binary matrix with the property U_f has a property U_f^* if its first f rows do not contain the zero vector c_f^0. A binary matrix with the property U_f has a property U_f^x, $x \neq 0$, if the vector c_f^x appears exactly once in its first f rows and, besides, these rows contain the zero vector c_f^0.

Construction ASL. Let $\mathbf{V}_0 \subset \Sigma_{r_0}$ be a *starting* k_0-set written in the matrix form so that $t_0 + m_0 = r_0$, the first t_0 rows are a $t_0 \times k_0$ matrix $\mathbf{V}_{t_0}$ and the last m_0 rows make an $m_0 \times k_0$

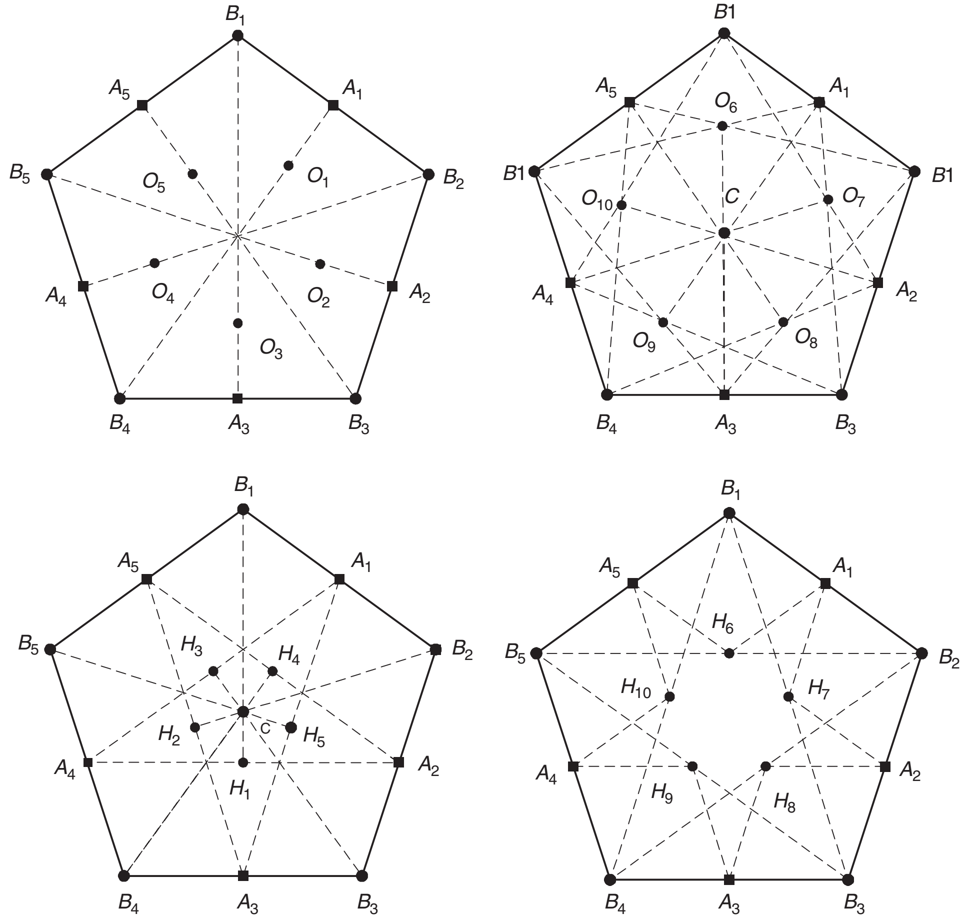

Fig. 1. Four 5-orbits of external points O_i and H_j of "Pentagon-C".

matrix $\mathbf{V}_{m_0}$. A new $(k_0 + 2^{m_0})$-set $\mathbf{V} \subset \Sigma_{r_0+1}$ has the form

$$\mathbf{V} = \left[\begin{array}{c|cccccccc} \mathbf{V}_{t_0} & c^g_{t_0} & c^g_{t_0} & c^g_{t_0} & c^g_{t_0} & c^g_{t_0} & \ldots & c^g_{t_0} & c^g_{t_0} \\ \hline & 0 & 0 & 0 & 0 & 0 & \ldots & 1 & 1 \\ & 0 & 0 & 0 & 0 & 0 & \ldots & 1 & 1 \\ & \vdots & \vdots & \vdots & \vdots & \vdots & \ldots & \vdots & \vdots \\ \mathbf{V}_{m_0} & 0 & 0 & 0 & 0 & 1 & \ldots & 1 & 1 \\ & 0 & 0 & 1 & 1 & 0 & \ldots & 1 & 1 \\ & 0 & 1 & 0 & 1 & 0 & \ldots & 0 & 1 \\ \hline \underbrace{0\,0\ldots 0}_{k_0} & \multicolumn{8}{c}{\underbrace{1 \quad 1 \quad 1 \quad 1 \quad 1 \quad \ldots \quad 1 \quad 1}_{2^{m_0}}} \end{array}\right] = [\,\mathbf{V}_l \;\; \mathbf{V}_r\,] \subset \Sigma_{r_0+1},$$

$$\begin{bmatrix} \mathbf{V}_{t_0} \\ \mathbf{V}_{m_0} \end{bmatrix} = \mathbf{V}_0 \subset \Sigma_{r_0}, \tag{10}$$

where the submatrices $\mathbf{V}_l$ and $\mathbf{V}_r$ consist, respectively, of the left k_0 and the right 2^{m_0} columns of $\mathbf{V}$. The columns of $\mathbf{V}_r$ are as follows. The first t_0 positions of every column is a binary representation of a nonnegative integer g. The next m_0 positions of these columns form a 2^{m_0}-set consisting of all the distinct binary m_0-positional columns including the zero column, i.e., it is the m_0-dimensional space of binary vectors. The last position is 1.

