Designs, Codes and Cryptography, 37, 61–80, 2005

Constructions of Small Complete Caps in Binary Projective Spaces

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Communicated by: J. W. P. Hirschfeld

Received January 13, 2003; Revised March 3, 2004; Accepted July 23, 2004

Abstract. In the binary projective spaces PG$(n, 2)$ k-caps are called *large* if $k > 2^{n-1}$ and *small* if $k \leq 2^{n-1}$. In this paper we propose new constructions producing infinite families of small binary complete caps.

Keywords: binary caps, complete caps, projective space, small complete caps

AMS Classification: 51E21, 51E22, 94B05

1. Introduction

In the binary projective space PG$(n, 2)$ k-caps are called *large* if $k > 2^{n-1}$ and *small* if $k \leq 2^{n-1}$ [2,3]. In this paper we consider new constructions of small complete caps.

A k-cap in PG$(n, 2)$ is a set of k points, no three of which are collinear. A k-cap in PG$(n, 2)$ is called complete if it is not contained in a $(k + 1)$-cap of PG$(n, 2)$. For an introduction to these geometric objects, see [10,11]. In a space PG$(n, 2)$ a complete cap, points of which are treated as $(n + 1)$-dimensional binary columns, defines a parity check matrix of a binary linear code with codimension $n+1$, Hamming distance $d \geq 4$, and covering radius 2 [2,5,11]. The only case with $d > 4$ is given by the complete 5-cap in PG$(3, 2)$ when $d = 5$. The codes mentioned are called quasi-perfect if $d = 4$ or perfect if $d = 5$ [12].

Relatively many facts on large complete caps in PG$(n, 2)$ are known. For example, in [5] all exact possible sizes and structure of complete k-caps with $k > 2^{n-1} + 1$ are obtained. Every such complete cap can be formed by repeated applying of the doubling construction to a "critical" complete $(2^{m-1}+1)$-cap of $PG(m,2)$, $m<n$. The structure and properties of critical caps are considered briefly in [1,3,5], and deeply in [2] where problems of critical caps structure are solved in main.

But our knowledge on small complete caps seems to be insufficient, see, e.g., [3, 4,6–9,11,14]. In $PG(n,2)$, $n\geq 6$, the smallest size and, in general, the spectrum of possible sizes of small caps are unknown. Relatively a few constructions of small complete caps are described in literature [3,4,9,14].

In this work we give some known results on small complete caps (Section 2) and propose new constructions of those (Sections 3 and 4).

2. Some Known Results on Small Complete Caps

The doubling construction (DC) or Plotkin construction is described and used in many works, see [1,4,5], and the references therein. From a complete k-cap in $PG(n,2)$ DC forms a complete $2k$-cap in $PG(n+1,2)$.

The black/white lift (BWL) construction proposed in [3] obtains, in the general case, a number of new complete $(2k-\delta_i)$-caps in $PG(n+1,2)$ from an "old" complete k-cap S in $PG(n,2)$ under condition that S has certain properties connected with points of $PG(n,2)\setminus S$. The values of δ_i are positive, they can be distinct for distinct new caps and depend on the structure of the cap S.

The papers [4] and [9] give two distinct constructions that obtain complete k-caps in $PG(n,2)$ with

$$k=2^{n+1-s}+2^s-3,\quad n\geq 3,\quad s=2,3,\ldots,\lfloor (n+1)/2\rfloor. \tag{1}$$

The construction of [9] uses methods from [13].

The smallest known complete k-caps in $PG(n,2)$, $n\geq 6$, with $k=f(n)$ are constructed in [9]. Here

$$\begin{aligned}&f(6)=21,\quad f(7)=28,\\ &f(2m)=23\times 2^{m-3}-3,\quad m\geq 4,\\ &f(2m-1)=15\times 2^{m-3}-3,\quad m\geq 5.\end{aligned} \tag{2}$$

For example, $f(8)=43$, $f(9)=57$, $f(10)=89$, $f(11)=117$, $f(12)=181$.

It is proved, see [7,8] and the references therein, that in $PG(n,2)$ for $n=2,3,4$ complete small caps do not exist and for $n=5$ there are only small complete 13-caps.

In the paper [14, Corrected version, Section 5, Propositions 7.1,8.1] the following sizes k of complete k-caps in $PG(n,2)$ are obtained:

$$\begin{aligned}&k=2^{n-v}+t(2^v-2)+1,\quad t=2,4,5,\ldots,2^{n-v-1},\ 2\leq v\leq n-2,\ n\geq 4;\\ &k=2^{n-v}+3\cdot 2^v-5,\quad n\geq 2v+3;\\ &k=2^{n-2}+t,\quad t=5,9,11,\ldots,2^{n-2}+1 \text{ if } n\geq 6,\ \text{besides } t=3,7 \text{ if } n\geq 7;\\ &2^{n-2}+8\leq k\leq 2^{n-1}-2,\ n\geq 6,\ k\neq 30 \text{ if } n=6.\end{aligned} \tag{3}$$

Table 1. The sizes of the known small complete caps in PG$(n, 2)$.

n	Sizes k of the known complete caps with $k \le 2^{n-1}$	References
5	$k = 13$	[4, 7–9, 14]
6	$21 \le k \le 31$, $k \ne 23, 30$	[4, 6, 9, 14]
7	$28 \le k \le 63$	[4, 6, 9, 14]
8	$43 \le k \le 127$	[4, 6, 9, 14]
9	$60 \le k \le 255$, $k = 57$	[4, 6, 9, 14]
10	$92 \le k \le 511$, $k = 89$	[6, 9, 14]
11	$133 \le k \le 1023$, $k = 117, 125, 126, 129, 130$	[6, 9, 14]
12	$196 \le k \le 2047$, $k = 181, 189, 190, 193, 194$	[6, 9, 14]

In Table 1 known sizes of small complete caps in PG$(n, 2)$, $5 \le n \le 12$, are written. Table 1 uses relations (1–3), Construction DC, works [4, 6–9, 14, Corrected version], in particular, computer results of [4, 6].

In [5, Remark 4, 6] and [9, p. 222], the following conjectures are proposed:

Conjecture 1. [5]. In the space PG$(n, 2)$ a complete 2^{n-1}-cap does not exist.

Conjecture 2. [6]. For $n \ge 7$ in the space PG$(n, 2)$ there exist complete caps of all sizes k with $f(n) \le k \le 2^{n-1} - 1$, where $f(n)$ is defined in equation (2).

Conjecture 3. [9]. In PG(6, 2) the smallest size of a complete cap is 21.

By equation (2) and Table 1, Conjecture 2 holds for $n = 7, 8$.

3. A Construction of Small Complete Caps

3.1. Spaces and Vectors

Let E_{n+1}, G_r, G_s, D_l, and D_m be spaces of binary $(n+1)$-positional vectors with dimensions $n+1, r, s, l$, and m, respectively, and let

$$E_{n+1} \supset G_r \supset G_s, \quad E_{n+1} \supset D_l \supset D_m, \quad G_r \cap D_l = \{0\}, \quad r > s, \quad l > m, \quad r + l = n + 1, \quad (4)$$

where 0 is the zero $(n+1)$-positional vector. The "main" space E_{n+1} and all its subspaces contain it. The asterisk denotes a space without the zero vector. We have $G_r^* \cap D_l^* = \emptyset$. A sum of two subsets A and B of E_{n+1} is, as usually, $A + B = \{a + b: a \in A, b \in B\}$. Then $G_r + D_l = E_{n+1}$.

We put $E_{n+1}^* = \mathrm{PG}(n, 2)$. Points of PG$(n, 2)$ are vectors of E_{n+1}^*.

We denote $G_s = \{g_0, g_1, \ldots, g_{2^s-1}\}$, $\quad D_m = \{d_0, d_1, \ldots, d_{2^m-1}\}$.

