Consider the problem of coding for a source with compound side-information $\mathfrak{S}_1 = (U, W_1, W_2)$ where $\mathcal{U} = \{1, 2\}, \mathcal{W} = \{1, 2, 3, 4\}$. The SI sets for the two constituent source-side-information pairs of random variables are as in Table I. This source is the dual of the channel considered in Section IV-C.

Observe that $S_1(a) \cap S_2(b) \neq \emptyset, \forall a, b \in \mathcal{U}$. The distinguishability graph for both source-side-information pairs is identical and is the complete graph on $\mathcal{U}$, shown in Fig. 1(a), which we denote by G. $\max_k R_{\rm w}(G_k) = R_{\rm w}(G) = 0$ bit/channel use. However, given any two sequences u^n and u'^n, there exists a possible side-information sequence w^n such that both (u^n, w^n) and (u'^n, w^n) are observable with probability greater than zero: at every coordinate i choose w_i from $S_1(u_i) \cap S_2(u'_i)$. Thus, we can never reliably distinguish any pair of source sequences based on the side-information, which implies that the minimum asymptotic rate for this source is $\log 2 = 1$ bit/sample, which is consistent with Theorem 3 since G_{12} and G_{21} are edge free graphs. Therefore, in general, $R(P_{UW_1 \ldots W_M})$ is strictly greater than $\max_k R(P_{UW_k})$. However, the minimum asymptotic rate is, somewhat surprisingly, indeed $\max_k R(P_{UW_k})$ [11], [12] if Bob (but not Alice) knows which of the M side-information sequences he is observing.

VI. The Compound Channel: Asymptotically Vanishing Error V. Zero Error

We observe that there are a number of differences between the conventional (asymptotically vanishing error) and zero-error scenarios. First, the conventional capacity does not increase if the decoder has side-information about the channel while the zero-error capacity does. The difference arises because, even without side-information, the decoder can almost reliably identify the channel in operation based on the channel output, which yields $C(\mathfrak{C}) = C_{\rm dec}(\mathfrak{C})$ in the conventional case. However, if we require zero-error, *almost* reliable identification is not sufficient and $C^0(\mathfrak{C}) \leq C^0_{\rm dec}(\mathfrak{C})$, possibly with strict inequality.

The aforementioned phenomenon also leads to another difference: in the conventional case, since the decoder can always effectively (almost reliably) know the channel in operation, $C_{\rm dec}(\mathfrak{C}) \leq C_{\rm enc}(\mathfrak{C})$. In the zero-error case, this inequality holds often but not always. The exception is encountered when some $G_{ss'}, s \neq s'$ in $G(\mathfrak{C})$ is the empty graph, and no $G_{ss}, s \in \mathcal{S}$ is empty. In this case, the encoder cannot convey its knowledge to the decoder and $C^0_{\rm enc}(\mathfrak{C})$ is zero while if the decoder knew the channel in operation, transmission at nonzero rates would be possible. Such a case arose in the example considered in Section IV. Although $C^0_{\rm enc} < C^0_{\rm dec}$ implies that $C^0_{\rm enc} = 0$, in general, we could have $C^0(\mathfrak{C}) < C^0_{\rm dec}(\mathfrak{C})$ with neither quantity being zero. For example, if $\mathcal{G} = \{G_{11}, G_{12}, G_{21}, G_{22}\}$ where G_{11} and G_{22} are both symmetrically directed complete graphs (edge set contains all ordered pairs of distinct vertices) on K vertices and G_{12} and G_{21} are identical symmetrically directed graphs on K vertices with a single pair of edges. From the discussion in Section III, there exists a compound channel $\mathfrak{C}$ whose characteristic set of graphs $\mathcal{G}(\mathfrak{C}) = \mathcal{G}$. For this channel, $C^0(\mathfrak{C}) = 1$ bit/channel use while $C^0_{\rm dec}(\mathfrak{C}) = \log_2 K$ bits/channel use.

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Locally Optimal (Nonshortening) Linear Covering Codes and Minimal Saturating Sets in Projective Spaces

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***Abstract*—A concept of locally optimal (LO) linear covering codes is introduced in accordance with the concept of minimal saturating sets in projective spaces over finite fields. An LO code is nonshortening in the sense that one cannot remove any column from a parity-check matrix without increasing the code covering radius. Several q^m-concatenating constructions of LO covering codes are described. Taking a starting LO code as a "seed", such constructions produce an infinite family of LO codes with the same covering radius. The infinite families of LO codes are designed using minimal saturating sets as starting codes. New upper bounds on the length function are given. New extremal and classification problems for linear covering codes are formulated and investigated, in particular, the spectrum of possible lengths of LO codes including the greatest possible length. The complete computer classification of the minimal saturating sets in small geometries and of the corresponding LO codes is obtained.**

***Index Terms*—Covering codes, covering density, covering radius, minimal saturating sets in projective geometry, nonshortening covering codes.**

I. Introduction

We consider linear covering codes, saturating sets in the projective spaces over finite fields, and connections between these objects. We

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introduce a new concept of locally optimal covering codes connected with the concept of minimal saturating sets.

Let F_q be the Galois field of q elements and let $F_q^* = F_q \setminus \{0\}$. A q-ary linear code with codimension r has *covering radius* R if every r-positional q-ary column is equal to a linear combination of at most R columns of a parity-check matrix of this code and R is the smallest value with such property [1], [4], [17].

Let PG(v,q) be the v-dimensional projective space over F_q [15], [16]. For an integer ϱ with $0 \le \varrho \le v$, a set of points $S \subseteq$ PG(v,q) is *ϱ-saturating* if for any point $x \in$ PG(v,q) there exist $\varrho+1$ points in S generating a subspace of PG(v,q) in which x lies and ϱ is the smallest value with such property [9]. The saturating sets are called "R-spanning sets" in [2], "thick sets" in [3], "saturated sets" in [6], [20].

A ϱ-saturating set S is called *minimal* if for every point $P \in S$ the set $S \setminus \{P\}$ is not ϱ-saturating [3], [9], [18], [20].

Definition 1: A linear covering code is called *locally optimal* (LO) if one cannot remove any column from the parity-check matrix of the code without increasing the covering radius. An LO code can be called also *nonshortening* in the sense mentioned.

Denote by $[n, n-r, d]_q R$ a q-ary linear code of length n, codimension r, minimum distance d, and covering radius R. In this notation one may omit d and R. The length function $l(r, R; q)$ is the smallest length of an $[n, n-r]_q R$ code [2].

The points of a ϱ-saturating n-set in PG$(r-1,q)$ can be considered as columns of a parity-check matrix of an $[n, n-r]_q R$ code with $R = \varrho + 1$ [2], [5], [6], [9], [11], [14], [18]. Points of a minimal saturating set form a parity-check matrix of an LO code.

The concept of LO covering codes essentially extends the region of combinatorial investigations of linear codes. It allows us to introduce new extremal and classification problems. For example, we propose to study the maximal possible length $m(r, R; q)$ of an $[n, n-r]_q R$ LO code and the spectrum of possible lengths of LO codes. The known extremal problems, e.g., the length function, also can be considered effectively in the framework of LO codes.

In Section II, we consider new q^m-concatenating constructions of LO codes. These constructions take an LO code as a starting point and produce an infinite family of LO codes of growing codimension with the same covering radius and almost the same covering density as the starting code. A parity-check matrix of a starting code is repeated q^m times in a parity-check matrix of a new code. Constructions described use ideas of works [4]–[6], [11] modifying them for LO codes.

In Section III, we consider new extremal problems connected with the maximal possible length $m(r, R; q)$ and the second greatest length $m'(r, R; q)$ of an $[n, n-r]_q R$ LO code. We give here also new constructions of LO codes aimed at these problems.

In Section IV, we consider new classification problems connected with the spectrum of possible lengths of LO codes. For $q \le 137$ we give lengths n of $[n, n-3]_q 2$ LO codes known from literature and obtained by computer in this work. With the help of computer we obtained also the complete classification of $[n, n-r, d]_q R$ LO codes for small r, q, R. To get the computer results mentioned we applied geometrical methods treating parity-check matrices of LO codes as minimal saturating sets. We propose also a geometrical construction for LO codes. Then, using the computer results and the constructions of Sections III as starting codes for q^m-concatenating constructions and for the geometrical construction, we design infinite families of LO codes thereby showing ways for obtaining LO codes with distinct lengths.

In Section V, we give new bounds on the length function. Using the classification of Section IV we get a few new exact bounds. Applying geometrical computer approaches we found short $[n, n-4]_q 3$ LO codes, $q \le 563$, giving new upper bounds on $l(4, 3; q)$. We use these codes as a starting point for a q^m-concatenating construction and design an infinite family of LO codes improving upper bounds on the length function $l(3t+1, 3; q)$.

Some results of this correspondence was represented without proofs in [8].

II. q^m-Concatenating Constructions of Locally Optimal Covering Codes

All matrices and columns below are q-ary. An element of F_{q^m} written in a q-ary matrix denotes an m-dimensional column that is a q-ary representation of this element, and vice versa, we can treat a q-ary m-dimensional column as an element of F_{q^m}.

We can treat a matrix as the set of its columns.

The following lemma is evident.

Lemma 1: For $q \ge 2$ an LO linear covering code always has the minimum distance $d \ge 3$.