Theorem 5. (i) *If a starting set* $\mathbf{V}_0$ *is a minimal* 1*-saturating* k_0*-set with the property* $U^*_{t_0}$ *and the* t_0*-peculiarity then the set* $\mathbf{V}$ *of* (10) *with* $g \neq 0$ *is a minimal* 1*-saturating* $(k_0 + 2^{m_0})$*-set with the same property* $U^*_{t_0}$ *and the* t_0*-peculiarity.*

(ii) *If a starting set* $\mathbf{V}_0$ *is a minimal* 1*-saturating* k_0*-set with the property* $U^x_{t_0}$ *and the* t_0*-peculiarity then the set* $\mathbf{V}$ *of* (10) *with* $g \neq x$ *is a minimal* 1*-saturating* $(k_0 + 2^{m_0})$*-set with the same property* $U^x_{t_0}$ *and the* t_0*-peculiarity.*

Proof. We denote by $\mathbf{T}_{u,0}$ the $(2^{u-1}-1)$-set consisting of all the distinct binary u-positional nonzero columns with zero on the last position. Let $\mathbf{T}_{u,1}$ be the 2^{u-1}-set consisting of all the distinct binary u-positional columns with unity on the last position. So, $\mathbf{T}_{u,0} \cup \mathbf{T}_{u,1} = \Sigma_u$.

As $\mathbf{V}_0$ is a 1-saturating set, $(\mathbf{V}_l + \mathbf{V}_l) \cup \mathbf{V}_l = \mathbf{T}_{r_0+1,0}$. Since the matrix $\mathbf{V}_0$ has the property U_{t_0} it holds for (i) and (ii) that $(\mathbf{V}_l + \mathbf{V}_r) \cup \mathbf{V}_r = \mathbf{T}_{r_0+1,1}$. So, $(\mathbf{V} + \mathbf{V}) \cup \mathbf{V} = \mathbf{T}_{r_0+1,0} \cup \mathbf{T}_{r_0+1,1} = \Sigma_{r_0+1}$ and $\mathbf{V}$ is a 1-saturating set for (i) and (ii).

Now we show that $\mathbf{V}$ is a *minimal* 1-saturating set or, in the other words, that $\mathbf{V}\backslash\{P\}$ is not a 1-saturating set for all points P of $\mathbf{V}$.

Let $P = P_l \in \mathbf{V}_l$. Then $P_l = (P'_l 0)$ where $P'_l \in \mathbf{V}_0$. Since $\mathbf{V}_0$ has the t_0-peculiarity, some point $B'_l \in \Sigma^*_{r_0,t_0}$ is unsaturated by $\mathbf{V}_0\backslash\{P'_l\}$. Hence the point $B_l = (B'_l 0) \in \Sigma^*_{r_0+1,t_0}$ is unsaturated by $\mathbf{V}_l\backslash\{P_l\}$. But $(\mathbf{V}_r + \mathbf{V}_r) \cap \Sigma^*_{r_0+1,t_0} = \emptyset$ as in $\mathbf{V}_r$ the first t_0 positions of all columns are equal. So, B_l is unsaturated by $\mathbf{V}\backslash\{P_l\}$ for the both cases (i) and (ii).

Let $P = P_r \in \mathbf{V}_r$.

(i) As $\mathbf{V}_0$ has the property $U^*_{t_0}$ the matrix $\mathbf{V}_{t_0}$ has not the zero column and $(\mathbf{V}_l+\mathbf{V}_r)\cap\mathbf{V}_r = \emptyset$. Hence P_r does not lie on any line inside $\mathbf{V}$ and P_r is unsaturated by $\mathbf{V}\backslash\{P_r\}$.

(ii) Let $c^x_{t_0} + c^g_{t_0} = c^y_{t_0}$. Then one point of the form $(c^y_{t_0} c^z_{m_0} 1)$ is unsaturated by $\mathbf{V}\backslash\{P_r\}$ since $c^x_{t_0}$ appears only once in the first t_0 rows of $\mathbf{V}_0$.

Now we show that $\mathbf{V}$ preserves the t_0-peculiarity. Note that $B_l \in \Sigma^*_{r_0+1,t_0}$. For (i) we have $P_r \in \Sigma^*_{r_0+1,t_0}$ as $g \neq 0$. For (ii) it holds that $(c^y_{t_0} c^z_{m_0} 1) \in \Sigma^*_{r_0+1,t_0}$ since $g \neq x$.