Let $b_{i,j}$ be a binary vector of length i that is the binary representation of a number j. If we are not interested in j we may write b_i. Denote by $\mathbf{0}^t$ the zero matrix (vector) with t rows (t positions) where "matrix" or "vector" are defined by context. Moreover, by $\mathbf{0}^t$ we will write only zeroes "necessary" for points representation that we put in this paper. We represent points of G_r, G_s, D_l, D_m, in the following form:

$$(b_r \mathbf{0}^{l-m} \mathbf{0}^m) \in G_r, \quad g_u = (\mathbf{0}^{r-s} b_{s,u} \mathbf{0}^{l-m} \mathbf{0}^m) \in G_s, \quad u = 0, 1, \ldots, 2^s - 1,$$
$$(\mathbf{0}^{r-s} \mathbf{0}^s b_l) \in D_l, \quad d_v = (\mathbf{0}^{r-s} \mathbf{0}^s \mathbf{0}^{l-m} b_{m,v}) \in D_m, \quad v = 0, 1, \ldots, 2^m - 1. \quad (5)$$

It should be emphasized that the formulas of equation (5) have been taken only for definiteness of a matrix representation of our geometrical objects and for finding a matrix form of needed caps $\mathbf{Z}$, see below. In general, a representation of points as elements of vector spaces can be arbitrary for constructions considered in this paper.

Let $g \in G_s^*,\ d \in D_m^*$. Hence, $g = g_a,\ a \neq 0,\ d = d_c,\ c \neq 0$.

We can treat a point of $\mathrm{PG}(n,2)$ as a column of a matrix, a subset of $\mathrm{PG}(n,2)$ as a matrix, and vice versa, we can consider a matrix as a set of points. We do not change notation for such treatment.

We denote by $\widehat{G}_r, \widehat{G}_s, \widehat{D}_l, \widehat{D}_m$, the corresponding spaces without "necessary" zeroes of the form $\mathbf{0}^{l-m}, \mathbf{0}^m, \mathbf{0}^{r-s}, \mathbf{0}^s$, see equation (5). So, $\widehat{G}_f$ (resp., $\widehat{D}_f$) is the f-dimensional space of binary f-positional vectors. We have $b_r \in \widehat{G}_r,\ \ b_s \in \widehat{G}_s$, $b_l \in \widehat{D}_l,\ \ b_m \in \widehat{D}_m$.

The spaces and points considered above can be represented in the matrix form, for example, as follows, see equation (5),

$$\left[\begin{array}{c|c|c|c|c|c}
G_r & D_l & G_s & g = g_a & D_m & d = d_c \\
\hline
\widehat{G}_r & \begin{array}{c}\mathbf{0}^{r-s}\\ ---\\ \mathbf{0}^{s}\end{array} & \begin{array}{c}\mathbf{0}^{r-s}\\ ---\\ \widehat{G}_s\end{array} & \begin{array}{c}\mathbf{0}^{r-s}\\ ---\\ b_{s,a}\end{array} & \begin{array}{c}\mathbf{0}^{r-s}\\ ---\\ \mathbf{0}^{s}\end{array} & \begin{array}{c}\mathbf{0}^{r-s}\\ ---\\ \mathbf{0}^{s}\end{array} \\
--- & --- & --- & --- & --- & --- \\
\begin{array}{c}\mathbf{0}^{l-m}\\ ---\\ \mathbf{0}^{m}\end{array} & \widehat{D}_l & \begin{array}{c}\mathbf{0}^{l-m}\\ ---\\ \mathbf{0}^{m}\end{array} & \begin{array}{c}\mathbf{0}^{l-m}\\ ---\\ \mathbf{0}^{m}\end{array} & \begin{array}{c}\mathbf{0}^{l-m}\\ ---\\ \widehat{D}_m\end{array} & \begin{array}{c}\mathbf{0}^{l-m}\\ ---\\ b_{m,c}\end{array}
\end{array}\right]. \tag{6}$$

3.2. *Construction S*

A point set $\mathbf{H}$ in $\mathrm{PG}(n,2)$ is defined as

$$\mathbf{H} = \mathbf{G} \cup \mathbf{D} \cup \mathbf{Z} \tag{7}$$

where $\mathbf{G}, \mathbf{D}, \mathbf{Z}$, are point sets of $\mathrm{PG}(n,2)$ formed as follows

$$\mathbf{G} = G_r \backslash G_s + \{d\}, \quad \mathbf{D} = D_l \backslash D_m + \{g\}, \tag{8}$$

$$\mathbf{Z} = \{z_1, z_2, \ldots, z_w\},\ \ z_i = g_{j_i} + d_{k_i},\ \ g_{j_i} \in G_s^*,\ \ d_{k_i} \in D_m^*,\ \ i = 1, 2, \ldots, w,\ w \geq 1. \tag{9}$$

By equations (4),(8),(9),

$$\mathbf{H} \subset E_{n+1}^*, \quad \mathbf{G} \cap \mathbf{D} = \emptyset, \quad \mathbf{G} \cap \mathbf{Z} = \emptyset, \quad \mathbf{D} \cap \mathbf{Z} = \emptyset. \tag{10}$$

Obviously, the size k of the set $\mathbf{H}$ is

$$\begin{aligned} k &= (2^r - 2^s) + (2^l - 2^m) + w = 2^r + 2^l - (2^s + 2^m - w) \\ &= 2^r + 2^{n+1-r} - (2^s + 2^m - w). \end{aligned} \tag{11}$$

We introduce the set

$$M = G_s^* + D_m^*. \tag{12}$$

By equation (9), $\mathbf{Z} \subseteq M$. The points of the set $\mathbf{Z}$ can be represented in the form, see equations (5),(6),(9),

$$z_i = g_{j_i} + d_{k_i} = (\mathbf{0}^{r-s} b_{s,j_i} \mathbf{0}^{l-m} b_{m,k_i}), \quad j_i \neq 0, \ k_i \neq 0, \ i = 1, 2, \dots, w. \tag{13}$$

Now we represent the set $\mathbf{H}$ in the matrix form, see equations (5),(6),(13),

$$\mathbf{H} = \left[\begin{array}{c|c|c} \mathbf{G} = G_r \backslash G_s + \{d\} & \mathbf{D} = D_l \backslash D_m + \{g\} & \mathbf{Z} = \{z_1, z_2, \dots, z_w\} \\ \hline \widehat{G}_r \backslash \widehat{G}_s & \mathbf{0}^{r-s} & \mathbf{0}^{r-s} \\ & ---- & ---- \\ & b_{s,a}\ b_{s,a} \dots b_{s,a} & b_{s,j_1}\ b_{s,j_2} \dots b_{s,j_w} \\ ---- & ---- & ---- \\ \mathbf{0}^{l-m} & & \mathbf{0}^{l-m} \\ ---- & \widehat{D}_l \backslash \widehat{D}_m & ---- \\ b_{m,c}\ b_{m,c} \dots b_{m,c} & & b_{m,k_1}\ b_{m,k_2} \dots b_{m,k_w} \end{array}\right]. \tag{14}$$

We introduce sets $G(\mathbf{Z})$ and $D(\mathbf{Z})$ and values $g(\mathbf{Z})$ and $d(\mathbf{Z})$ such that

$$\begin{aligned} G(\mathbf{Z}) &= \{g_{j_i} : z_i = g_{j_i} + d_{k_i}, \ i = 1, 2, \dots, w\}, \\ D(\mathbf{Z}) &= \{d_{k_i} : z_i = g_{j_i} + d_{k_i}, \ i = 1, 2, \dots, w\}. \end{aligned} \tag{15}$$

$$g(\mathbf{Z}) = \begin{cases} 1 & \text{if } g \in G(\mathbf{Z}) \\ 0 & \text{if } g \notin G(\mathbf{Z}) \end{cases}, \quad d(\mathbf{Z}) = \begin{cases} 1 & \text{if } d \in D(\mathbf{Z}) \\ 0 & \text{if } d \notin D(\mathbf{Z}) \end{cases}. \tag{16}$$

Conditions on $\mathbf{Z}$ sufficient for $\mathbf{H}$ to be a complete cap.

$\mathscr{A}$. $\mathbf{Z} \cap (\mathbf{Z} + \mathbf{Z}) = \emptyset$, i.e., $\mathbf{Z}$ is a cap.