Further we consider only *linear combinations* of *nonzero distinct* q-ary *columns* with *nonzero* q-ary *coefficients*. Such linear combinations have the form $L = \sum_i c_i h_i$ where $c_i \in F_q^*$, $h_i \in F_{q^r}^*$, $r \ge 1$, $h_i \ne h_j$ if $i \ne j$.

Definition 2: Let $\boldsymbol{H}$ be a parity-check matrix of an $[n, n-r]_q R$ code.

i) Suppose that $1 \le \gamma \le R$. If a column $f \in F_{q^r}$ is equal to a linear combination L of γ columns from $\boldsymbol{H}$ then L is a *generating γ-combination* for f. If the exact value of γ is not important one may say simply "a *generating combination* for f''. But always, by definition, the number of summands of a generating combination lies in the region $1, \dots, R$.
ii) A partition of the column set of the matrix $\boldsymbol{H}$ into nonempty subsets is called an *R-partition* if for every column of $F_{q^r}^*$ there is a generating combination from columns of $\boldsymbol{H}$ belonging to *distinct subsets*.
iii) A partition of $\boldsymbol{H}$ into n one-element subsets is called *trivial*.

We give a general view of a q^m-concatenating construction based on the ideas of [4]–[6], [11]. Then we describe concrete variants taking into consideration the local optimality.

Construction CC. We use a *starting* $[n_0, n_0 - r_0]_q R$ LO code V_0 with a parity-check matrix $\boldsymbol{H}_0 = [h_1 h_2 \dots h_{n_0}]$ where columns $h_j \in F_{q^{r_0}}$. Let $m \ge 1$ be an integer parameter. We suppose that $\boldsymbol{H}_0$ has an R-partition $\mathcal{P}_0$ to p_0 subsets. For every column h_j we assign an *indicator* $\beta_j \in F_{q^m}$ so that if columns h_i and h_j belong to distinct subsets of $\mathcal{P}_0$ then $\beta_i \ne \beta_j$. If h_i and h_j belong to the same subset we may assign either $\beta_i = \beta_j$ or $\beta_i \ne \beta_j$ as well. We denote $\mathcal{B} = \{\beta_1, \beta_2, \dots, \beta_{n_0}\}$. Let $\boldsymbol{A}$ be a matrix with $r_0 + Rm$ rows. The parity-check matrix $\boldsymbol{H}_V$ of a new $[n, n-(r_0+Rm)]_q R_V$ code V has the form

$$\boldsymbol{H}_V = [\boldsymbol{A}\boldsymbol{B}_\Sigma], \quad \boldsymbol{B}_\Sigma = [\boldsymbol{B}_1 \boldsymbol{B}_2 \dots \boldsymbol{B}_{n_0}] \tag{1}$$

$$\boldsymbol{B}_j = \begin{bmatrix} h_j & h_j & \cdots & h_j \\ \xi_1 & \xi_2 & \cdots & \xi_{q^m} \\ \beta_j \xi_1 & \beta_j \xi_2 & \cdots & \beta_j \xi_{q^m} \\ \beta_j^2 \xi_1 & \beta_j^2 \xi_2 & \cdots & \beta_j^2 \xi_{q^m} \\ \cdots & \cdots & \cdots & \cdots \\ \beta_j^{R-1}\xi_1 & \beta_j^{R-1}\xi_2 & \cdots & \beta_j^{R-1}\xi_{q^m} \end{bmatrix} \tag{2}$$

where $\{\xi_1, \xi_2, \dots, \xi_{q^m}\} = F_{q^m}, \xi_1 = 0$.

Let $1 \le \delta \le R$. If the zero r_0-dimensional column has a generating δ-combination from columns of $\boldsymbol{H}_0$ belonging to distinct subsets of $\mathcal{P}_0$ we say that $t_0 = \delta$. Else $t_0 = 0$.

We use the following notations: $w_{m,q} = (q^m-1)/(q-1)$, $\boldsymbol{W}_m$ is a parity-check matrix of the $[w_{m,q}, w_{m,q}-m]_q 1$ LO Hamming code, $\mathbf{0}_k$ is the zero matrix with k rows.

Construction CC$_1$. Here $R=2$, $q^m > p_0$, $\mathcal{B} \subseteq F^*_{q^m}$, $t_0 = 0$, $n = n_0 q^m + 2w_{m,q}$

$$\boldsymbol{A} = \left[\begin{array}{c|c} \mathbf{0}_{r_0} & \mathbf{0}_{r_0} \\ \boldsymbol{W}_m & \mathbf{0}_m \\ \mathbf{0}_m & \boldsymbol{W}_m \end{array}\right].$$

Theorem 1: The code V of Construction CC$_1$ has covering radius $R_V = 2$. It is an $[n, n-(r_0+2m)]_q 2$ LO code. Its parity-check matrix $\boldsymbol{H}_V$ has a 2-partition into p_0 subsets.

Proof: To prove $R_V = 2$ we show that an arbitrary nonzero column $\mathcal{C} = (a, b_1, b_2)$ with $a \in F_{q^{r_0}}$, $b_1, b_2 \in F_{q^m}$, is equal to a linear combination of at most two columns of $\boldsymbol{H}_V$.

Case 1) $a = \sum_{i=1}^{2} s_i h_{j_i} \neq 0$, $s_1, s_2 \in F^*_q$, columns h_{j_1}, h_{j_2} belong to distinct subsets of $\mathcal{P}_0$. We have

$$\mathcal{C} = \sum_{i=1}^{2} s_i (h_{j_i}, x_i, \beta_{j_i} x_i)$$

where x_i values are found from the equation system

$$\sum_{i=1}^{2} s_i x_i \beta_{j_i}^{u-1} = b_u, \qquad u = 1, 2. \tag{3}$$

Case 2) $a = s_1 h_{j_1} \neq 0$, $s_1 \in F^*_q$.
We consider two variants.

$$\mathcal{C} = s_1(h_{j_1}, x_1, \beta_{j_1} x_1) + v(0,0,w), \quad x_1 = b_1/s_1.$$
$$\mathcal{C} = s_1(h_{j_1}, x'_1, \beta_{j_1} x'_1) + v'(0,w',0), \quad x'_1 = b_2/s_1\beta_{j_1}.$$

Here, $v, v' \in F_q$, $w, w' \in \boldsymbol{W}_m$.

Case 3) $a = 0$.
Here $(0, b_1, b_2) = v_1(0, w_1, 0) + v_2(0, 0, w_2)$ where $v_1, v_2 \in F_q$, $w_1, w_2 \in \boldsymbol{W}_m$.

Now we show that V is an LO code. As the starting code V_0 is LO, every column h_j of $\boldsymbol{H}_0$ necessarily takes part in either Case 1 or Case 2. If we remove a column $(h_j, \xi, \beta_j \xi)$ from $\boldsymbol{B}_j$ then there exist values of b_1, b_2 for which the system of (3) gives $x_1 = \xi$. Besides, for $b_1 = \xi s_1$, $b_2 = \xi s_1 \beta_{j_1}$, we have $x_1 = x'_1 = \xi$ in Case 2. As result, the linear combinations for $\mathcal{C}$ used in Cases 1 and 2 become impossible. So, all columns of $\boldsymbol{B}_\Sigma$ are essential.

Since $0 \notin \mathcal{B}$ the column $(0, b, 0)$, $b \neq 0$, can be represented only as $(0, b, 0) = v(0, w, 0)$, $v \in F^*_q$, $w \in \boldsymbol{W}_m$. Similarly, $(0,0,b) = v(0,0,w)$. So, we need also all columns of $\boldsymbol{A}$.

We design a 2-partition $\mathcal{P}_V$ of the matrix $\boldsymbol{H}_V$. We partition the submatrix $\boldsymbol{B}_\Sigma$ into p_0 subsets corresponding to the 2-partition $\mathcal{P}_0$. For every j, all the columns of $\boldsymbol{B}_j$ belong to the same subset of $\mathcal{P}_V$. It is possible as above distinct columns of $\boldsymbol{B}_j$ never take part in the same linear combination for $\mathcal{C}$. The columns of $\boldsymbol{B}_i$ and $\boldsymbol{B}_j$ belong to the same subsets of $\mathcal{P}_V$ if and only if columns h_i and h_j belong to the same subsets of $\mathcal{P}_0$. Further, taking into account two variants of Case 2, we can inscribe the columns of the left part of $\boldsymbol{A}$ to some subset of $\boldsymbol{B}_\Sigma$ and columns of the right part of $\boldsymbol{A}$ to another subset of $\boldsymbol{B}_\Sigma$. □

Construction CC$_2$. Here $R = 2$, $n_0 \geq q^m \geq p_0$, $\mathcal{B} = F_{q^m}$, $t_0 = 0$, $n = n_0 q^m + w_{m,q}$

$$\boldsymbol{A} = \begin{bmatrix} \mathbf{0}_{r_0+m} \\ \boldsymbol{W}_m \end{bmatrix}.$$

Theorem 2: The code V of Construction CC$_2$ has covering radius $R_V = 2$. It is an $[n, n-(r_0+2m)]_q 2$ LO code. Its parity-check matrix $\boldsymbol{H}_V$ has a 2-partition into $2p_0 + 1$ subsets.

Proof: The condition $\mathcal{B} = F_{q^m}$ is possible since $n_0 \geq q^m$. One can prove the theorem similarly to Theorem 1. In Case 3 for $b_1 \neq 0$ the indicator $\beta_i = b_2/b_1$ always belongs to $\mathcal{B} = F_{q^m}$. We put $(0, b_1, b_2) = (h_i, b_1, \beta_i b_1) - (h_i, 0, 0)$ including the situation $b_2 = 0$.