The properties $U^*_{t_0}$ for (i) and $U^x_{t_0}$ for (ii) are preserved due to $g \neq 0$ and $g \neq x$. □

Construction ASL can be applied iteratively. As $\sum_{i=0}^{d-1} 2^i = 2^d - 1$, we have

Corollary 2. *If* $r_0 = t_0 + m_0$ *and in the space* Σ_{r_0} *there is a minimal* 1*-saturating* k_0*-set* $\mathbf{V}_0$ *with the* t_0*-peculiarity and with one of the properties either* $U^*_{t_0}$ *or* $U^x_{t_0}$ *then in the space* Σ_{r_0+d} *there is a minimal* 1*-saturating* k_d*-set* $\mathbf{V}_d$ *with the* t_0*-peculiarity and with the same property* $U^*_{t_0}$ *or* $U^x_{t_0}$ *so that*

$$\mathbf{V}_d \subset \Sigma_{r_0+d}, \quad r_0 = t_0 + m_0, \quad k_d = k_0 + 2^{m_0} \cdot (2^d - 1), \quad d \geqslant 0. \tag{11}$$

If the starting sets for ASL are obtained by Constructions A–C we do not get new sizes but we can obtain projectively distinct sets. To get new sizes one can obtain the starting sets, e.g., by Constructions D,E,"Polygon", by computer, by using the known parity check matrices of covering codes of [1,2,6,14].

Example 3. (a) The 11-set "Pentagon-C" of (8) has the 3-peculiarity and the property U_3^x, $x \neq 4$. We have $t_0 = 3$ and, by (11), we obtain the infinite family of k-sets $\mathbf{V}$ with

$$\mathbf{V} \subset \Sigma_{v+1}, \quad k = 2^{v-2} + 7, \quad v \geqslant 4. \tag{12}$$

(b) In the process of the complete classification of Section 6 we got a minimal 1-saturating 16-set $\mathbf{V}_0 \subset \Sigma_6$ with the 2-peculiarity and the property U_2^3.

$$\mathbf{V}_0 = \begin{bmatrix} 0000000 & 0000 & 1111 & 1 \\ 0000000 & 1111 & 0000 & 1 \\ 0001111 & 0111 & 0011 & 1 \\ 0010001 & 0001 & 0001 & 1 \\ 0100010 & 0001 & 0100 & 0 \\ 1000111 & 0011 & 0111 & 0 \end{bmatrix} \subset \Sigma_6.$$

We have $t_0 = 2$, $m_0 = 4$, and, by (11), obtain the infinite family of k-sets $\mathbf{V}$ with

$$\mathbf{V} \subset \Sigma_{v+1}, \quad k = 2^{v-1}, \quad v \geqslant 5. \tag{13}$$

(c) Codes corresponding to 1-saturating k_0-sets $\mathbf{V}_0 \subset \Sigma_{r_0}$ with $r_0 = 2b$, $k_0 = 7\cdot 2^{b-2} - 2$, are obtained in [14, Theorem 2]. We conjecture that if we move the last b rows of the matrix $\mathbf{V}_0$ of [14, Formulas (31),(35)] to the top we get a minimal 1-saturating set with the b-peculiarity and with the property U_b^x. By computer we verified it for $b = 4$, $r_0 = 8$, $t_0 = m_0 = 4$, $k_0 = 26$. By (11), we obtain the infinite family of minimal 1-saturating k-sets $\mathbf{V}$ with

$$\mathbf{V} \subset \Sigma_{v+1}, \quad k = 2^{v-3} + 10, \quad v \geqslant 7. \tag{14}$$

5. "Addition of space lift" for complete caps

Definition 2. A binary matrix has a property L_p^* if its last p rows contain *all* distinct *nonzero* p-positional columns and the zero p-positional column is absent in these rows.

Theorem 6. (i) *If in Construction ASL of* (10) *a starting set* $\mathbf{V}_0 \subset \Sigma_{r_0}$ *is a complete cap with the property* $U_{t_0}^*$ *then the set* $\mathbf{V} \subset \Sigma_{r_0+1}$ *with* $g \neq 0$ *is a complete cap with the same property* $U_{t_0}^*$.

(ii) *If the set* $\mathbf{V}_0$ *of* (10) *has the property* $L_{h_0}^*$, $h_0 \leqslant m_0$, *then* $\mathbf{V}$ *has the property* $L_{h_0+1}^*$.

Proof. (i) As $\mathbf{V}_0$ is a cap, $(\mathbf{V}_l + \mathbf{V}_l) \cap \mathbf{V}_l = \emptyset$. By (10) and the property $U_{t_0}^*$, $(\mathbf{V}_r + \mathbf{V}_r) \cap \mathbf{V}_l = \emptyset$, $(\mathbf{V}_l + \mathbf{V}_r) \cap \mathbf{V}_r = \emptyset$. So, $\mathbf{V}$ is a cap.