$\mathscr{B}$. $M \subseteq \mathbf{Z} \cup (\mathbf{Z} + \mathbf{Z})$, i.e., $G_s^* + D_m^* \subseteq \mathbf{Z} \cup (\mathbf{Z} + \mathbf{Z})$.

$\mathscr{C}$. $G(\mathbf{Z}) \cup \{g\} = G_s^*$, $D(\mathbf{Z}) \cup \{d\} = D_m^*$.

$\mathscr{D}$. $r \geq s + 2 - d(\mathbf{Z})$, $l \geq m + 2 - g(\mathbf{Z})$.

In examples below boldface $\mathbf{0}$ denotes the zero from a region of "necessary" zeroes connected with the representation of spaces and points taken in this paper, see equations (5),(6),(13),(14).

EXAMPLE 1. Let $r = 3$, $s = 2$, $l = 4$, $m = 2$, $w = 6$, $g = g_1 = (\mathbf{0}^1 b_{2,1} \mathbf{0}^2 \mathbf{0}^2) = (\mathbf{0}\ 01\ \mathbf{00}\ \mathbf{00})$, $d = d_3 = (\mathbf{0}^1 \mathbf{0}^2 \mathbf{0}^2 b_{2,3}) = (\mathbf{0}\ \mathbf{00}\ \mathbf{00}\ 11)$, and let $\mathbf{Z} = \{(g_2 + d_1), (g_2 + d_2),$

$(g_2+d_3), (g_3+d_1), (g_3+d_2), (g_3+d_3)\}$. Then, see equation (14),

$$\mathbf{H}=\left[\begin{array}{c|c|c} 1111 & \mathbf{0000\ 0000\ 0000} & \mathbf{000\ \ 000} \\ 0011 & 0000\ 0000\ 0000 & 111\ \ 111 \\ 0101 & 1111\ 1111\ 1111 & 000\ \ 111 \\ -\ - & -\ -\ -\ -\ -\ -\ - & -\ -\ -\ - \\ \mathbf{0000} & 0000\ 1111\ 1111 & \mathbf{000\ \ 000} \\ \mathbf{0000} & 1111\ 0000\ 1111 & \mathbf{000\ \ 000} \\ 1111 & 0011\ 0011\ 0011 & 011\ \ 011 \\ 1111 & 0101\ 0101\ 0101 & 101\ \ 101 \end{array}\right]. \tag{17}$$

The first 4 columns are points of $G_r \backslash G_s + \{d\}$, the next 12 columns are $D_l \backslash D_m + \{g\}$, and the last 6 columns are $\mathbf{Z}$. The form of $\mathbf{Z}$ will be explained later in Construction Z_2. By above, $G(\mathbf{Z}) = \{g_2, g_3\}$, $G(\mathbf{Z}) \cup \{g\} = G_s^*$, $g(\mathbf{Z}) = 0$ as $g \notin G(\mathbf{Z})$, $D(\mathbf{Z}) = \{d_1, d_2, d_3\} = D_m^*$, $d(\mathbf{Z}) = 1$ as $d \in D(\mathbf{Z})$. So, the conditions $\mathscr{C}$ and $\mathscr{D}$ hold. One can check directly that the conditions $\mathscr{A}$ and $\mathscr{B}$ hold too.

THEOREM 1. *Under conditions $\mathscr{A} - D$ the point set $\mathbf{H}$ in equation (7) is a complete cap.*

Proof. We show that $\mathbf{H}$ is a cap, i.e., $\mathbf{H} \cap (\mathbf{H}+\mathbf{H}) = \emptyset$ and that the cap $\mathbf{H}$ is complete, i.e., $\mathbf{H} \cup (\mathbf{H}+\mathbf{H}) \supseteq E_{n+1}^* = \mathrm{PG}(n,2)$. By equation (7),

$$\mathbf{H}+\mathbf{H} = (\mathbf{G}+\mathbf{G}) \cup (\mathbf{D}+\mathbf{D}) \cup (\mathbf{Z}+\mathbf{Z}) \cup (\mathbf{G}+\mathbf{D}) \cup (\mathbf{G}+\mathbf{Z}) \cup (\mathbf{D}+\mathbf{Z}). \tag{18}$$

By equations (4),(8),(9), and the condition $\mathscr{A}$,

$$\mathbf{H} \cap (\mathbf{Z}+\mathbf{Z}) = \emptyset. \tag{19}$$

(a) Let

$$g(\mathbf{Z}) = d(\mathbf{Z}) = 0, \text{ i.e., } g \notin G(\mathbf{Z}),\ d \notin D(\mathbf{Z}). \tag{20}$$

By equation (20) and the condition $\mathscr{D}$**,** we have $r \geq s+2$, $l \geq m+2$. Hence,

$$\mathbf{G}+\mathbf{G} = G_r, \quad \mathbf{D}+\mathbf{D} = D_l. \tag{21}$$

One can see in Example 1 the relation $\mathbf{D}+\mathbf{D} = D_l$ where $l = m+2$. But in Example 1 $r = s+1$ and the relation $\mathbf{G}+\mathbf{G} = G_r$ does not hold. So, for equation (21) the conditions $r \geq s+2$, $l \geq m+2$ are necessary.

Since $G_r \backslash G_s + \{g\} = G_r \backslash G_s$ and $D_l \backslash D_m + \{d\} = D_l \backslash D_m$, again see Example 1, we have

$$\mathbf{G}+\mathbf{D} = G_r \backslash G_s + D_l \backslash D_m. \tag{22}$$

By equation (20) and the condition $\mathscr{C}$, we have $G(\mathbf{Z}) = G_s^* \backslash \{g\}$ and $D(\mathbf{Z}) = D_m^* \backslash \{d\}$. Hence

$$\mathbf{G}+\mathbf{Z} = G_r \backslash G_s + D_m^* \backslash \{d\}, \quad \mathbf{D}+\mathbf{Z} = D_l \backslash D_m + G_s^* \backslash \{g\}. \tag{23}$$

From equations (6),(7),(10),(18),(19),(21)–(23), it follows that $\mathbf{H}\cap(\mathbf{H}+\mathbf{H})=\emptyset$, i.e., $\mathbf{H}$ is a cap.

Taking into account the condition $\mathscr{B}$ one can see that $\mathbf{H}$ is a complete cap. In fact,

$$\mathbf{H}\cup(\mathbf{H}+\mathbf{H})\supseteq(G_r\backslash G_s+\{d\})\cup(D_l\backslash D_m+\{g\})\cup G_r\cup D_l\cup(G_r\backslash G_s+D_l\backslash D_m)\cup$$
$$(G_r\backslash G_s+D_m^*\backslash\{d\})\cup(D_l\backslash D_m+G_s^*\backslash\{g\})\cup(G_s^*+D_m^*).$$

Note that

$$(G_r\backslash G_s+\{d\})\cup(G_r\backslash G_s+D_m^*\backslash\{d\})=G_r\backslash G_s+D_m^*,$$
$$(D_l\backslash D_m+\{g\})\cup(D_l\backslash D_m+G_s^*\backslash\{g\})=D_l\backslash D_m+G_s^*,$$
$$(G_r\backslash G_s+D_m^*)\cup(D_l\backslash D_m+G_s^*)\cup(G_r\backslash G_s+D_l\backslash D_m)=(G_r\backslash G_s+D_l^*)\cup$$
$$(D_l\backslash D_m+G_r^*),$$
$$(G_r\backslash G_s+D_l^*)\cup D_l=(G_r\backslash G_s^*+D_l^*)\cup\{0_{n+1}\},$$
$$(D_l\backslash D_m+G_r^*)\cup G_r=(D_l\backslash D_m^*+G_r^*)\cup\{0_{n+1}\},$$

where 0_{n+1} is the zero $(n+1)$-positional vector. Now we can write

$$\mathbf{H}\cup(\mathbf{H}+\mathbf{H})\supseteq(G_r\backslash G_s^*+D_l^*)\cup(D_l\backslash D_m^*+G_r^*)\cup(G_s^*+D_m^*)=E_{n+1}^*=\mathrm{PG}(n,2).$$

(b) Let

$$g(\mathbf{Z})=d(\mathbf{Z})=1,\ \text{ i.e., } g\in G(\mathbf{Z}),\ d\in D(\mathbf{Z}). \tag{24}$$

Hence $r\geq s+1,\ \ l\geq m+1$, see the condition $\mathscr{D}$.