To get a 2-partition $\mathcal{P}_V$ of the matrix $\boldsymbol{H}_V$, in the beginning we partition $\boldsymbol{B}_\Sigma$ into p_0 subsets in the same manner as in Theorem 1 except for inscribing columns of $\boldsymbol{A}$. Then we partition every subset $\mathcal{T}$ obtained into two subsets so that the first one consists of the columns $(h_i, 0, 0)$ of all the submatrices $\boldsymbol{B}_i$ belonging to $\mathcal{T}$. The last subset of $\mathcal{P}_V$ is $\boldsymbol{A}$.□

Construction CC$_3$. Here $R = 3$, $q^m > p_0$, $\mathcal{B} \subseteq F^*_{q^m}$, $t_0 = 0$, $n = n_0 q^m + 3w_{m,q}$

$$\boldsymbol{A} = \left[\begin{array}{c|c|c} \mathbf{0}_{r_0} & \mathbf{0}_{r_0} & \mathbf{0}_{r_0} \\ \boldsymbol{W}_m & \mathbf{0}_m & \mathbf{0}_m \\ \mathbf{0}_m & \boldsymbol{W}_m & \mathbf{0}_m \\ \mathbf{0}_m & \mathbf{0}_m & \boldsymbol{W}_m \end{array}\right].$$

Theorem 3: The code V of Construction CC$_3$ has covering radius $R_V = 3$. It is an $[n, n-(r_0+3m)]_q 3$ LO code.

Proof: We consider a nonzero column $\mathcal{C} = (a, b_1, b_2, b_3)$, $a \in F_{q^{r_0}}$, $b_1, b_2, b_3 \in F_{q^m}$.

Case 1) $a = \sum_{i=1}^{3} s_i h_{j_i} \neq 0$, all $s_i \in F^*_q$, all columns h_{j_i} belong to distinct subsets of $\mathcal{P}_0$.
We have

$$\mathcal{C} = \sum_{i=1}^{3} s_i (h_{j_i}, x_i, \beta_{j_i} x_i, \beta_{j_i}^2 x_i)$$

where x_i values are found from the linear system

$$\sum_{i=1}^{3} s_i x_i \beta_{j_i}^{u-1} = b_u, \qquad u = 1, 2, 3. \tag{4}$$

Case 2) $a = \sum_{i=1}^{2} s_i h_{j_i} \neq 0$, $s_1, s_2 \in F^*_q$, columns h_{j_1}, h_{j_2} belong to distinct subsets of $\mathcal{P}_0$.
We consider three variants with $k = 1, 2, 3$, respectively. For $v_k \in F_q$, $g_k \in \boldsymbol{A}$, , $w_k \in \boldsymbol{W}_m$, we have

$$\mathcal{C} = \sum_{i=1}^{2} s_i \left(h_{j_i}, x_i^{(k)}, \beta_{j_i} x_i^{(k)}, \beta_{j_i}^2 x_i^{(k)}\right) + v_k g_k$$
$$\sum_{i=1}^{2} s_i x_i^{(k)} \beta_{j_i}^{u-1} = b_u, \quad u = u_1^{(k)}, u_2^{(k)} \tag{5}$$

where $\{u_1^{(k)}, u_2^{(k)}\} = \{1,2,3\} \setminus \{k\}$,
$g_1 = (0, w_1, 0, 0)$, $g_2 = (0, 0, w_2, 0)$, $g_3 = (0, 0, 0, w_3)$. In the kth variant we solve the system of two equations and find $x_i^{(k)}$. The systems for $k = 1, 3$ can be solved always. If $\beta_{j_1}^2 = \beta_{j_2}^2$ the system for $k = 2$ has no solution.

Case 3) $a = s_1 h_{j_1} \neq 0$, $s_1 \in F^*_q$.
We consider three variants with $k = 1, 2, 3$, respectively. For $v_t^{(k)} \in F_q$, $g_t^{(k)} \in \boldsymbol{A}$, $w_s^{(k)} \in \boldsymbol{W}_m$, we have

$$\mathcal{C} = s_1 \left(h_{j_1}, x_1^{(k)}, \beta_{j_1} x_1^{(k)}, \beta_{j_1}^2 x_1^{(k)}\right) + v_1^{(k)} g_1^{(k)} + v_2^{(k)} g_2^{(k)}$$

where $x_1^{(k)} = b_1/s_1 \beta_{j_1}^{k-1}$

$$g_1^{(1)} = \left(0, 0, w_2^{(1)}, 0\right), \quad g_2^{(1)} = \left(0, 0, 0, w_3^{(1)}\right)$$
$$g_1^{(2)} = \left(0, w_1^{(2)}, 0, 0\right), \quad g_2^{(2)} = \left(0, 0, 0, w_3^{(2)}\right)$$
$$g_1^{(3)} = \left(0, w_1^{(3)}, 0, 0\right), \quad g_2^{(3)} = \left(0, 0, w_2^{(3)}, 0\right).$$

Case 4) $a = 0$.
For $v_i \in F_q$, $w_i \in \boldsymbol{W}_m$, we have $\mathcal{C} = (0, b_1, b_2, b_3) = v_1(0, w_1, 0, 0) + v_2(0, 0, w_2, 0) + v_3(0, 0, 0, w_3)$.

So, $R_V = 3$. Now we show that V is an LO code. Every column h_j of $\boldsymbol{H}_0$ takes part in Case 1, or Case 2, or Case 3. If we remove a column $(h_j, \xi, \beta_j\xi, \beta_j^2\xi)$ from $\boldsymbol{B}_j$ then there are b_1, b_2, b_3 for which the system of (4) gives $x_1 = \xi$. Besides, it can be shown that there exist b_1, b_2, b_3 such that the systems of (5) give $x_1^{(1)} = x_1^{(2)} = x_1^{(3)} = \xi$. In Case 3 we have $x_1^{(1)} = x_1^{(2)} = x_1^{(3)} = \xi$ if $b_1 = \xi s_1$, $b_2 = \xi s_1\beta_{j_1}$, $b_3 = \xi s_1\beta_{j_1}^2$. As result, the linear combinations for $\mathcal{C}$ used in Cases 1, 2, 3 become impossible. So, all columns of $\boldsymbol{B}_\Sigma$ are essential.

For the variant of Case 4 we need all columns of $\boldsymbol{A}$. In fact the only other possibility is $\mathcal{C} = (0, b_1, b_2, b_3) = v_1 g + v_2(h_t, x, \beta_t x, \beta_t^2 x) - v_2(h_t, 0, 0, 0)$, $v_i \in F_q$, $g \in \boldsymbol{A}$, $\beta_t \in \mathcal{B}$, $x \neq 0$. For a vector $(0, b, 0, 0)$, $b \neq 0$, such variant needs either $\beta_t = 0 \notin \mathcal{B}$ or $\mathcal{C} = v_1(0, w, 0, 0)$. For vectors $(0, 0, b, 0)$ or $(0, 0, 0, b)$ it is possible only $\mathcal{C} = v_1(0, 0, w, 0)$ or $\mathcal{C} = v_1(0, 0, 0, w)$. So, we come back to the variant of Case 4 and see again that all columns of $\boldsymbol{A}$ are necessary. □

Construction CC$_4$. Here $R = 3, q^m > p_0, \mathcal{B} \subseteq F_{q^m}\backslash\{1\}, t_0 = 3$. Besides, there exists a column $\bar{h} \in \boldsymbol{H}_0$ having the only generating combination $L = \bar{h}$, the indicator assigned to $\bar{h}$ is $\bar{\beta} = 0$

$$\boldsymbol{A} = \left[\begin{array}{c|c} \mathbf{0}_{r_0+m} & \mathbf{0}_{r_0+m} \\ \boldsymbol{W}_m & \mathbf{0}_m \\ \mathbf{0}_m & \boldsymbol{W}_m \end{array}\right], \quad n = n_0 q^m + 2w_{m,q}.$$

Theorem 4: The code V of Construction CC$_4$ has covering radius $R_V = 3$. It is an $[n, n-(r_0+3m), 3]_q 3$ LO code.

Proof: One can prove the theorem similarly to Theorem 3 considering a column $\mathcal{C} = (a, b_1, b_2, b_3)$, $a \in F_{q^{r_0}}$, $b_i \in F_{q^m}$. The situation with $a \neq 0$ and regarding the necessity of all the columns of $\boldsymbol{B}_\Sigma$ is the same as in Theorem 3 without variants with columns of the form $(0, w, 0, 0)$. As $t_0 = 3$, the case $a = 0$ is included in Case 1 of Theorem 3. Therefore to prove the necessity of $\boldsymbol{A}$ we consider the case $a = s_1\bar{h}$. We put $\mathcal{C} = s_1(\bar{h}, b_1/s_1, 0, 0) + v_1(0, 0, w_1, 0) + v_2(0, 0, 0, w_2)$, $v_i \in F_q$, $w_i \in \boldsymbol{W}_m$. If we do not use columns of $\boldsymbol{A}$ the only possible variant is $\mathcal{C} = s_1(\bar{h}, x_1, 0, 0) + v(h_t, x_2, \beta_t x_2, \beta_t^2 x_2) - v(h_t, 0, 0, 0)$, $v \in F_q^*$, $\beta_t \in \mathcal{B}$, $x_2 \neq 0$. If $b_2 = b_3$ then $\beta_t = 1 \notin \mathcal{B}$. So, the variant without $\boldsymbol{A}$ is impossible. □

III. On New Extremal Problems

We consider extremal problems connected with the maximal possible length $m(r, R; q)$ of an $[n, n-r]_q R$ LO code and with the second greatest length $m'(r, R; q)$.