As $\mathbf{V}_0$ is a complete cap, $(\mathbf{V}_l + \mathbf{V}_l) \cup \mathbf{V}_l = \mathbf{T}_{r_0+1,0}$ where $\mathbf{T}_{u,0}$ and $\mathbf{T}_{u,1}$ are introduced in the proof of Theorem 5. Since the matrix $\mathbf{V}_0$ has the property $U_{t_0}^*$ we have $(\mathbf{V}_l + \mathbf{V}_r) \cup \mathbf{V}_r = \mathbf{T}_{r_0+1,1}$. So, $(\mathbf{V} + \mathbf{V}) \cup \mathbf{V} = \mathbf{T}_{r_0+1,0} \cup \mathbf{T}_{r_0+1,1} = \Sigma_{r_0+1}$. Hence, $\mathbf{V}$ is a complete cap.

For $\mathbf{V}$ the property $U_{t_0}^*$ holds directly by (10) due to $g \neq 0$.

(ii) It holds by the structure of (10). Since $h_0 \leqslant m_0$, sets of rows providing the properties $U_{t_0}^*$ and $L_{h_0}^*$ are distinct and they do not intersect. □

Corollary 3. *If $r_0 = t_0 + m_0$ and in Σ_{r_0} there is a complete k_0-cap $\mathbf{V}_0$ with the property $U^*_{t_0}$ then in Σ_{r_0+d} there is a complete k_d-cap $\mathbf{V}_d$ with the same property $U^*_{t_0}$ so that*

$$\mathbf{V}_d \subset \Sigma_{r_0+d}, \quad r_0 = t_0 + m_0, \quad k_d = k_0 + 2^{m_0} \cdot (2^d - 1), \quad d \geqslant 0. \tag{15}$$

Remark 1. Let us suppose that in the space Σ_{r_0}, $r_0 = t_0 + h_0 + \delta$, $t_0 \geqslant 1$, $h_0 \geqslant 1$, $\delta \geqslant 0$, there exists a complete k_0-cap $\mathbf{V}_0$ with the properties $U^*_{t_0}$ and $L^*_{h_0}$. Due to $\delta \geqslant 0$, in a $(t_0+h_0+\delta) \times k_0$ matrix $\mathbf{V}_0$, the sets of rows providing the properties $U^*_{t_0}$ and $L^*_{h_0}$ are distinct and they do not intersect. So, one can rearrange rows and obtain a matrix with properties $U^*_{h_0}$ and $L^*_{t_0}$. This change preserves the property of the matrix $\mathbf{V}_0$ to be a complete cap. We can use this approach in an iterative process applying ASL on every step. It is easy to see that for a complete cap $\mathbf{V}_d$, obtained in the iterative process by d steps, the size k_d and the pair of properties $U^*_{h_0+a}$, $U^*_{t_0+b}$ depend only on the fact: how many times (say, $i \leqslant d$) the properties $U^*_{h_0+a}$ based on h_0 were used. (Respectively, the properties $U^*_{t_0+b}$ based on t_0 were used $d - i$ times.) The order of using properties $U^*_{h_0+a}$ and $U^*_{t_0+b}$ is not important. Properties $U^*_{h_0+a}$, $U^*_{t_0+b}$ appear due to Theorem 6, part (ii), and rearranging rows.

Corollary 4. *We suppose that there exists a complete k_0-cap $\mathbf{V}_0 \subset \Sigma_{r_0}$ with the properties $U^*_{t_0}$ and $L^*_{h_0}$, where $r_0 = t_0+h_0+\delta$, $t_0 \geqslant 1$, $h_0 \geqslant 1$, $\delta \geqslant 0$. Then there exist complete $k_{d,i}$-caps $\mathbf{V}_{d,i}$ with the properties $U^*_{t_0+i}$, $L^*_{h_0+d-i}$ such that*

$$\mathbf{V}_{d,i} \subset \Sigma_{r_0+d}, \quad k_{d,i} = k_0 + (2^i - 1) \cdot 2^{t_0+\delta} + (2^{d-i} - 1) \cdot 2^{h_0+\delta}, \quad d \geqslant i \geqslant 0. \tag{16}$$

Proof. We may rearrange rows as in Remark 1. Relation (16) can be proved by induction. □

Example 4. We take k_0-caps of [14, Theorem 4, Remark 2] as starting caps $\mathbf{V}_0 \subset \Sigma_{2v}$ with the properties U^*_v and L^*_v. Here $k_0 = 15 \cdot 2^{v-3} - f(v)$, $v \geqslant 4$, $f(4) = 2$, $f(v) = 3$ if $v \geqslant 5$.

For $v = 4$ we modify the 28-cap of [14, Formula (51)]. We add the sum of two last rows to the 4th row and obtain the starting complete 28-cap $\mathbf{V}_0$ with the properties U^*_4, L^*_4.

For $v \geqslant 5$ we directly take as $\mathbf{V}_0$ the matrix U^{2v} of [14, Formulas (31),(39)–(42)] with $e_i \neq 0$ in the matrix Y of [14, Formula (31)]. Such matrix U^{2v} gives a complete $(15 \cdot 2^{v-3} - 3)$-cap in Σ_{2v} with the properties U^*_v and L^*_v. In the matrix U^{2v} we put $\beta = (11\dots1)$, $\gamma = (101\dots1)$, $\delta = \beta + \gamma = (010\dots0)$, $w_1 = (0\dots01) \neq 0$, $w_2 = (0\dots010) \neq 0$, $w_3 = w_1 + w_2 = (0\dots011)$, and $e_i = (010\dots0)$ in Y, see [14, p. 223, Example] for $v = 5$.