The relation (22) holds in the case (b).

By equation (24) and the condition $\mathscr{C}$, we have $G(\mathbf{Z})=G_s^*$ and $D(\mathbf{Z})=D_m^*$. Hence

$$\mathbf{G}+\mathbf{Z}=G_r\backslash G_s+D_m\backslash\{d\},\quad \mathbf{D}+\mathbf{Z}=D_l\backslash D_m+G_s\backslash\{g\}, \tag{25}$$

cf. with equation (23). Note that $0_{n+1}\in D_m\backslash\{d\}$ and $0_{n+1}\in G_s\backslash\{g\}$.

Now we consider situations connected with correlation between r and s, l and m.

In the beginning let $r=s+1,\ \ l=m+1$. Then

$$\mathbf{G}+\mathbf{G}=G_s,\quad \mathbf{D}+\mathbf{D}=D_m. \tag{26}$$

One can see in Example 1, where $r=s+1$, the relation $\mathbf{G}+\mathbf{G}=G_s$. Again for equation (26) the conditions $r=s+1,\ \ l=m+1$, are necessary.

Using equations (25),(26), similarly to the case (a) we see that $\mathbf{H}$ is a cap.

From equations (7),(12),(18),(22),(25),(26) and the conditions $\mathscr{A}$ and $\mathscr{B}$ it follows that $\mathbf{H}$ is a complete cap. We have

$$(G_r\backslash G_s+\{d\})\cup(G_r\backslash G_s+D_m\backslash\{d\})\cup(D_l\backslash D_m+\{g\})\cup(D_l\backslash D_m+G_s\backslash\{g\})\cup$$
$$(G_r\backslash G_s+D_l\backslash D_m)=(G_r\backslash G_s+D_l)\cup(D_l\backslash D_m+G_r),$$
$$G_s\cup D_m\cup(G_s^*+D_m^*)=G_s+D_m.$$

Hence

$$\mathbf{H}\cup(\mathbf{H}+\mathbf{H})\supseteq(G_r\backslash G_s+D_l)\cup(D_l\backslash D_m+G_r)\cup(G_s+D_m)\supseteq E^*_{n+1}=\mathrm{PG}(n,2).$$

Now let $r\geq s+2,\ l=m+1$. Then

$$\mathbf{G}+\mathbf{G}=G_r,\quad \mathbf{D}+\mathbf{D}=D_m, \tag{27}$$

cf. with equations (21) and (26). We change equation (26) by (27) and again similarly to the case (a) we see that **H** is a cap. Since $G_s\subset G_r$ the change mentioned retains **H** as a complete cap.

Finally, for the situation $r\geq s+2,\ l\geq m+2$, we obtain the relation (21) instead of equations (26) or (27), and, as $G_s\subset G_r,\ D_m\subset D_l$, we see, by above, that **H** is a complete cap.

(c) Let

$$g(\mathbf{Z})=1,\ d(\mathbf{Z})=0,\ \text{i.e.,}\ g\in G(\mathbf{Z}),\ d\notin D(\mathbf{Z}). \tag{28}$$

Hence $r\geq s+2,\ \ l\geq m+1$, see the condition $\mathscr{D}$.

The relation (22) holds in the case (c).

By equation (28) and the condition $\mathscr{C}$**,** we have $G(\mathbf{Z})=G^*_s$ and $D(\mathbf{Z})=D^*_m\backslash\{d\}$. Hence

$$\mathbf{G}+\mathbf{Z}=G_r\backslash G_s+D^*_m\backslash\{d\},\quad \mathbf{D}+\mathbf{Z}=D_l\backslash D_m+G_s\backslash\{g\}, \tag{29}$$

cf. with equations (23) and (25). Note that $0_{n+1}\in G_s\backslash\{g\}$.

In the beginning we put $r\geq s+2,\ l=m+1$. Then the relation (27) holds.

Similarly to the case (a) one can see that **H** is a cap.

From equations (7),(12),(18),(22),(27),(29) and the conditions $\mathscr{A}$ and $\mathscr{B}$ it follows that **H** is a complete cap. In fact,

$$\begin{aligned}&(G_r\backslash G_s+\{d\})\cup(G_r\backslash G_s+D^*_m\backslash\{d\})\cup(D_l\backslash D_m+\{g\})\cup(D_l\backslash D_m+G_s\backslash\{g\})\cup\\&\quad(G_r\backslash G_s+D_l\backslash D_m)=(G_r\backslash G_s+D^*_l)\cup(D_l\backslash D_m+G_r),\\&D_m\cup(G^*_s+D^*_m)=(G_s+D^*_m)\cup\{0_{n+1}\}.\end{aligned}$$

Hence

$$\begin{aligned}&\mathbf{H}\cup(\mathbf{H}+\mathbf{H})\supseteq(G_r\backslash G_s+D^*_l)\cup(D_l\backslash D_m+G_r)\cup G_r\cup\\&\quad(G_s+D^*_m)\supseteq E^*_{n+1}=PG(n,2).\end{aligned}$$

Now let $r\geq s+2,\ l\geq m+2$. We obtain the relation (21) instead of (27), and, as $D_m\subset D_l$, we see, by above, that **H** is a complete cap.

(d) The case $g(\mathbf{Z})=0,\ d(\mathbf{Z})=1$, can be considered similarly to the previous cases. ■

Note that the condition $\mathscr{A}$ is necessary for a set **H** to be a cap. Without the condition $\mathscr{A}$ the relation (19) does not hold. The conditions $\mathscr{B},\mathscr{C}$, and $\mathscr{D}$ are needed for $\mathbf{H}$ to be a complete cap. In the proof without the term $G_s^* + D_m^*$ connected with the condition $\mathscr{B}$ the requirement $\mathbf{H} \cup (\mathbf{H}+\mathbf{H}) \supseteq E_{n+1}^*$ does not hold. Similarly, without the condition $\mathscr{C}$ the sets $\mathbf{G}+\mathbf{Z}$ and $\mathbf{D}+\mathbf{Z}$ do not have the form of equations (23),(25), or (29), without the condition $\mathscr{D}$ the sets $\mathbf{G}+\mathbf{G}$ and $\mathbf{D}+\mathbf{D}$ do not have the form of equations (21),(26), or (27), and again the condition $\mathbf{H} \cup (\mathbf{H}+\mathbf{H}) \supseteq E_{n+1}^*$ will not be true. Of course, we can put $r \geq s+2$, $l \geq m+2$, independently of $d(\mathbf{Z})$ and $g(\mathbf{Z})$, but this does not allow us to get some sizes of caps.

4. Constructions of Caps Z

4.1. On Infinite Families of Small Complete Caps

We consider examples of distinct constructions of the cap $\mathbf{Z}$. Every construction generates infinite families of complete caps with distinct sizes since parameters r and l (and therefore $n = r + l - 1$) are bounded only from below. For the given construction of $\mathbf{Z}$ the dimension n of the space $\mathrm{PG}(n, 2)$, where the obtained cap $\mathbf{H}$ lies, can tend to infinity. Moreover, for a fixed n every construction of $\mathbf{Z}$ generates many distinct sizes of complete caps since n is a sum of r and l and, besides, there exist parameters s and m which can change and which are bounded only from below too. Finally, an iterative process, when complete caps obtained by Construction S are used to create new caps $\mathbf{Z}$, also gives new families of sizes.

Of course, the set of constructions of $\mathbf{Z}$ described here is not complete. One can form other constructions of $\mathbf{Z}$ and get new sizes of caps by Construction S.