We give the direct sum (DS) construction [1], [4, Sec. 3.2], [20, Lemma 10] for LO codes.

Lemma 2: Let $\boldsymbol{H}_1$ and $\boldsymbol{H}_2$ be the parity-check matrices of an $[n_1, n_1 - r_1]_q R_1$ LO code V_1 and an $[n_2, n_2 - r_2]_q R_2$ LO code V_2, respectively. Then the $(r_1 + r_2) \times (n_1 + n_2)$ matrix

$$\boldsymbol{H} = \left[\begin{array}{c|c} \boldsymbol{H}_1 & \mathbf{0}_{r_1} \\ \mathbf{0}_{r_2} & \boldsymbol{H}_2 \end{array}\right] \tag{6}$$

is a parity-check matrix of an $[n_1 + n_2, n_1 + n_2 - (r_1 + r_2)]_q R$ LO code with $R = R_1 + R_2$.

Let $\boldsymbol{I}_m$ be the identity matrix of the order m and let $\boldsymbol{H}^*$ be a parity-check matrix of an $[n^*, n^* - r^*]_q R^*$ LO code. By Lemma 2, the matrix

$$\boldsymbol{H} = \left[\begin{array}{c|c} \boldsymbol{I}_{R-R^*} & \mathbf{0}_{R-R^*} \\ \mathbf{0}_{r^*} & \boldsymbol{H}^* \end{array}\right] \tag{7}$$

is a parity-check matrix of an $[n, n-r]_q R$ LO code with $n = n^* + R - R^*$, $r = r^* + R - R^*$.

Construction of (7) is useful for distinct estimates. For example, by (7)

$$\begin{aligned} m(r, R; q) &\geq m(r^*, R^*; q) + R - R^* \\ m'(r, R; q) &\geq m'(r^*, R^*; q) + R - R^* \end{aligned} \tag{8}$$

where the second inequality holds if $m(r^*, R^*; q) \neq l(r^*, R^*; q)$.

Theorem 5: For the maximal possible length $m(r, R; q)$ of an $[n, n-r]_q R$ LO code it holds that

$$m(r, R; q) \geq M_q(r, R) = (q^{r-R+1} - 1)/(q-1) + R - 1 \tag{9}$$

where $r \geq R \geq 2, q \geq 2$. For $R = 2$ and $r = R$, we have the equalities $m(r, 2; q) = M_q(r, 2)$ and $m(R, R; q) = M_q(R, R)$.

Proof: As $m(r^*, 1; q) = (q^{r^*} - 1)/(q-1)$, the inequality in (9) follows from (8). The equality $m(r, 2; q) = M_q(r, 2)$ is proved in [9, Corollary 1] for minimal 1-saturating sets. □

Note that the complete classification in Section IV, Table II, and Table III, gives

$$\begin{aligned} m(r, 3; q) = M_q(r, 3) \text{ for } \quad & r = 4,\ q = 2, 3, 4, 7; \\ & r = 5,\ q = 2, 3; \\ & r = 6,\quad q = 2. \\ m(r, 4; q) = M_q(r, 4), \text{ for } \quad & r = 5,\ q = 2, 3, 4; \\ & r = 6,\ q = 2, 3; \\ & r = 7,\ q = 2. \end{aligned}$$
$$m(R+1, R; 5) = M_5(R+1, R) + 1, \text{ for } R = 3, 4. \tag{10}$$

By (8) and (10)

$$m(R+1, R; 5) \geq M_5(R+1, R) + 1 = R + 6, \quad R \geq 5. \tag{11}$$

Let $n_r(t)$ be the smallest integer such that any $[n, n-r, d]_2 R$ code with $d \geq 3$, $n \geq n_r(t)$, has $R \leq t$. Evidently, $m(r, R; 2) \leq n_r(R)$. Hence, the upper bounds on $n_r(R)$ of [4, Ch. 18], [21] are valid also for $m(r, R; 2)$. Approaches of the works mentioned with some modifications are useful for our goals. For example, similar to [21] we have.

Lemma 3: It holds that $m(R+1, R; 2) = M_2(R+1, R) = R+2$.

Proof: A parity-check matrix of an $[n, n-(R+1)]_2 R$ LO code can be represented in the form $[\boldsymbol{I}_{R+1}\boldsymbol{U}]$ where $\boldsymbol{U}$ is an $(R+1) \times t$ matrix with $t \geq 1$. The matrix $\boldsymbol{U}$ contains columns only with weight 2 or 3. Otherwise, if a column of weight 4 or greater is present in $\boldsymbol{U}$, the covering radius becomes to be equal to $R - 1$. It is sufficiently for covering radius R to have in $\boldsymbol{U}$ one arbitrary column of weight 2 or 3. So, $t = 1$. □

Definition 3: Let $\boldsymbol{H} = [h_1 h_2 \ldots h_n]$ be a parity-check matrix of an $[n, n-r]_q 2$ code. Here $h_k \in F_{q^r}$ is a column. If a column $f \in F_{q^r}$ has the only generating combination $c_1 h_{i_1} + c_2 h_{i_2} = f$, $c_1, c_2 \in F_q^*$, $h_{i_1}, h_{i_2} \in \boldsymbol{H}$, then $\{h_{i_1}, h_{i_2}\}$ is called a *critical pair of columns*.

Construction A. Let $\boldsymbol{H}_0 = [H_1 H_2]$ be a parity-check matrix of a *starting* $[n_0, n_0 - r_0]_q 2$ LO code V_0. Here $\boldsymbol{H}_j = [h_1^{(j)} h_2^{(j)} \ldots h_{n_j}^{(j)}]$ is an $r_0 \times n_j$ submatrix, $j = 1, 2$, $n_j \geq 1$, $h_t^{(j)} \in F_{q^{r_0}}$ is a column. We assume that the following conditions hold.

a) Every column $f \in F_{q^{r_0}}$ has a generating combination of the form either $c_1 h_{k_j}^{(j)} = f$ or $c_1 h_{k_j}^{(j)} + c_2 h_u^{(2)} = f$ where $c_1, c_2 \in F_q^*$, $j \in \{1, 2\}$, $k_j \in \{1, 2, \ldots, n_j\}$, $u \in \{1, 2, \ldots, n_2\}$.

b) Every column $h_u^{(2)}$ belongs to a critical pair of columns of the form $\{h_s^{(1)}, h_u^{(2)}\}$, $s \in \{1, 2, \ldots, n_1\}$.

c) There is a column $h' \in \boldsymbol{H}_1$ which has not a generating combination of the form $c_1 h_{k_j}^{(j)} + c_2 h_u^{(2)} = h'$ where $c_1, c_2 \in F_q^*$, $j \in \{1, 2\}$, $k_j \in \{1, 2, \ldots, n_j\}$, $u \in \{1, 2, \ldots, n_2\}$.

Let $m \geq 1$ be an integer parameter. The parity-check matrix $\boldsymbol{H}_V$ of a new $[n, n-(r_0+m)]_q R_V$ code V with $n = (q^m-1)/(q-1) + n_1 + q^m n_2$ has the form

$$\boldsymbol{H}_V = [\boldsymbol{L}\boldsymbol{K}_1\boldsymbol{K}_2 \ldots \boldsymbol{K}_{n_2}], \quad \boldsymbol{L} = \begin{bmatrix} \mathbf{0}_{r_0} & \boldsymbol{H}_1 \\ \boldsymbol{W}_m & \mathbf{0}_m \end{bmatrix}$$
$$\boldsymbol{K}_u = \begin{bmatrix} h_u^{(2)} & h_u^{(2)} & \ldots & h_u^{(2)} \\ \xi_1 & \xi_2 & \ldots & \xi_{q^m} \end{bmatrix} \quad (12)$$

where $\{\xi_1, \xi_2, \ldots, \xi_{q^m}\} = F_{q^m}$, $\xi_1 = 0$, $u = 1, 2, \ldots, n_2$.

Theorem 6: The code V of Construction A is an $[n, n-(r_0+m)]_q 2$ LO code.

Proof: We consider a nonzero column $\mathcal{C} = (a, b)$, $a \in F_{q^{r_0}}$, $b \in F_{q^m}$. Let $v \in F_q^*$, $w \in \boldsymbol{W}_m$.

Case 1) $a = 0$.
Here $b \neq 0$ and $\mathcal{C} = v(0, w)$. Another way is $\mathcal{C} = (h_u^{(2)}, b) - (h_u^{(2)}, \xi_1)$ as $\xi_1 = 0$.