One can apply Corollary 4 with $t_0 = h_0 = v$, $\delta = 0$, $r_0 = 2v$, and $k_0 = 15 \cdot 2^{v-3} - f(v)$. For $v \geqslant 4$ we obtain complete $k_{d,i}$-caps $\mathbf{V}_{d,i}$ with the properties U^*_{v+i}, L^*_{v+d-i} such that

$$\begin{aligned} &\mathbf{V}_{d,i} \subset \Sigma_{2v+d}, \quad k_{d,i} = (2^{i+3} + 2^{d-i+3} - 1) \cdot 2^{v-3} - f(v), \\ &d \geqslant i \geqslant 0, \quad v \geqslant 4. \end{aligned} \tag{17}$$

Now for $v \geqslant 5$ we rearrange rows in the matrix U^{2v} with parameters $\beta, \gamma, \delta, w_j, e_i$, as above. We write the 3rd, $(v+1)$th, and $(v+2)$th rows as the first three ones and obtain a complete $(15 \cdot 2^{v-3} - 3)$-cap in Σ_{2v} with the property U_3^*. One can illustrate this process considering [14, p. 223, Example]. We can apply Corollary 3 with $r_0 = 2v$, $t_0 = 3$, $k_0 = 15 \cdot 2^{v-3} - 3$ for $v \geqslant 5$. We obtain complete k_d-caps $\mathbf{V}_d$ with the property U_3^* so that

$$\mathbf{V}_d \subset \Sigma_{2v+d}, \quad k_d = (2^{v+d} - 2^v + 15) \cdot 2^{v-3} - 3, \quad d \geqslant 0, \ v \geqslant 5. \tag{18}$$

Example 5. For (10), as starting complete caps $\mathbf{V}_0 \subset \Sigma_{2b+1}$ with the properties U_b^*, L_{b+1}^* we take complete k_0-caps of [14, Theorem 5, Remark 2] where $k_0 = 23 \cdot 2^{b-3} - 3$, $b \geqslant 4$.

To get the needed form we modify the matrix of the cap Ψ^{2m-1} in [14, Formula (50)] writing the 1st row as the last one and putting $e_u = (11\ldots1) \neq 0$ in the matrix J. We take $e_i = (010\ldots0) \neq 0$ in the matrix Y of [14, Formula (31)] . In the matrix $U^{2(m-1)}$ of [14, Theorem 5] we put parameters β, γ, δ, w_j as in Example 4. For $b \geqslant 5$, we obtain a complete $(23 \cdot 2^{b-3} - 3)$-cap $\mathbf{V}_0 \subset \Sigma_{2b+1}$ with the properties U_b^*, L_{b+1}^*.

It is noted in [14, Remark 2] that $b = 4$ is a special case. We put in the matrix $U^{2\cdot 4}$ of [14, Formulas (39),(50)] $\beta = (111)$, $\gamma = (101)$, $\delta = (010)$, $w_1 = w_2 = w_3 = 1$, $e_i = (0100)$, $e_u = (1111)$. We write the 1st row as the last one and obtain the modified matrix $\Psi^{2\cdot 5-1}$. We examined by computer that it is a complete cap with the properties U_4^* and L_5^*.

We apply Corollary 4 with $t_0 = b$, $h_0 = b + 1$, $\delta = 0$, $r_0 = 2b + 1$, $k_0 = 23 \cdot 2^{b-3} - 3$. We obtain complete $k_{d,i}$-caps $\mathbf{V}_{d,i}$ with the properties U_{b+i}^*, $L_{b+1+d-i}^*$ such that

$$\mathbf{V}_{d,i} \subset \Sigma_{2b+d+1}, \quad k_{d,i} = (2^{i+3} + 2^{d-i+4} - 1) \cdot 2^{b-3} - 3, \quad d \geqslant i \geqslant 0, b \geqslant 4. \tag{19}$$

Now for $b \geqslant 4$ we rearrange rows in the modified matrix Ψ^{2m-1} with parameters β, γ, δ, w_j, of Example 4. We write the 3rd, $(b+1)$th, and $(b+2)$th rows as the first three ones and obtain a complete $(23 \cdot 2^{b-3} - 3)$-cap in Σ_{2b+1} with the property U_3^*. We apply Corollary 3 with $r_0 = 2b + 1$, $t_0 = 3$, $k_0 = 23 \cdot 2^{b-3} - 3$, and obtain complete k_d-caps $\mathbf{V}_d$ with the property U_3^* so that

$$\mathbf{V}_d \subset \Sigma_{2b+d+1}, \quad k_d = (2^{b+d+1} - 2^{b+1} + 23) \cdot 2^{b-3} - 3, \\ d \geqslant 0, \ b \geqslant 4. \tag{20}$$