Construction Z_1

We put $s = m = 1$. Then $G_s^* = G_1^* = \{g_1\}$, $g_1 = (\mathbf{0}^{r-1} b_{1,1} \mathbf{0}^{l-1} \mathbf{0}^1)$, $D_m^* = D_1^* = \{d_1\}$, $d_1 = (\mathbf{0}^{r-1} \mathbf{0}^1 \mathbf{0}^{l-1} b_{1,1})$, see equations (5),(6). Obviously, $g = g_1$, $d = d_1$, $w = 1$, $\mathbf{Z} = \{z_1\}$, $z_1 = g + d$, $M = \{g\} + \{d\}$, $\mathbf{Z} = M$, $G(\mathbf{Z}) = G_1^*$, $D(\mathbf{Z}) = D_1^*$, $g(\mathbf{Z}) = d(\mathbf{Z}) = 1$, see equation (16), $r \geq s+1, l \geq m+1$. Since $\mathbf{Z} = M$, the condition $\mathscr{B}$ holds. We have, see equation (7),

$$\mathbf{H} = \mathbf{G} \cup \mathbf{D} \cup \mathbf{Z} = (G_r \backslash G_1 + \{d_1\}) \cup (D_l \backslash D_1 + \{g_1\}) \cup \{z_1\}.$$

If $r = l = 3$, we obtain $n = 5$, $k = 13$,

$$\mathbf{H} = \left[\begin{array}{c|c|c} 001111 & \mathbf{000000} & \mathbf{0} \\ 110011 & \mathbf{000000} & \mathbf{0} \\ 010101 & 111111 & 1 \\ - \; - \; - & - \; - \; - & - \\ \mathbf{000000} & 001111 & \mathbf{0} \\ \mathbf{000000} & 110011 & \mathbf{0} \\ 111111 & 010101 & 1 \end{array}\right]. \tag{30}$$

By equation (11), the size k of the complete cap $\mathbf{H} \subset \mathrm{PG}(n, 2)$ containing the cap $\mathbf{Z}$ of Construction Z_1 is $k = 2^r + 2^l - 3 = 2^r + 2^{n+1-r} - 3$, $r \geq 2$, $l \geq 2$, $n \geq r+1$. It is easy to see that Construction S in the particular case with the cap $\mathbf{Z}$ of Construction Z_1 gives the same complete cap as in [9, Theorem 3], cf. equation (1) and the last formula for k.

4.2. *Modified Notation. Caps Z'_0 and Z'*

Now we will construct the caps $\mathbf{Z}$ not considering "necessary" zeroes of the form $\mathbf{0}^{r-s}$ and $\mathbf{0}^{l-m}$, see equations (5),(6),(13),(14).

We denote $t = s + m - 1$. Let E_{t+1} be the $(t+1)$-dimensional space of binary $(t+1)$-positional vectors. We put $E^*_{t+1} = \mathrm{PG}(t, 2)$.

In E_{t+1} we introduce vector subspaces G'_s, D'_m, a subset $M' = G'^*_s + D'^*_m$, and a point set $\mathbf{Z}'$, that are obtained from G_s, D_m, M, and $\mathbf{Z}$ by removing "necessary" zeroes of the form $\mathbf{0}^{r-s}$ and $\mathbf{0}^{l-m}$. Respectively we introduce points g'_u, d'_v, g', d', z'. Now, cf. equations (5),(9),(13),

$$\begin{aligned} &G'_s = \{g'_0, g'_1, \ldots, g'_{2^s-1}\}, \ \ g'_u = (b_{s,u}\mathbf{0}^m), \ \ u = 0, 1, \ldots, 2^s - 1, \\ &D'_m = \{d'_0, d'_1, \ldots, d'_{2^m-1}\}, \ \ d'_v = (\mathbf{0}^s b_{m,v}), \ \ v = 0, 1, \ldots, 2^m - 1. \end{aligned} \tag{31}$$

$$\begin{aligned} &\mathbf{Z}' = \{z'_1, z'_2, \ldots, z'_w\} \subset E^*_{t+1}, \\ &z' = g'_{j_i} + d'_{k_i} = (b_{s,j_i} b_{m,k_i}), \quad g'_{j_i} \in G'^*_s, \quad d'_{k_i} \in D'^*_m, \quad i = 1, 2, \ldots, w. \end{aligned} \tag{32}$$

The functions $G'(\mathbf{Z}'), D'(\mathbf{Z}'), g'(\mathbf{Z}')$, and $d'(\mathbf{Z}')$, are introduced similarly to equations (15),(16), with change z_i by z'_i and so on, again cf. equations (5),(9),(13) with equations (31),(32). Clearly, $g'(\mathbf{Z}') = g(\mathbf{Z})$ and $d'(\mathbf{Z}') = d(\mathbf{Z})$. Finally, the conditions $\mathscr{A}', \mathscr{B}', \mathscr{C}', \mathscr{D}'$ are perfectly analogous to those $\mathscr{A}, \mathscr{B}, \mathscr{C}, \mathscr{D}$ after an addition of upper primes.

Clearly, $\mathbf{Z}$ and $\mathbf{Z}'$ are in one-to-one correspondence and directly determine one another.

By the condition $\mathscr{A}'$, the point set $\mathbf{Z}'$ is a cap in $\mathrm{PG}(t, 2)$.

We will find a needed caps $\mathbf{Z}'$ in a matrix form using a matrix form of a *starting* complete cap $\mathbf{Z}'_0$ in $\mathrm{PG}(t, 2)$. We call an s-region (resp., an m-region) the first s (resp., the last m) rows of matrices corresponding to $\mathbf{Z}'$ and $\mathbf{Z}'_0$.

If for $\mathbf{Z}'_0$ the conditions $\mathscr{A}', \mathscr{B}', \mathscr{C}', \mathscr{D}'$ hold we can put $\mathbf{Z}' = \mathbf{Z}'_0$. To change parameters or to provide the conditions $\mathscr{A}', \mathscr{B}', \mathscr{C}', \mathscr{D}'$, we can form sums of rows in $\mathbf{Z}'_0$ (to support the condition $\mathscr{C}'$) and remove columns from $\mathbf{Z}'_0$ with s zeroes in the s-region, of the form $(b_{s,0}b_m)$, or with m zeroes in the m-region, of the form $(b_s b_{m,0})$, because $g'_{j_i} \in G'^*_s, d'_{k_i} \in D'^*_m$. Such columns can present in the beginning of the process and can appear after summing rows in $\mathbf{Z}'_0$. The removed columns (points) do not belong to $M' = G'^*_s + D'^*_m$ and therefore they are not required to be saturated with respect to $\mathbf{Z}'$. The operations mentioned preserve the property of $\mathbf{Z}'_0$ to be a cap. Hence the condition $\mathscr{A}'$ always holds.

4.3. *Using the Greatest Binary Complete Cap*

In Constructions Z_2 and Z_3 as the starting complete cap $\mathbf{Z}'_0$ we use the greatest complete 2^t-cap A_t in the space $\mathrm{PG}(t, 2)$ that is the complement to some hyperplane L of $\mathrm{PG}(t, 2)$, i.e., A_t consists of the affine space $\mathrm{PG}(t, 2)\backslash L$, see [1,11]. In the matrix form we can represent the cap A_t by a $(t+1) \times 2^t = (s+m) \times 2^{s+m-1}$-matrix such that the first row consists of 2^{s+m-1} ones, the other $t = s + m - 1$ rows contain numbers $0, 1, \ldots, 2^{s+m-1} - 1$ written as columns in the lexicographical order. In Construction Z_4 we modify the greatest complete 2^{t-1}-cap $A_{t-1} \subset \mathrm{PG}(t-1, 2)$ to get the starting complete $(2^{t-1}+1)$-cap $\mathbf{Z}'_0 \subset \mathrm{PG}(t, 2)$.

Remark 1. Every point of $\mathrm{PG}(t,2)\backslash A_t$ lies on 2^{t-1} bisecants of the cap A_t. If we remove $j < 2^{t-1}$ points from A_t to get a cap $A_{t,j}$ then every point of $\mathrm{PG}(t,2)\backslash A_t$ lies at least on one bisecant of $A_{t,j}$, i.e., all points of $\mathrm{PG}(t,2)\backslash A_t$ are saturated.