Case 2) $a \neq 0$.

i) If $a = ch_u^{(2)}$, $c \in F_q^*$, then $\mathcal{C} = c(h_u^{(2)}, b/c)$. Another way is $\mathcal{C} = c(h_u^{(2)}, \xi_i) + v(0, w), \xi_i \neq b/c$.

ii) If $a = ch_s^{(1)}$, $c \in F_q^*$, then $\mathcal{C} = c(h_s^{(1)}, 0)$ for $b = 0$ or $\mathcal{C} = c(h_s^{(1)}, 0) + v(0, w)$, for $b \neq 0$. For $h_s^{(1)} = h'$ such way is the only, hence all columns of $\boldsymbol{W}_m$ are necessary.

iii) If $a \neq ch_u^{(2)}$ and $a \neq ch_s^{(1)}$ then, by the condition a), $\mathcal{C} = c_1(h_{k_j}^{(j)}, 0) + c_2(h_u^{(2)}, b)$.

So, $R_V = 2$. We need all columns of the form $(h_s^{(1)}, 0)$ and $(h_u^{(2)}, \xi_1)$ as V_0 is an LO code. By the condition b), all columns of all submatrices $\boldsymbol{K}_u$ are necessary. □

Construction B. Let $\{\xi_1, \xi_2, \ldots, \xi_q\} = F_q$, $\xi_1 = 0$, $q \geq 3$. We design a matrix with columns h_i.

$$\boldsymbol{H} = \left[\begin{array}{c|cccccc} 1 & 1 & 0 & 0 & \ldots & 0 \\ 0 & 0 & 1 & 1 & \ldots & 1 \\ 0 & 1 & \xi_2 & \xi_3 & \ldots & \xi_q \end{array}\right] = [h_1 h_2 \ldots h_{q+1}]. \quad (13)$$

Theorem 7: The matrix $\boldsymbol{H}$ of (13) is a parity-check matrix of a $[q+1, q-2]_q 2$ LO code. All pairs of columns of the form $\{h_1, h_i\}$, $i \geq 2$, are critical.

Proof: We consider a nonzero column $\mathcal{C} = (a, b, c)$, $a, b, c \in F_q$, putting $v_i \in F_q^*$, $\gamma \in \{1, 2\}$.

Case 1) $a = 0$. Here $\mathcal{C} = \sum_{i=1}^{\gamma} v_i h_{j_i}$, $j_i \geq 3$. For $\gamma = 2$ the two needed columns h_{j_1}, h_{j_2} exist as $q \geq 3$.

Case 2) $a \neq 0$.

i) Let $b = 0$. Then $\mathcal{C} = \sum_{i=1}^{\gamma} v_i h_{j_i}$, $j_i \leq 2$. If $\mathcal{C} \notin \{(v_1, 0, 0), (v_2, 0, v_2)\}$ we have $\gamma = 2$ and the representation considered is unique. So, $\{h_1, h_2\}$ is a critical pair of columns.

ii) Let $b, c \neq 0$. Then $\mathcal{C} = ah_1 + bh_j$, $b\xi_{j-1} = c$, $j \geq 3$. For $a = c$, $b \neq 0$, it is the only possibility. Changing b we see that all pairs $\{h_1, h_j\}$, $j \geq 3$, are critical.

iii) If $b \neq 0$, $c = 0$, then the only possibility is $\mathcal{C} = ah_2 + bh_j$, $j \geq 3$, $a + b\xi_{j-1} = 0$. □

Corollary 1: For the second greatest length $m'(3, 2; q)$ of an $[n, n-3]_q 2$ LO code it holds that

$$m'(3, 2; q) = q + 1, \quad q \geq 3. \quad (14)$$

Remark 1: The geometrical forms of Construction B and Theorem 7 are given in [3, p. 116], [18, Theorem 3] without a proof. The geometrical conclusion corresponding to Corollary 1 is written in [18, Corollary 3]. Theorem 7 here gives the matrix proof of the assertions of [3], [18]. By the way, note that the proof of [9, Theorem 3] where similar problems are considered is correct only for PG$(2, q)$. Hence the assertions of [9, Theorem 3, Corollary 2] are right only in PG$(2, q)$.

Corollary 2: There exists an $[n, n-r]_q 2$ LO code with $n = q^{r-2} + (q^{r-3}-1)/(q-1) + 1$, $r \geq 3$, $q \geq 3$.

Proof: As the matrix $\boldsymbol{H}_0$ of Construction A we use the matrix of (13) putting $n_1 = 1$. From (13) and Theorem 7 it follows that the conditions a)–c) of Construction A hold. □

Corollary 3: For the second greatest length $m'(r, R; q)$ of an $[n, n-r]_q R$ LO code we have

$$m'(r, R; q) \geq T_q(r, R) = q^{r-R} + \frac{q^{r-R-1}-1}{q-1} + R - 1 \quad (15)$$

where $q \geq 3, r > R \geq 2$. For $r = 3$, $R = 2$, the equality $m'(3, 2; q) = T_q(3, 2)$ holds.

Proof: By Corollary 2, it holds that $m'(r^*, 2; q) \geq q^{r^*-2} + (q^{r^*-3}-1)/(q-1) + 1$. Then we use (8). The case $r = 3$, $R = 2$, is considered in Corollary 1. □

Note that the complete classification in Section IV, Table II and Table III, gives

$$\begin{aligned} m'(R+1, R; q) &= T_q(R+1, R), && \text{for } R = 3,\ q = 3, 4, 7; \\ & && R = 4,\ q = 3, 4. \\ m'(R+2, R; q) &= T_q(R+2, R), && \text{for } R = 3, 4, \quad q = 3. \\ m'(R+1, R; 5) &= T_5(R+1, R) + 1, && \text{for } R = 3, 4. \end{aligned} \quad (16)$$

By (8) and (16), it holds that

$$m'(R+1, R; 5) \geq T_5(R+1, R) + 1 = R + 5, \qquad R \geq 3. \quad (17)$$

For binary LO codes, by results of [13] for complete caps and by (8) and Theorem 5, we have

$$\begin{aligned} 2^{r-1} &> m'(r, 2; 2) \geq 5 \cdot 2^{r-4}, \\ m'(r, R; 2) &\geq \Theta_2(r, R) = 5 \cdot 2^{r-R-2} + R - 2. \end{aligned} \quad (18)$$

By results of [12] for minimal 1-saturating sets and the complete classification in Section IV, Table II, and Table III, we have

$$\begin{aligned} m'(R+4, R; 2) &= \Theta_2(R+4, R), && \text{for } R = 2 \\ m'(r, R; 2) &= \Theta_2(r, R) + 1, && \text{for } r = R+2, R+3 \\ & && R = 2, 3, 4. \end{aligned} \quad (19)$$

By (8) and (19), it holds that

$$m'(r, R; 2) \geq \Theta_2(r, R) + 1, \quad r = R+2, R+3, \qquad R \geq 2. \quad (20)$$

IV. New Classification Problems. Spectrum of Possible Lengths of Locally Optimal Codes

We consider classification problems connected with the spectrum of possible lengths of LO codes. Constructions of Section III and of [5], [6], [9], [14] advance these problems and give starting codes for q^m-concatenating constructions forming infinite code families.

For spectrum problems we can also use computer. We give examples of LO codes obtained as minimal saturating sets by geometrical computer methods. These codes are interesting themselves and can be applied as starting ones in q^m-concatenating constructions. They are

TABLE I
LENGTHS n OF THE KNOWN $[n, n-3]_q 2$ LO CODES WITH $n \le q$

q	lengths n	q	lengths n
7.	$6 \le n \le 7$	61	$20 \le n \le 57$
8.	$6 \le n \le 8$	64	$19 \le n \le 61,\ n \ne 20, 21$
9.	$6 \le n \le 9$	67	$23 \le n \le 63$
11.	$7 \le n \le 11$	71	$22 \le n \le 67$
13.	$8 \le n \le 13$	73	$24 \le n \le 69$
16.	$9 \le n \le 16$	79	$26 \le n \le 74$
17	$10 \le n \le 17$	81	$26 \le n \le 76,\ n \ne 70$
19	$10 \le n \le 19$	83	$26 \le n \le 79,\ n \ne 71, 72$
23	$10 \le n \le 23,\ n \ne 11$	89	$28 \le n \le 84$
25	$12 \le n \le 25$	97	$29 \le n \le 91$
27	$12 \le n \le 26$	101	$30 \le n \le 95,\ n \ne 86, 87$
29	$13 \le n \le 28$	103	$30 \le n \le 97$
31	$14 \le n \le 30$	107	$31 \le n \le 100,\ n \ne 94$
32	$13 \le n \le 31$	109	$31 \le n \le 102,\ n \ne 95$
37	$16 \le n \le 36$	113	$32 \le n \le 106,\ n \ne 98, 99$
41	$16 \le n \le 39$	121	$32 \le n \le 113,\ n \ne 103, 107$
43	$16 \le n \le 41$	125	$34 \le n \le 117,\ n \ne 108-110$
47	$18 \le n \le 45$	127	$35 \le n \le 119,\ n \ne 110-113$
49	$18 \le n \le 47$	128	$34 \le n \le 120,\ n \ne 110-115$
53	$18 \le n \le 50,\ n \ne 19$	131	$35 \le n \le 123,\ n \ne 113, 114$
59	$20 \le n \le 56$	137	$36 \le n \le 129,\ n \ne 108-120$

useful also for estimates and exact answers in extremal and classification problems.

For computer search, we use randomized greedy algorithms [9, Sec. 6], [10], and a breadth-first algorithm [18, Sec. 3]. The first way is convenient for large q and for obtaining different sizes of geometrical objects. The second approach has been applied to obtain classifications of minimal saturating sets and to establish the smallest size of minimal saturating sets.