Example 6. We use complete caps of [20] as starting sets. Denote by $\mathbf{M}_w^r \subset \Sigma_r$ the set of all $\binom{r}{w}$ points represented by binary vectors whose Hamming weight is equal to w. In [20, Theorem 2.3] it is proved that the union $\mathbf{V}_0^{(s)} = \bigcup_{w=2s+1}^{3s+1} \mathbf{M}_w^{3s+1} \subset \Sigma_{3s+1}$ is a complete cap for $s \geqslant 1$. It is easy to see that $\mathbf{V}_0^{(s)}$ has properties U_{s+1}^*, L_{s+1}^*. Corollary 4 with $t_0 = h_0 = s + 1$, $\delta = s - 1$, gives complete $k_{s,d,i}$-caps $\mathbf{V}_{s,d,i}$ with properties U_{s+1+i}^*, $L_{s+1+d-i}^*$ so that

$$\mathbf{V}_{s,d,i} \subset \Sigma_{3s+d+1}, \ k_{s,d,i} = \sum_{w=2s+1}^{3s+1} \binom{3s+1}{w} + 2^{2s}(2^i + 2^{d-i} - 2), \\ s \geqslant 1, \quad d \geqslant i \geqslant 0. \tag{21}$$

6. On classification and spectrum of possible sizes of minimal 1-saturating sets

In Table 1 we give the complete classification of the minimal 1-saturating k-sets, including complete caps, in $PG(v,2)$, $v \leqslant 5$, for all k, and in $PG(6,2)$ for $k \leqslant 20$. The minimal 1-saturating sets with inner lines are noted by "inl" in the column "type". The value n is the number of objects of type noted. In the column "stab. group - constr." the order of the stabilizer group of a minimal 1-saturating set is written and the construction obtaining the set with this stabilizer group is indicated if it is known. A stabilizer group order up to 24 has two indexes. The superscript is the ordinal number of the structure of the group with such order in [16, Table 2.3]. The subscript is the number of groups with the same order and structure in our table. For example, 8_3^4 notes three groups $\mathbf{D}_4$. A subscript of the group order greater than 24 is the number of group with the same order in our table, here subscripts "1" are not written. Constructions A–E are indicated by the corresponding letter. Other letters mean the following: P—Construction "Polygon", H—the complement to the hyperplane, L_j—Construction ASL with parameters from formula (j), UL—Construction ASL used iteratively with a starting set obtained by Construction U, E_t—Construction E with parameters from formula (t). If equivalent sets can be obtained from distinct constructions we note only one. A construction written for a group order with a subscript greater than one provides only one of the sets with such order.

We obtained the classification in this work using an exhaustive computer search with backtracking algorithms considered in [8,22]. Note that in [19] complete caps of $PG(v,2)$, $v \leqslant 6$, are considered. However, sets with inner lines are not studied in [19].

We use the following notations [8] for the space $PG(v,2)$: $l(v,2,1)$ is the smallest size of a minimal 1-saturating set; $m(v,2,1)$, $m'(v,2,1)$, and $m''(v,2,1)$ are the sizes of the largest, the 2nd largest, and the 3rd largest minimal 1-saturating set, respectively; $t_2(v,2)$ is the smallest size of the complete cap. By [8, Corollary 1], $m(v,2,1) = 2^v$. By Table 1,

$$l(2,2,1) = m''(2,2,1) = m'(2,2,1) = 4;$$
$$l(3,2,1) = m''(3,2,1) = 5, \quad m'(3,2,1) = 6,$$

$$l(4,2,1) = 9, \; m''(4,2,1) = 10, \; m'(4,2,1) = 11,$$

$$l(5,2,1) = 13, \; m''(5,2,1) = 18, \; m'(5,2,1) = 20; \quad l(6,2,1) = 19.$$

Note that in $PG(6,2)$ there is a complete 21-cap [14, Theorem 3] but there are not complete k-caps with $k \leqslant 20$, see Table 1 and [19]. Hence $t_2(6,2) = 21$. Such conjecture has been done in [14, p. 222]. The values of $l(v,2,1)$, $v \leqslant 6$, are given also in [1, Table 2].

In [11, Remark 5, p. 271] five distinct complete 17-caps in $PG(5,2)$ are constructed and the conjecture that this list is complete has been done. This conjecture is proved by an exhaustive computer search in this work (see Table 1, $k = 17$, type "cap") and in [19]. That allows us to obtain all nonequivalent complete $17 \cdot 2^{v-5}$-caps in $PG(v,2)$, $v \geqslant 6$, by $(v-5)$-fold applying Construction D to complete 17-caps in $PG(5,2)$ [11].

The stabilizer groups in Table 1 are obtained by computer except when Construction A and complement to hyperplane (the notations A and H) are used, see below.