Let W be a matrix form of a point set in $\mathrm{PG}(f+p-1, 2)$ where f and p are nonnegative integers, $f+p \geq 3$. Every $(f+p)$-positional column of W represents a point of $\mathrm{PG}(f+p-1, 2)$. We say that the matrix W has a property $U_{f,h}$ if $f \geq 1$ and the first f rows of W contain all distinct nonzero f-positional columns except some h nonzero columns and furthermore the zero f-positional column is present in these rows. If the zero f-positional column is absent the property is denoted by $U^*_{f,h}$. Respectively we introduce properties $L_{p,h}$ and $L^*_{p,h}$ for the last p rows of the matrix W.

Remark 2. Let parameters s and m are given. If a matrix form of a cap $\mathbf{Z}'$ has the property $U^*_{s,0}$ then $G'(\mathbf{Z}') = G'^*_s$, the 1st part of the condition $\mathscr{C}'$ holds, $g'(\mathbf{Z}') = 1$. To satisfy the 2nd part of the condition $\mathscr{D}'$ we must put $l \geq m+1$. If $\mathbf{Z}'$ has the property $U^*_{s,1}$ then $G'(\mathbf{Z}') = G'^*_s \setminus \{g'_i\}$ with $i \neq 0$. To satisfy the 1st part of the condition $\mathscr{C}'$ one must take $g' = g'_i$. For such g' we have $g'(\mathbf{Z}') = 0$. To satisfy the 2nd part of the condition $\mathscr{D}'$ we must put $l \geq m+2$. Respectively, for the property $L^*_{m,0}$ we have that $D'(\mathbf{Z}') = D'^*_m$, the 2nd part of the condition $\mathscr{C}'$ holds, $d'(\mathbf{Z}') = 1$. To satisfy the 1st part of the condition $\mathscr{D}'$ one must put $r \geq s+1$. For the property $L^*_{m,1}$ it holds that $D'(\mathbf{Z}') = D'^*_m \setminus \{d'_j\}$, $j \neq 0$. To satisfy the 2nd part of the condition $\mathscr{C}'$ one must take $d' = d'_j$. For such d' we have $d'(\mathbf{Z}') = 0$. To satisfy the 1st part of the condition $\mathscr{D}'$ we must put $r \geq s+2$.

Construction Z_2

We put $s=2$, $m \geq 2$, $t \geq 3$, $w = 2^{m+1} - 2$. Obviously, the matrix A_t has the properties $U^*_{2,1}$ and $L_{m,0}$. From the matrix A_t we remove $j=2$ columns $(b_{2,2}b_{m,0})$ and $(b_{2,3}b_{m,0})$ with m zeroes in the m-region. We put that the matrix obtained is $\mathbf{Z}'$. Clearly, $G'(\mathbf{Z}') = \{g'_2, g'_3\}$, $D'(\mathbf{Z}') = D'^*_m$. We take $g' = g'_1 = (b_{2,1}\mathbf{0}^m)$. Then $G'(\mathbf{Z}') \cup \{g'\} = G'^*_s$ and $g'(\mathbf{Z}') = 0$. Let $d' = (\mathbf{0}^2 b_{m,v})$, $v \neq 0$. Then $d'(\mathbf{Z}') = 1$ as $D'(\mathbf{Z}') = D'^*_m$. We put $r \geq s+1 = 3$, $l \geq m+2 \geq 4$. Now the conditions $\mathscr{C}'$ and $\mathscr{D}'$ hold. Since $2^{t-1} \geq 4 > j$ all points of $\mathrm{PG}(t,2)\backslash A_t$ are saturated, see Remark 1. The removed two columns (points) are not saturated but they do not belong to M'. So, $M' \subset \mathbf{Z}' \cup (\mathbf{Z}' + \mathbf{Z}')$. The condition $\mathscr{B}'$ holds. As an example with $m=2$, $t=3$, see the 3rd section of the matrix in (17) without "necessary" boldface **0.** By (11), the size k of the complete cap $\mathbf{H} \subset PG(n, 2)$ obtained with the help of $\mathbf{Z}'$ of Construction Z_2 is

$$\begin{aligned} k &= 2^r + 2^l + 2^m - 6 \\ &= 2^r + 2^{n+1-r} + 2^m - 6, \quad r \geq 3,\ m \geq 2,\ l \geq m+2,\ n \geq r+m+1. \end{aligned} \tag{33}$$

Construction Z_3

We put $s \geq 3,\ \ m \geq 2,\ \ t \geq 4,\ \ w = 2^{s+m-1} - 2^{s-1} - 2^{m-1}$. In the matrix A_t we add the $(s+1)$-th row to the 1st row. Now the matrix A_t has the properties $U_{s,0}$ and $L_{m,0}$. If $s = m = 3$ we obtain the matrix

$$\left[\begin{array}{cccccccc} 1111 & 0000 & 1111 & 0000 & 1111 & 0000 & 1111 & 0000 \\ 0000 & 0000 & 0000 & 0000 & 1111 & 1111 & 1111 & 1111 \\ 0000 & 0000 & 1111 & 1111 & 0000 & 0000 & 1111 & 1111 \\ \hline 0000 & 1111 & 0000 & 1111 & 0000 & 1111 & 0000 & 1111 \\ 0011 & 0011 & 0011 & 0011 & 0011 & 0011 & 0011 & 0011 \\ 0101 & 0101 & 0101 & 0101 & 0101 & 0101 & 0101 & 0101 \end{array}\right].$$

Then we remove 2^{m-1} columns with s zeroes in the s-region and 2^{s-1} columns with m zeroes in the m-region. The removed columns have the form $(b_{s,0}b_{m,v}),\ v = 2^{m-1}, 2^{m-1}+1, \ldots, 2^m - 1$, and $(b_{s,u}b_{m,0}),\ u = 2^{s-1}, 2^{s-1}+1, \ldots, 2^s - 1$. As result we obtain the matrix $A_{t,j}$ with $j = 2^{m-1} + 2^{s-1}$ and put $\mathbf{Z}' = A_{t,j}$. For $s \geq 3,\ \ m \geq 2$, we have $2^{t-1} = 2^{s+m-2} > j = 2^{m-1} + 2^{s-1}$. Hence all points of $\mathrm{PG}(t,2) \setminus A_t$ are saturated, see Remark 1. Again, as in Construction Z_2, the removed columns (points) are not saturated but they do not belong to M' and the condition $\mathscr{B}'$ holds. It is easy to see that $G'(\mathbf{Z}') = G_s^{\prime *},\ D'(\mathbf{Z}') = D_m^{\prime *}$. Therefore we need to assume $r \geq s+1$, $l \geq m+1$. By (11), the size k of the complete cap $\mathbf{H} \subset \mathrm{PG}(n,2)$ obtained with the help of $\mathbf{Z}'$ of Construction Z_3 is

$$k = 2^r + 2^l + 2^{s+m-1} - 3(2^{s-1} + 2^{m-1}) = 2^r + 2^{n+1-r} + 2^{s+m-1} - 3(2^{s-1} + 2^{m-1}),$$
$$s \geq 3,\quad m \geq 2,\quad r \geq s+1,\quad l \geq m+1,\quad n \geq r+m. \tag{34}$$