In Table I, we give lengths of the known $[n, n-3]_q 2$ LO codes with $n \le q$. The codes with $n = q+1, q+2$ always exist, see Theorems 5 and 7. In Table I, we use results of [7], [9, Tables 2, 3, 4], and computer search done in this work. The dot means that all possible lengths n for the given q are known. Besides, for $q = 3, 4, 5$ there are only $n = q+1, q+2$ [9, Table 2].

Note that, as complete caps are minimal 1-saturating sets, results of works [7], [10], [12], [13], [15], [16] on possible sizes of complete caps can be used for the problem of spectrum of possible lengths of $[n, n-r]_q 2$ LO codes and then with the help of (6), (7) for LO codes with $R \ge 3$.

In Tables II and III, for small r, q, we give the complete classification of $[n, n-r, d]_q R$ LO codes, $R = 3, 4$. In Table IV, we describe the classification of $[n, n-r, d]_q R$ LO codes of the smallest possible length, $R = 3, 4$. The codes are obtained by the geometrical computer methods as minimal saturating sets. In the tables t is the number of distinct $[n, n-r, d]_q R$ LO codes, "stabilizer group order" is the order of the stabilizer group of the minimal saturating set corresponding to a parity-check matrix of a code.

It should be noted that the stabilizer group connected with parity-check matrix columns treated as points of the projective space PG$(r-1, q)$ is a subgroup of the full collineation group of PG$(r-1, q)$. A collineation of PG(v, q) is a bijective function that sends lines in lines and preserves the incidences [15], [17, Sec.4, Appendix B]. It implies that every subspace is mapped onto a subspace of the same dimension. The fundamental theorem of projective geometry [15, Sec. 2.1] states that the full collineation group of PG(v, q) is equal to $P\Gamma L(v+1, q)$ that is the group of the collineations of the form σM where σ is an automorphism of the field F_q and M is an invertible $(v+1) \times (v+1)$ matrix over F_q. Two sets of points S and S_1 are equivalent if a collineation f exists such that $f(S) = S_1$. The stabilizer group of a set S is $G = \{f \in P\Gamma L(v+1, q) \mid f(S) = S\}$.

In Tables II–IV a stabilizer group order up to 24 has two indexes. The superscript is the ordinal number of the structure of the group with such order in [15, Table 2.3]. The subscript is the number of groups with the same order and structure in our table, e.g., 12_2^3 notes two groups $\boldsymbol{D}_6$. The subscript of a group order greater than 24 is the number of groups with the same order in our table, here the subscript "1" is not written. In Table IV, for some R, q, r the smallest length of the LO code is found but the values d and/or t are not obtained.

By [9, Tables. 1, 2], [12], Tables I–III, and Theorems 5 and 7 of this correspondence, we have the following.

Theorem 8: It holds that

i) For $R = 2$, $2 \le q \le 16$, $R = 3$, $2 \le q \le 7$, and $R = 4$, $2 \le q \le 5$, there exist $[n, n-(R+1)]_q R$ LO codes of all possible lengths n in the region $l(r, R; q) \le n \le m(r, R; q)$.

ii) For $q = 2$, $R = 2, 3, 4$, $r = R+2, R+3$, and $q = 2$, $R = 2$, $r = R+4$, and $q = 3$, $R = 3, 4$, $r = R+2$, there are not $[n, n-r]_q R$ LO codes of all possible lengths n in the region $l(r, R; q) \le n \le m(r, R; q)$.

Now we demonstrate using codes of Tables I–IV and Section III as the starting LO codes in q^m-concatenating constructions to obtain LO codes of distinct lengths.

Example 1: We use Construction CC_1. As the starting $[n_0, n_0 - 3]_q 2$ code we take a code either of Table I with $n_0 \le q$ or of Theorems 5, 7 with $n_0 = q+2, q+1$. We apply the trivial 2-partition $\mathcal{P}_0$ and we obtain an infinite family of $[n, n-r]_q R$ LO codes V_m with parameters

$$R = 2, \quad r = 3 + 2m, \quad n = n_0 q^m + 2(q^m - 1)/(q-1)$$
$$m \ge 1 \text{ if } n_0 < q, \qquad m \ge 2 \text{ if } n_0 \ge q. \tag{21}$$

By Theorem 1, the parity-check matrix of the code V_m has a 2-partition into n_0 subsets. We use codes V_m as a starting point for Construction CC_2 with $m = m_1$ and for every code V_m we obtain a few new LO codes V_{m,m_1}. By Theorem 2, the parity-check matrix of the code V_{m,m_1} has a 2-partition into $2n_0 + 1$ subsets. Now we use codes V_{m,m_1} as a starting point for Construction CC_2 with $m = m_2$ and for every code V_{m,m_1} we obtain new $[n, n-r]_q R$ LO codes with

$$R = 2, \quad r = 3 + 2(m + M)$$
$$n = n_0 q^{m+M} + (2q^{m+M} - q^M - 1)/(q-1) \tag{22}$$

where $M = m_1 + m_2$, $m \ge m_1 \ge 2$, $m + m_1 \ge m_2 \ge 2$, $m \ge 2$. Similarly, applying Construction CC_2 iteratively we obtain an infinite chain of LO codes with parameters of the form (22). Now one can again use Construction CC_1 taking codes of this chain as starting.

Example 2: By (13) and the Proof of Theorem 7, $\mathcal{P}_0 = \{h_1\}, \{h_2\}, \{h_3, h_4\}, \{h_5, \ldots, h_{q+1}\}$ is a 2-partition of the matrix $\boldsymbol{H}$ for $q \ge 5$. As the starting code for Construction CC_2 with $m = 1$ we take the code with $\boldsymbol{H}$ of (13) and we obtain the $[n, n-5]_q 2$ LO code, $n = (q^3 - 1)/(q-1)$. By Theorem 2, the parity-check matrix of this code has a 2-partition into nine subsets. Using Construction CC_2 iteratively we obtain an infinite chain of $[n, n-r]_q R$ LO codes with

$$R = 2, \quad q \ge 5, \quad r = 2m + 1, \quad n = (q^{m+1} - 1)/(q-1). \tag{23}$$

By Construction CC_1, one can design an infinite family of LO codes for every code of (23).

TABLE II
COMPLETE CLASSIFICATION OF $[n, n-r, d]_q 3$ LO CODES FOR SMALL $r,\ q$

q	r	n	d	t	stabilizer group order
2	4	5	3 4	1 1	12_1^3 24_1^4
2	5	6	5 6	1 1	120 720
2	5	7	3	2	$72, 144$
2	5	9	3 4	1 1	336 1344
2	6	7	7	1	5040
2	6	8	3 5	1 1	720 144
2	6	9	3	1	1296
2	6	10	3 4	1 2	144 $336, 1008$
2	6	11	3 4	10 3	$8_1^4, 12_1^3, 48_2, 192_2, 384,$ $1008, 4032, 8064$ $48, 1920_2$
2	6	12	3 4	1 1	10_1^2 120
2	6	17	3 4	1 1	40320 322560
3	4	5	4 5	1 1	48 120
3	4	6	3	1	192
3	5	8	3 4	8 3	$12_1^3, 24_1^7, 36, 48_2, 72,$ $144, 1152$ $4_1^2, 8_1^4, 12_1^3$
3	5	9	3 4	14 11	$2_1^1, 4_1^2, 6_1^1, 8_3^3, 16_1^8, 24_1^7,$ $36, 48, 72, 144, 288, 4608$ $4_1^2, 8_1^3, 12_3^3, 24_1^4, 24_1^7, 48_3,$ 384

q	r	n	d	t	stabilizer group order
3	5	10	3	3	$12_1^3, 16_2^8$
3	5	11	3 4	7 2	$4_2^2, 8_1^3, 24_2^7, 64, 192$ $72, 2880$
3	5	12	3	2	$24_1^7, 216$
3	5	15	3	1	44928
4	4	5	5	1	240
4	4	6	3	1	36
4	4	7	3 4	2 2	$144, 2160$ $336, 2160$
5	4	6	3 4 5	1 2 1	12_1^3 $8_1^2, 8_1^4$ 120
5	4	7	3 4	6 2	$8_1^2, 16_1^8, 24_2^{12}, 32_2$ $24_1^4, 480$
5	4	8	3	1	3840
5	4	9	3	1	72
7	4	7	3 4	15 54	$1_2^1, 2_5^1, 3_1^1, 4_2^2, 6_2^1, 6_1^2, 9_1^2, 36$ $1_{19}^1, 2_{19}^1, 4_1^1, 4_8^2, 6_2^2, 12_2^3, 24_1^4,$ $72, 216$
7	4	8	3 4 5	174 3 1	$1_{59}^1, 2_{72}^1, 3_2^1, 4_{13}^2, 6_7^1, 8_4^4, 9_1^2,$ $12_5^2, 12_3^3, 12_1^5, 16_1^8, 24_1^{10}, 36_5$ $2_1^1, 8_1^2, 48$ 336
7	4	9	3 4	38 1	$1_1^1, 2_1^1, 4_3^2, 6_{15}^1, 8_1^3, 12_{10}^2,$ $18_2^2, 24_1^2, 24_1^3, 48, 72, 144$ 2016
7	4	10	3	5	$2_1^1, 3_1^1, 18_1^1, 144, 24192$