Table 1
Complete classification of minimal 1-saturating k-sets for all k in $PG(v, 2)$, $v \leqslant 5$, and for $k \leqslant 20$ in $PG(6, 2)$

v	k	Type	n	Stab. group - constr.	v	k	Type	n	Stab. group - constr.
2	4	Arc Inl	1 1	24_1^4-H 6_1^2-A	5	14	Inl	19	$8_2^3, 12_1^3, 24_5^4, 32_5$, 96, 168, 192, 576, 1152, 56448-A
3	5	Cap	1	120-B	5	15	Inl	14	$4_1^2, 8_2^3, 8_3^4$-L_{12}, 12_3^3-P, $16_1^8, 24_1^7, 32_2$, 72
3	6	Inl	1	72-A	5	16	Inl	16	$1_1^1, 2_6^1, 4_4^2, 8_2^3, 12_2^3, 16_1^8$-$L_{13}$
3	8	Cap Inl	1 1	1344-H 168-A	5	17	Cap Inl	5 48	384, 576-L_{21}, 720, 11520, 40320-L_{21} $2_7^1, 3_1^1, 4_7^2, 6_2^2, 8_2^3, 8_3^4, 10_5^2$-$E_9$, 16_1^8, $20_2^3, 24_1^4, 32_2, 48_2$, 64, 96_5, 120-P, 192, 384-BL, 1140_2, 1152, 8064-B
4	9	Cap Inl	1 1	336-L_{21} 144-B	5	18	Cap Inl	1 108	10752-D $2_{14}^1, 4_3^1, 4_1^2, 6_{12}^2, 8_{12}^3, 8_4^4, 12_1^3$, $16_{21}^8, 16_4, 32_3, 48_{14}, 64_5$, 96, 128, 144, $192_2, 384_3, 768_2$-E_B, 1152-AL, 2688_2, 120960-A
4	10	Cap Inl	1 6	1920-D $8_1^4, 12_1^3, 48_2$, 192-E_B, 1008-A	5	20	Cap Inl	1 1	184320-D 9216-E_B
4	11	Inl	1	10_1^2-P	5	32	Cap Inl	1 1	319979520-H 9999360-A
4	16	Cap Inl	1 1	322560-H 20160-A	6	19	Inl	5	32,120,480,1440,5760
5	13	Cap Inl	1 7	1152-L_{21} 32, 48, 96_3, 1152-B, 4032-D_A	6	20	Inl	36	$4_3^2, 8_2^3, 8_1^4, 16_6^5, 16_1^8, 32_4$, 64, 96, $128_4, 144_2, 192_4$, 384_3-C, 1152_3, 2880

Let $GL(t, 2)$ be the group of nonsingular linear transformations of the t-dimensional binary vector space. Its order is $|GL(t, 2)| = \prod_{u=0}^{t-1} (2^t - 2^u)$ [5, 21, Appendix B].

For representation (2), where $r_1 + r_2 = r$, we consider the set $S = \Sigma_{r_1} \cup \Sigma_{r_2} \subset \Sigma_r$ in correspondence to relation (3) of Construction A. It can be shown that for $r_1 = r_2 = r/2$ the stabilizer group G_1 of S has the cardinality $|G_1| = 2 \cdot |GL(r_1, 2)|^2$. If $r_1 \neq r_2$ for the stabilizer group G_2 of S it holds: $G_2 \cong GL(r_1, 2) \times GL(r_2, 2)$, $|G_2| = |GL(r_1, 2)| \cdot |GL(r_2, 2)|$.

Theorem 7. *Let C be the binary cap of maximal size in $PG(v,2)$. The stabilizer group of C is $AGL(v,2)$ and its order is* $2^v \prod_{i=0}^{v-1}(2^v - 2^i)$.

Proof. The stabilizer group of C stabilizes also the complement of C that is a hyperplane. The subgroup of $GL(v+1,2)$, stabilizing an hyperplane, is isomorph to $AGL(v,2)$ [5]. □

Many sets obtained in the process of the classification given in Table 1 have interesting symmetrical structure connected with inner lines and orbits of a stabilizer group.

Example 7. In $PG(6,2)$ we consider a minimal 1-saturating 19-set S with the 3-peculiarity and the property U_3^* and describe its properties obtained by computer.

$$S = \left[\begin{array}{c|c} 011 & 0001\ 0100\ 0001\ 1011 \\ 101 & 0000\ 0011\ 0110\ 0101 \\ 000 & 1111\ 1111\ 1111\ 1111 \\ 000 & 0000\ 0000\ 1111\ 1111 \\ 000 & 0000\ 1111\ 0000\ 1111 \\ 000 & 0011\ 0011\ 0011\ 0011 \\ 000 & 0101\ 0101\ 0101\ 0101 \end{array}\right] = Q_1 \cup Q_2. \tag{22}$$

The stabilizer group of S has the size 5760 and partitions $PG(6,2)$ into a 3-orbit Q_1, a 16-orbit Q_2, a 48-orbit Q_3, and a 60-orbit Q_4. The first three columns of (22) are Q_1, the rest ones are Q_2. The only inner line is Q_1. The 16-orbit Q_2 is an incomplete cap, points of which form a parity check matrix of a code D. We have $Q_1 + Q_2 = Q_3$, $Q_2 + Q_2 = Q_4$, where $Q_i + Q_j$ is the sum of sets. All 48 bisecants corresponding to $Q_1 + Q_2$ are critical. Every point of Q_4 lies on two of 120 bisecants corresponding to $Q_2 + Q_2$. We denote by M the set of all linear dependent fours of points of Q_2. We have $|M| = 20$. Every pair of points of Q_2 belongs to one four of M. So, M is a 2-(16,4,1) design or the Steiner system $S(2,4,16)$ [21, Section 2.5]. We can treat the system $S(2,4,16)$ as an affine plane of order 4 [21, Section 2.5]. Note that the last five rows of the right part of the matrix in (22) can be considered as an affine plane of order 4 if in the last four rows the pairs 00, 01, 10, 11 are treated as elements of the field F_4. Clearly, it is another plane than that connected with the Steiner system. Besides, the five rows mentioned form the maximal 2^4-cap in $PG(4,2)$.