Construction Z_4

We put $s \geq 3,\ \ m \geq 2,\ \ w = 2^{s+m-2} + 1,\ \ t = s + m - 1$, take the complete 2^{t-1}-cap $A_{t-1} \subset \mathrm{PG}(t-1,2)$ and insert at the top a new row of 2^{s+m-2} zeroes. Then we remove 2^{m-2} columns $t_i = (01b_{s-2,0}01b_{m-2,i}),\ i = 0,1,\ldots,2^{m-2}-1$, and 2^{s-2} columns $u_j = (01b_{s-2,j}b_{m,0}),\ j = 0,1,\ldots,2^{s-2}-1$. We put $e = (11b_{s-2,0}b_{m-1,0}1)$ and insert the following $2^{m-2} + 2^{s-2} + 1$ columns into the matrix: $t_i' = e + t_i,\ i = 0,1,\ldots,2^{m-2}-1,\ u_j' = e + u_j,\ j = 0,1,\ldots,2^{s-2}-1$, and e. We take the obtained matrix as $\mathbf{Z}_0' \subset \mathrm{PG}(t,2)$. If $s = 4,\ m = 3$, we have

$$\mathbf{Z}_0' = \left[\begin{array}{cccc|cc|c} 00000 & 0000000 & 0000000 & 0000000 & 11 & 1111 & 1 \\ 11111 & 1111111 & 1111111 & 1111111 & 00 & 0000 & 1 \\ 00000 & 0000000 & 1111111 & 1111111 & 00 & 0011 & 0 \\ 00000 & 1111111 & 0000000 & 1111111 & 00 & 0101 & 0 \\ \hline 01111 & 0001111 & 0001111 & 0001111 & 00 & 0000 & 0 \\ 00011 & 0110011 & 0110011 & 0110011 & 11 & 0000 & 0 \\ 10101 & 1010101 & 1010101 & 1010101 & 10 & 1111 & 1 \end{array}\right].$$

By construction, $\mathbf{Z}_0'$ is a cap, e.g., $e = t_i + t_i'$, but columns t_i are removed. Moreover, $\mathbf{Z}_0'$ is a complete cap. Columns of the form $(00b_{s+m-2})$ are saturated since for $s \geq 3,\ m \geq 2$, we have $2^{s-2} + 2^{m-2} < 2^{s+m-3}$, see Remark 1. Columns $(01b_{s+m-2})$ either belong to $\mathbf{Z}_0'$ or can be obtained as $t_i = e + t_i',\ u_j = e + u_j'$. Columns $(10b_{s+m-2})$ either belong to $\mathbf{Z}_0'$, see t_i' and u_j', or can be obtained as $f + e$ where

f is a column from the left submatrix of $\mathbf{Z}'_0$. Finally, columns $(11b_{s+m-2}) \neq e$ can be obtained as $f + t'_i$ or $f + u'_j$.

Note that complete $(2^v + 1)$-caps of considered structure are described in [5, formula (18)] and researched in [2, Section 4].

Now we add the $(s+1)$-th and the $(s+2)$-th rows of $\mathbf{Z}'_0$ to the 1st and the 2nd rows respectively and obtain the cap $\mathbf{Z}'$ with the properties $U^*_{s,0}$ and $L^*_{m,0}$. If $s = 4$, $m = 3$, then

$$\mathbf{Z}' = \left[\begin{array}{cccc|cc|c} 01111 & 0001111 & 0001111 & 0001111 & 11 & 1111 & 1 \\ 11100 & 1001100 & 1001100 & 1001100 & 11 & 0000 & 1 \\ 00000 & 0000000 & 1111111 & 1111111 & 00 & 0011 & 0 \\ 00000 & 1111111 & 0000000 & 1111111 & 00 & 0101 & 0 \\ --- & ---- & ---- & ---- & - & -- & - \\ 01111 & 0001111 & 0001111 & 0001111 & 00 & 0000 & 0 \\ 00011 & 0110011 & 0110011 & 0110011 & 11 & 0000 & 0 \\ 10101 & 1010101 & 1010101 & 1010101 & 10 & 1111 & 1 \end{array}\right].$$

We put $r \geq s+1$, $l \geq m+1$, see Remark 2. All conditions $\mathscr{A}', \mathscr{B}', \mathscr{C}', \mathscr{D}'$ hold. By (11), the size k of the complete cap $\mathbf{H} \subset \mathrm{PG}(n, 2)$ obtained with the help of $\mathbf{Z}'$ of Construction Z_4 is

$$k = 2^r + 2^l + 2^{s+m-2} + 1 - 2^s - 2^m = 2^r + 2^{n+1-r} + 2^{s+m-2} + 1 - 2^s - 2^m,$$
$$s \geq 3, \quad m \geq 2, \quad r \geq s+1, \quad l \geq m+1, \quad n \geq r+m. \tag{35}$$

4.4. Iterative Constructing of Z'

In Constructions Z_5–Z_9 we consider an iterative process when a complete cap $\mathbf{H}$ obtained by Construction S is used to create the complete starting cap $\mathbf{Z}'_0$. Suppose by Construction S we got a family of complete k_0-caps $\mathbf{H}_0$ with fixed parameters s_0, m_0, Δ_0, c_r, c_l, so that

$$\mathbf{H}_0 \subset \mathrm{PG}(n_0, 2), \; n_0 = r_0 + l_0 - 1, \; k_0 = 2^{r_0} + 2^{l_0} + \Delta_0, \; r_0 \geq s_0 + c_r, \; l_0 \geq m_0 + c_l, \tag{36}$$

where $c_r, c_l \in \{1, 2\}$. By above, every complete cap obtained by Construction S belongs to a family of such form. Changing parameters mentioned we obtain another family. Distinct values of r_0, l_0 give distinct caps $\mathbf{H}_0$ of the same family.

By equations (7–9),(14), and the condition $\mathscr{C}$, the complete cap $\mathbf{H}_0$ has the properties $U^*_{r_0,0}$ and $L^*_{l_0,0}$. Hence we can put $\mathbf{H}_0 = \mathbf{Z}'$ with $s = r_0$, $m = l_0$, $G'(\mathbf{Z}') = G'^*_s$, $D'(\mathbf{Z}') = D'^*_m$, $w = k_0$. Taking into account that $\mathbf{H}_0$ is a *complete* cap, all conditions $\mathscr{A}', \mathscr{B}', \mathscr{C}'$ hold. In order to satisfy the condition $\mathscr{D}'$ we put $r \geq s+1 = r_0 + 1$, $l \geq m + 1 = l_0 + 1$, and by Construction S we obtain a new complete cap $\mathbf{H}$ of the size $k = 2^r + 2^l - (2^{r_0} + 2^{l_0} - k_0) = 2^r + 2^l + \Delta_0$. Comparing this k with k_0 of equation (36) we see that such a direct method does not yield new sizes. But applying the doubling construction (DC) to $\mathbf{H}_0$ we can obtain a cap $\mathbf{Z}'_0$ providing a new family of sizes. It should be noted that we use DC only for obtaining $\mathbf{Z}'$ and then we obtain a new complete cap $\mathbf{H}$ by Construction S.

Construction Z_5

We apply DC to the complete cap $\mathbf{H}_0$ with parameters (36). To do this we repeat the matrix $\mathbf{H}_0$ two times and insert at the top a new row consisting of sequences of k_0 zeroes and k_0 ones [5]. We obtain a complete $2k_0$-cap $\mathbf{Z}'_0$ in $\mathrm{PG}(n_0+1,2)=\mathrm{PG}(t,2)$ and put $\mathbf{Z}'=\mathbf{Z}'_0$ with $s=r_0+1,\ m=l_0,\ w=2k_0=2^{r_0+1}+2^{l_0+1}+2\Delta_0,\ t=r_0+l_0$. So,

$$\mathbf{Z}'=\mathbf{Z}'_0=\left[\begin{array}{c|c} 00\ldots0 & 11\ldots1 \\ --- & --- \\ \mathbf{H}_0 & \mathbf{H}_0 \end{array}\right]. \tag{37}$$

Since the cap $\mathbf{H}_0$ has the properties $U^*_{r_0,0}$ and $L^*_{l_0,0}$, the cap $\mathbf{Z}'$ has the properties $U^*_{s,1}=U^*_{r_0+1,1}$ and $L^*_{m,0}=L^*_{l_0,0}$. The (r_0+1)-positional column $(10\ldots0)$ is absent in the first $s=r_0+1$ rows of $\mathbf{Z}'$. This means one must take $g'=g'_{2^{r_0}}=(b_{s,2^{r_0}}\mathbf{0}^m)\notin G'(\mathbf{Z}')$. So, $g'(\mathbf{Z}')=0$. To satisfy the condition $\mathscr{D}'$ we should put $r\geq s+2=r_0+3$, $l\geq m+1=l_0+1$, see Remark 2. Taking into account that $\mathbf{Z}'$ is a *complete* cap, all conditions $\mathscr{A}',\mathscr{B}',\mathscr{C}',\mathscr{D}'$ hold. By equations (11),(36), the size k of the complete cap $\mathbf{H}\subset\mathrm{PG}(n,2)$ obtained with the help of $\mathbf{Z}'$ of Construction Z_5 is

$$k=2^r+2^l+2^{l_0}+2\Delta_0=2^r+2^{n+1-r}+2^{l_0}+2\Delta_0,\ \ r\geq r_0+3,\ \ l\geq l_0+1,\ \ n\geq r+l_0. \tag{38}$$