Example 3: We use Construction CC_3. As a starting code V_0 we take any $[n_0, n_0 - r_0, d]_q 3$ code of Tables II and IV with $d \geq 4$ and use the trivial 3-partition. For $d \geq 4$ all 3-partitions have $t_0 = 0$. We put $m \geq \lceil \log_q(n_0 + 1) \rceil$. For every starting code we obtain an infinite family of LO codes. Taking as V_0 an $[n_0, n_0 - 4, 4]_7 3$ code of Table II we obtain three infinite families of $[n, n-r]_q R$ LO codes with parameters

$$R = 3, \quad q = 7, \quad r = 4 + 3m, \quad n = ((2n_0 + 1) \cdot 7^m - 1)/2$$
$$n_0 = 7, 8, 9, \qquad m \geq 2.$$

Example 4: We use Construction CC_4. As a starting code V_0 we take an $[n_0, n_0 - r_0, 3]_q 3$ code of Tables II and IV with a parity-check matrix $\boldsymbol{H}_0$ such that there is a column $\bar{h} \in \boldsymbol{H}_0$ having the only generating combination $L = \bar{h}$. We use the trivial 3-partition. For $d = 3$ the trivial partition has $t_0 = 3$. We put $m \geq \lceil \log_q(n_0 + 1) \rceil$. For every starting code we obtain an infinite family of LO codes. Using the $[6, 2, 3]_5 3$ code of Table II as V_0 we obtain the family of $[n, n-r]_q R$ LO codes with parameters

$$R = 3, \quad q = 5, \quad r = 4 + 3m, \quad n = (13 \cdot 5^m - 1)/2, \quad m \geq 2.$$

The following construction gives infinite families of LO codes in a geometrical form.

Construction C. We consider a minimal 1-saturating set S_0 in the projective plane $\mathrm{PG}(2, q)$ such that all its points are placed on two lines l_1 and l_2, i.e., $S_0 \subset \{l_1 \cup l_2\} \subset \mathrm{PG}(2, q)$. We suppose that at least one point V of $\mathrm{PG}(2, q) \setminus l_2$ is not saturated by $S_0 \setminus \{P\}$ for all points $P \in S_0$. We denote the point $a = l_1 \cap l_2$ and the subsets $S_0^{(1)}, S_0^{(2)}$ with $S_0^{(1)} \cup S_0^{(2)} = S_0$, $S_0^{(1)} \subseteq l_1$, $S_0^{(2)} \subseteq l_2$.

TABLE III
COMPLETE CLASSIFICATION OF $[n, n-r, d]_q 4$ LO CODES FOR SMALL r, q

q	r	n	d	t	stabilizer group order	q	r	n	d	t	stabilizer group order
2	5	6	3 4	1 1	36 48	3	6	8	4	2	$192, 2304$
3	5	6	4 5 6	1 1 1	192 240 720	3	6	9	3 4	12 2	$24^7_2, 48, 72_2, 96_2, 144, 288_2, 2304, 5760$ $8^3_1, 16^8_1$
3	5	7	3	1	1152	3	6	10	3 4	15 11	$4^2_1, 8^3_1, 12^2_1, 16^5_3, 16^8_1, 32, 72$ $96, 192, 288_2, 1152, 18432$ $8^3_1, 16^5_1, 24^7_3, 48_2,$ $96_2, 192, 1536$
4	5	6	5 6	1 1	720 1440	3	6	11	3	3	$24^7_1, 32, 64$
4	5	7	3	1	216	3	6	12	3 4 5	7 2 1	$16^8_2, 32, 96_2, 256, 768$ $144, 11520$ 432
4	5	8	3 4	2 2	$864, 19440$ $1008, 12960$	3	6	13	3	2	$96, 864$
5	5	6	6	1	720	3	6	16	3	1	269568
5	5	7	3 4 5	1 3 1	48 $32_2, 48$ 480	2	7	8	7 8	1 1	5040 40320
5	5	8	3 4	6 3	$64_2, 192_2, 256_2$ $96, 384, 3480$	2	7	9	3 4 5 6	2 1 1 1	$720, 4320$ 2880 144 1296
5	5	9	3	1	46080	2	7	10	3	2	$1728, 1296$
5	5	10	3	1	288	2	7	11	3 4	1 2	288 $672, 1008$
2	6	7	5 6	1 1	240 720	2	7	12	3 4	10 5	$16^8_1, 24^7_1, 96_2, 192, 384_2,$ $2016, 4032, 8064$ $48, 1536, 1920, 3840, 32256$
2	6	8	3 4	2 1	144_2 1152	2	7	13	3 4	1 1	20^3_1 120
2	6	10	3 4	1 1	1008 2688	2	7	18	3 4	1 1	120960 645120
3	6	7	7	1	5040						

In a hyperplane PG$(v-1, q)$ of the projective space PG(v, q), $v \geq 3$, we pass all possible lines $L_1, L_2, \ldots, L_\theta$ through a point $A \in$ PG$(v-1, q)$. Clearly, $\theta = (q^{v-1}-1)/(q-1)$. Let L be a line through points A and B where $B \in$ PG$(v, q) \setminus$ PG$(v-1, q)$ and let π_i be a plane through B and L_i, $i = 1, 2, \ldots, \theta$. All planes π_i intersect at the line L. In every plane π_i we build a minimal 1-saturating set S_i with the same structure as S_0 so that the line L is l_1, the line L_i is l_2, and the point A is a. We obtain the set $S = \bigcup_{i=1}^{\theta} S_i$ with the cardinality $|S| = |S_0^{(1)}| + \theta |S_0^{(2)}|$ if $a \notin S_0$, $|S| = |S_0^{(1)}| + \theta(|S_0^{(2)}| - 1)$ if $a \in S_0$.

Theorem 9: The set S of Construction C is a minimal 1-saturating set in PG(v, q), $v \geq 3$. The points of S form a parity-check matrix of a $[|S|, |S| - (v+1)]_q 2$ LO code.

Proof: As $\bigcup_{i=1}^{\theta} \pi_i =$ PG(v, q) and every plane π_i is saturated by the "own" 1-saturated set S_i, the set S is 1-saturated. To prove minimality note that every bisecant of S covering the affine geometry $AG(v, q) =$ PG$(v, q) \setminus$ PG$(v-1, q)$ is a line through two points K_i and M with $K_i \in L_i$, $M \in L$. Every such bisecant lies in the plane π_i and it is a bisecant of the set S_i. So, in the geometry $AG(v, q)$ the set S has not other ("new") bisecants than these belonging to the sets S_i. All new bisecants of S through points of distinct sets S_i and S_j lie in the hyperplane PG$(v-1, q)$. If we remove off an arbitrary point of S_i then at least one point V_i of $\pi_i \setminus L_i$ becomes unsaturated by bisecants of S_i. But the point $V_i \in AG(v, q)$ and it cannot be saturated by new bisecants. So, all points of S are necessary. □

Example 5: Many codes from Table I have the structure convenient for the set S_0 of Construction C, e.g., the $[12, 9]_{17} 2$ LO code with the parity-check matrix

$$\boldsymbol{H} = \left[\begin{array}{ccccc|ccccccc} 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\ 1 & 6 & 8 & 10 & 12 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 2 & 5 & 7 & 8 & 9 & 11 & 16 \end{array}\right] = [h_1 h_2 \ldots h_{12}].$$

We treat h_i as points of S_0. The columns $h_1, \ldots, h_5$ belong to the first line, the rest of points belong to the second one. Considering these lines as l_1, l_2 and l_2, l_1 we obtain two infinite families of $[n, n-r]_q R$ LO codes with parameters

$$R = 2, \quad q = 17, \quad r \geq 3$$
$$n = (7 \cdot 17^{r-2} + 73)/16, \quad n = (5 \cdot 17^{r-2} + 107)/16.$$

TABLE IV
CLASSIFICATION OF $[n, n-r, d]_q R$ LO CODES OF THE SMALLEST LENGTH

R	q	r	n	d	t	stabilizer group order
3	8	4	7	3	3	$1_1^1, 2_1^1, 6_1^1$
				4	19	$1_6^1, 2_3^1, 4_3^2, 6_2^1, 8_1^3, 24_1^4, 24_2^8, 168$
				5	1	18_1^4
3	9	4	7	4	27	$1_5^1, 2_7^1, 4_3^1, 4_4^2, 8_1^2, 8_2^4, 12_1^3, 16_2^8, 40, 960$
3	11	4	8	3	≥ 1	
				4	≥ 1	
				5	≥ 1	
3	4	5	9	3	1	144
				4	21	$4_4^2, 8_1^3, 8_5^4, 16_1^8, 32, 48_4, 96, 240, 360, 384, 672$
3	5	5	10			
3	3	6	11	3	8	$12_2^3, 72_4, 144, 432$
4	7	5	7	4	1	18_1^4
				5	3	$4_2^2, 24_1^{15}$
4	8	5	7	5	1	6_1^1
4	2	8	9	9	1	362880

Example 6: We treat columns of the parity-check matrix of [6, eq. (30)] as points of the set S_0 of Construction C and we obtain two infinite families of $[n, n-r]_q R$ LO codes with parameters

$$R = 2, \quad q = p^2, \quad r \geq 3, \quad n = p + (2p-1)(q^{r-2} - 1)/(q-1)$$
$$n = 2p - 1 + p(q^{r-2} - 1)/(q-1).$$

Example 7: As S_0 of Construction C we use the set of (13) and obtain $[n, n-r]_q R$ LO codes with parameters

$$R = 2, \quad q \geq 3, \quad r \geq 3, \quad n = q^{r-2} + 1,$$
$$n = (2q^{r-2} + q^2 - 2q - 1)/(q-1).$$

To obtain LO codes of distinct lengths one can use also Construction DS of (6) taking as V_1, V_2 codes from infinite families above, codes from Tables I–V, the Hamming code.