The code D and its dual code $D^\perp$ have the symmetrical weight spectrums, respectively, $w_0 = w_{16} = 1$, $w_4 = w_{12} = 20$, $w_6 = w_{10} = 160$, $w_8 = 150$, and $w_0 = w_{16} = 1$, $w_6 = w_{10} = 48$, $w_8 = 30$, where w_i is the number of codewords of weight i. The code D words form 2-(16,4,1), 2-(16,6,20), 2-(16,10,60), 2-(16,12,11), and 3-(16,8,15) designs. The code $D^\perp$ words form 2-(16,6,6), 2-(16,10,18), and 3-(16,8,3) designs.

We use the set of (22) as the set S_0 in Construction E with $S_0^{(2)} = Q_2$. By above, conditions (a)–(c) of Construction E hold. By (6), we obtain the infinity family of k-sets S:

$$S \subset \Sigma_{7+t}, \quad k = 2 + 17 \cdot 2^t, \quad t \geqslant 1. \tag{23}$$

Table 2
Sizes of the known minimal 1-saturating sets in $PG(v,2)$, $v \geqslant 6$

v	$l(v,2,1)$	Sizes of the known minimal 1-saturating sets in $PG(v,2)$	m
6	19	**19**-L_{24}, **20**-C, 21, 22, **23**-PL_{12}, 24-29, **30**, 31, **32**-L_{13}, 33, 34, 36, 40, 64	64
7	$\geqslant 25$	**26**-L_{14}, **27**, 28-63, **64**-L_{13}, 65, 66, 68, 72, 80, 128	128
8	$\geqslant 34$	**39**-**41**, **42**-L_{14}, 43-127, **128**-L_{13}, 129, 130, 132, 136, 144, 160, 256	256
9	$\geqslant 47$	**51**-**56**, 57, **58**, **59**, 60-255, **256**-L_{13}, 257, 258, 260, 264, 272, 288, 320, 512	512

Finally, by (11), we obtain the infinite family of minimal 1-saturating k-sets $\mathbf{V}$ with

$$\mathbf{V} \subset \Sigma_{v+1}, \quad k = 2^{v-2} + 3, \quad v \geqslant 6. \tag{24}$$

In Table 2 sizes of the known minimal 1-saturating sets in $PG(v,2)$, $v \geqslant 6$, are given. We denote $m = m(v,2,1) = 2^v$. For the lower bounds on $l(v,2,1)$, see [1, Table 2].

As complete caps are minimal 1-saturating sets we wrote (using the usual font) the known sizes of binary complete caps from [7,9,11,14,19,20,24]. The sizes not coinciding with the known sizes of complete caps are marked by the bold font. All they are obtained by minimal 1-saturating sets with *inner lines*. We used results of Sections 2–4 and relation (24). Notations for constructions are the same as in Table 1. Some bold sizes are obtained in this work by an exhaustive computer search with backtracking algorithms of [8,22] or by randomized greedy algorithms of [8,9] with starting sets from Sections 2–4 and [14,18]. The size 2^{v-1} of sets with inner lines of (13) is interesting as there is the conjecture [11, Remark 4] that in $PG(v,2)$ complete 2^{v-1}-caps do not exist. Note that, by (19), we obtained also the complete 59-cap in $PG(9,2)$.

Clearly many sizes of known complete caps can be obtained also by sets with inner lines, see Sections 2–4, Table 1, and Remark 2 below. In particular, sets of Construction "Polygon", sets obtained in the process of the complete classification, and the known parity check matrices of covering codes can be used as starting sets for Constructions E and ASL, see Construction E_B, Examples 3, 7, and (9), (23).

Remark 2. We consider the open problem of the sizes of *large* minimal 1-saturating k-sets with inner lines. Here $k > 2^{v-1}$. For complete caps the problem is solved in [11, Theorems 1, 2, Corollaries 1, 4], see also [3, 17, Theorem 4.3ii]. Namely, in $PG(v,2)$ for each $g = 0, 2, 3, \ldots, v-1$ there is a complete $(2^{v-1} + 2^{v-1-g})$-cap and *large* complete caps of any other sizes do not exist.

If one take $r_1 = 1$ in (3) and $r_1 = 2$ in (4) and (7) then Constructions A, B, and E_B provide that in $PG(v,2)$ for each $g = 0, 2, 3, \ldots, v-1$ there is a minimal 1-saturating $(2^{v-1} + 2^{v-1-g})$-set with inner lines. However for $v \geqslant 6$ we do not know if large minimal 1-saturating sets with inner lines exist for other sizes. For $v = 5$ the answer "they do not exist" is given by Table 1. For $v \geqslant 6$, by computer, we have not found large sets of sizes other than $2^{v-1} + 2^{v-1-g}$ and it is possible that they do not exist really.

Finally, we conjecture that in $PG(v,2)$ there exists, up the projective equivalence, an unique minimal 1-saturating 2^v-set with inner lines.

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