V. NEW BOUNDS ON THE LENGTH FUNCTION

By Tables III and IV, we have new exact values, cf. [11, Table II], [14, Table I], [19, p. 303]

$$l(5,3;4) = 9, \quad l(5,3;5) = 10, \quad l(4,3;11) = 8$$
$$l(5,4;4) = l(5,4;5) = 6, \quad l(5,4;7) = l(5,4;8) = 7.$$

In Table V, we used results of [11] and computer search done in this correspondence. By computer, we found parity-check matrices of $[l_q, l_q - 4, d]_q 3$ LO codes as minimal 2-saturating l_q-sets in PG$(3,q)$. By [11], an $[n, n-4, d]_q 3$ LO code has $d \in \{3,4,5\}$ and its parity-check matrix corresponds to a complete n-arc in PG$(3,q)$ if $d = 5$ or an incomplete n-cap if $d = 4$. In Table V the subscript indicates the distance d of the code. Entries "3, 4, 5," "3, 4, "... mean that distinct types of 2-saturating sets give the same result. The dot indicates the exact bounds with $l(4,3;q) = l_q$. For all codes with $d = 3$ in the parity-check matrix there is a column $\bar{h}$ having the only generating combination $L = \bar{h}$.

TABLE V
UPPER BOUNDS l_q ON THE LENGTH FUNCTION $l(4,3;q)$, $q \leq 563$

q	l_q	q	l_q	q	l_q	q	l_q
2	$5_{3,4}$.	81	17_4	227	$27_{3,5}$	379	$33_{3,5}$
3	$5_{4,5}$.	83	17_4	229	$27_{3,5}$	383	$33_{3,5}$
4	5_5.	89	$18_{3,5}$	233	$27_{3,5}$	389	33_4
5	$6_{3,4,5}$.	97	$19_{3,5}$	239	$27_{3,5}$	397	$34_{3,5}$
7	$7_{3,4}$.	101	19_5	241	$28_{3,5}$	401	$34_{3,5}$
8	$7_{3,4,5}$.	103	19_5	243	$28_{3,5}$	409	$34_{3,5}$
9	7_4.	107	19_4	251	$28_{3,5}$	419	34_3
11	$8_{3,4,5}$.	109	$20_{3,5}$	256	$28_{3,5}$	421	34_3
13	$8_{4,5}$	113	$20_{3,5}$	257	$28_{3,5}$	431	$35_{3,5}$
16	$9_{3,4,5}$	121	20_4	263	$28_{3,5}$	433	35_3
17	$9_{3,4,5}$	125	$21_{3,5}$	269	$29_{3,5}$	439	$35_{3,5}$
19	$9_{4,5}$	127	$21_{3,5}$	271	$29_{3,5}$	443	$35_{3,5}$
23	$10_{3,4,5}$	128	$21_{3,5}$	277	$29_{3,5}$	449	$35_{3,5}$
25	$11_{3,4,5}$	131	$21_{3,5}$	281	$29_{3,5}$	457	35_4
27	$11_{3,4,5}$	137	$22_{3,5}$	283	$29_{3,5}$	461	$36_{3,5}$
29	$11_{3,4,5}$	139	$22_{3,5}$	289	29_4	463	36_3
31	11_4	149	22_5	293	29_4	467	36_3
32	$12_{3,4,5}$	151	22_4	307	$30_{3,5}$	479	36_3
37	$12_{4,5}$	157	$23_{3,5}$	311	30_4	487	$36_{3,5}$
41	$13_{3,4,5}$	163	23_5	313	30_4	491	36_4
43	$13_{4,5}$	167	$24_{3,5}$	317	30_4	499	$37_{3,5}$
47	$14_{3,4,5}$	169	$24_{3,5}$	331	$31_{3,5}$	503	$37_{3,5}$
49	$14_{3,4,5}$	173	$24_{3,5}$	337	31_3	509	$37_{3,5}$
53	$15_{3,4,5}$	179	24_5	343	31_4	512	37_5
59	$15_{3,4,5}$	181	24_4	347	$32_{3,5}$	521	37_4
61	15_4	191	$25_{3,5}$	349	$32_{3,5}$	523	38_3
64	$16_{3,4,5}$	193	$25_{3,5}$	353	$32_{3,5}$	529	38_5
67	$16_{3,4,5}$	197	$25_{3,5}$	359	$32_{3,5}$	541	38_5
71	$16_{4,5}$	199	25_5	361	32_3	547	38_4
73	16_4	211	$26_{3,5}$	367	32_4	557	39_5
79	$17_{3,5}$	223	$27_{3,5}$	373	$33_{3,5}$	563	39_5

Many results of Table V are better than ones in [11, Tables II, III], cf. [11, Theorem 2], and (24). Results for $q \geq 347$ in Table V are new. Many codes with $d = 3$ appear in Table V for the first time. Using Table V, one can obtain new upper bounds.

Theorem 10: For the length function $l(4,3;q)$ it holds that

$$l(4,3;q) \leq b_q \sqrt[3]{q}, \qquad b_q \leq 4 \text{ if } q \leq 83$$
$$b_q \leq 4.5 \text{ if } q \leq 343, \quad b_q \leq 5 \text{ if } q \leq 563. \tag{24}$$

We denote by Q the set of q values for which in Table V there is a code with $d = 3$. We use the $[l_q, l_q - 4, 3]_q 3$ codes of Table V as the starting codes for Construction CC$_4$. Then we apply the trivial 3-partition and put $t = m + 1$. As result we obtain the infinite family of $[n, n-r, 3]_q R$ LO codes with parameters

$$R = 3, \quad r = 3t + 1, \quad n = l_q q^{t-1} + 2(q^{t-1} - 1)/(q-1)$$
$$t \geq 2 \text{ if } q \geq 8, \quad t \geq 3 \text{ if } q \leq 7, \quad q \in Q. \tag{25}$$

We compare codes of (25) with those of [11, eqs. (10), (11)] where the starting code length l_q is denoted by $n_{q,3}$. Always $l_q \leq n_{q,3}$. The code length in (25) is smaller than that in [11] even if $l_q = n_{q,3}$. Moreover, from [11, Tables II, III] and Table V, it follows that for many q, including $q \in Q$, we have $l_q < n_{q,3}$. The restrictions for t in (25) are better. So, the codes of (25) give new upper bounds on the length function $l(3t+1, 3; q)$, $q \in Q$.

If $q \notin Q$ one can use [11, eq. (10)] changing $n_{q,3}$ by l_q from Table V. By above, we again obtain new upper bounds on the length function $l(3t+1, 3; q)$.

Finally, we can obtain a relatively good upper bounds on the length function $l(r_1 + r_2, R_1 + R_2; q)$ if in Construction DS of (6) the codes V_1, V_2 are taken from Tables I–V.

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Some Results for Linear Binary Codes With Minimum Distance 5 and 6

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***Abstract*—We prove that a linear binary code with parameters $[34, 24, 5]$ does not exist. Also, we characterize some codes with minimum distance 5 and 6.**

***Index Terms*—Algorithm, linear binary code, optimal code.**

I. INTRODUCTION

Let F_2^n be the n-dimensional vector space over the Galois field $F_2 = \mathrm{GF}(2)$. The Hamming distance between two vectors of F_2^n is defined to be the number of coordinates in which they differ. A linear binary $[n, k, d]$-code is a k-dimensional linear subspace of F_2^n with minimum Hamming distance d. The *weight* of the vector $c(wt(c))$ is the number of nonzero entries in c.

A central problem in coding theory is that of optimizing one of the parameters n, k, and d for given values of the other two. Three versions are as follows:

Problem 1: Find $n_2(k, d)$, the largest value of n for which a binary $[n, k, d]$-code exists.

Problem 2: Find $d_2(n, k)$, the largest value of d for which a binary $[n, k, d]$-code exists.

Problem 3: Find $k_2(n, d)$, the largest value of k for which a binary $[n, k, d]$-code exists.

These three functions are closely connected.

A lower bound on $n_2(k, d)$ is the Griesmer bound [6] given by

$$n_2(k,d) \geq g_2(k,d) = \sum_{i=0}^{k-1} \lceil d/2^i \rceil. \quad (1)$$

For fixed k and sufficiently large d, the lower bound is achieved, i.e., there is a constant $D_0(k)$ such that $n_2(k,d) = g_2(k,d)$ for $d \geq D_0(k)$ [1].

Bounds for $d_2(n, k)$ were presented in [4].

In this correspondence, we consider mostly $k_2(n, d)$. The exact values of $k_2(n, d)$ are known for $d \leq 4$ and for $d = 5, n \leq 33$. This is the reason to consider the following problem: Are there linear binary $[34, 24, 5]$ codes? This is the first open case for the function $k_2(n, d)$. We know that $k_2(34, 5) \geq 23$.

We call the codes with parameters $[n, k_2(n,d), d]$ *optimal*. Another important problem related to $k_2(n, d)$ is

Problem 4: Characterize all binary $[n, k_2(n,d), d]$ codes for given values of n and d.

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