Construction Z_6

We proceed similarly to Construction Z_5 but insert the new row at the bottom. Then

$$\mathbf{Z}'=\mathbf{Z}'_0=\left[\begin{array}{c|c} \mathbf{H}_0 & \mathbf{H}_0 \\ --- & --- \\ 00\ldots0 & 11\ldots1 \end{array}\right], \tag{39}$$

$s=r_0,\ m=l_0+1,\ w=2k_0,\ t=r_0+l_0,\ r\geq s+1=r_0+1,\ l\geq m+2=l_0+3$. The size k of the complete cap $\mathbf{H}\subset\mathrm{PG}(n,2)$ obtained with the help of $\mathbf{Z}'$ of Construction Z_6 is

$$\begin{aligned} k&=2^r+2^l+2^{r_0}+2\Delta_0\\ &=2^r+2^{n+1-r}+2^{r_0}+2\Delta_0, \quad r\geq r_0+1,\ \ l\geq l_0+3,\ \ n\geq r+l_0+2. \end{aligned} \tag{40}$$

Construction Z_7

Applying DC of equation (39) to the cap of equation (37) we obtain the complete cap $\mathbf{Z}'_0$ in $\mathrm{PG}(n_0+2,2)=\mathrm{PG}(t,2)$, and again we put $\mathbf{Z}'=\mathbf{Z}'_0$. We have

$$\mathbf{Z}'=\mathbf{Z}'_0=\left[\begin{array}{c|c|c|c} 00\ldots0 & 11\ldots1 & 00\ldots0 & 11\ldots1 \\ --- & --- & --- & --- \\ \mathbf{H}_0 & \mathbf{H}_0 & \mathbf{H}_0 & \mathbf{H}_0 \\ --- & --- & --- & --- \\ 00\ldots0 & 00\ldots0 & 11\ldots1 & 11\ldots1 \end{array}\right], \tag{41}$$

$s=r_0+1,\ m=l_0+1,\ w=4k_0=2^{r_0+2}+2^{l_0+2}+4\Delta_0$. Since the cap $\mathbf{H}_0$ has the properties $U^*_{r_0,0}$ and $L^*_{l_0,0}$, the cap $\mathbf{Z}'$ has the properties $U^*_{s,1}=U^*_{r_0+1,1}$ and $L^*_{m,1}=L^*_{l_0+1,1}$. The (r_0+1)-positional column $(10\ldots0)$ is absent in the first s rows of $\mathbf{Z}'$ and the (l_0+1)-positional column $(0\ldots01)$ is absent in the last m rows. Hence one must take $g'=(b_{s,2^{r_0}}\mathbf{0}^m)\notin G'(\mathbf{Z}')$, $d'=(\mathbf{0}^s b_{m,1})\notin D'(\mathbf{Z}')$, and put $r\geq s+2=r_0+3$,

$l \geq m+2=l_0+3$, cf. Construction Z_5 and Remark 2. All conditions $\mathscr{A}', \mathscr{B}', \mathscr{C}', \mathscr{D}'$ hold. By equations (11), (36), the size k of the complete cap $\mathbf{H} \subset \mathrm{PG}(n,2)$ obtained with the help of $\mathbf{Z}'$ of Construction Z_7 is

$$k=2^r+2^l+2^{r_0+1}+2^{l_0+1}+4\Delta_0=2^r+2^{n+1-r}+2^{r_0+1}+2^{l_0+1}+4\Delta_0,$$
$$r \geq r_0+3, \quad l \geq l_0+3, \quad n \geq r+l_0+2. \tag{42}$$

Construction Z_8

We consider the complete cap $\mathbf{Z}_0' \subset \mathrm{PG}(n_0+1,2)=\mathrm{PG}(t,2)$ of equation (37). Since $\mathbf{H}_0$ is a complete cap, every point of $\mathrm{PG}(t,2)\backslash \mathbf{Z}_0'$ lies on at least two bisecants of $\mathbf{Z}_0'$. Therefore if we remove one point from $\mathbf{Z}_0'$ all points of $\mathrm{PG}(t,2)\backslash \mathbf{Z}_0'$ are saturated.

We add the first row of equation (37) to the (m_0+1)-th row from the bottom and obtain another matrix form of $\mathbf{Z}_0'$, say $\mathbf{Z}_{0,a}'$. The left part of equation (37) does not change but in the region $\mathbf{D}$ of the right part exactly one column with l_0 zeroes in the last l_0 rows appears. Before it was the column $(b_{1,1}b_{r_0}b_{l_0,2^{m_0}})$. If $\mathbf{H}_0$ is taken from equation (30), where $m_0=1,\ l_0=3$, then

$$\mathbf{Z}_{0,a}' = \left[\begin{array}{c|c|c|c|c|c}
000000 & 000000 & 0 & 111111 & 111111 & 1 \\
--- & --- & - & --- & --- & - \\
001111 & 000000 & 0 & 001111 & 000000 & 0 \\
110011 & 000000 & 0 & 110011 & 000000 & 0 \\
010101 & 111111 & 1 & 010101 & 111111 & 1 \\
--- & --- & - & --- & --- & - \\
000000 & 001111 & 0 & 000000 & 001111 & 0 \\
000000 & 110011 & 0 & \mathbf{111111} & \mathbf{001100} & \mathbf{1} \\
111111 & 010101 & 1 & 111111 & 010101 & 1
\end{array}\right], \tag{43}$$

where boldface shows the values changed. If $\mathbf{H}_0$ is taken from equation (17), where $m_0=2,\ l_0=4$, then the right (changed) part of $\mathbf{Z}_{0,a}'$ has the form

$$\left[\begin{array}{c|c|c}
1111 & 1111\ 1111\ 1111 & 111\ 111 \\
-- & --\ --\ -- & --\ -- \\
1111 & 0000\ 0000\ 0000 & 000\ 000 \\
0011 & 0000\ 0000\ 0000 & 111\ 111 \\
0101 & 1111\ 1111\ 1111 & 000\ 111 \\
-- & --\ --\ -- & --\ -- \\
0000 & 0000\ 1111\ 1111 & 000\ 000 \\
\mathbf{1111} & \mathbf{0000\ 1111\ 0000} & \mathbf{111\ 111} \\
1111 & 0011\ 0011\ 0011 & 011\ 011 \\
1111 & 0101\ 0101\ 0101 & 101\ 101
\end{array}\right]. \tag{44}$$

We remove the column with l_0 zeroes in the last l_0 rows and take the obtained matrix as $\mathbf{Z}'$. We put $s=r_0+1,\ m=l_0,\ w=2k_0-1$, cf. Construction Z_5. The cap $\mathbf{Z}'$ has the properties $U_{s,1}^*$ and $L_{m,0}^*$, as in Construction Z_5. Therefore $r \geq s+2=r_0+3,\ l \geq m+1=l_0+1$. The removed column does not belong to M' and it may fail to be saturated. All conditions $\mathscr{A}', \mathscr{B}', \mathscr{C}', \mathscr{D}'$ hold. By equations (11),(36), the size k of the complete cap $\mathbf{H} \subset \mathrm{PG}(n,2)$ obtained with the help of $\mathbf{Z}'$ of Construction Z_8 is

$$k=2^r+2^l+2^{l_0}+2\Delta_0-1=2^r+2^{n+1-r}+2^{l_0}+2\Delta_0-1,$$
$$r \geq r_0+3, \quad l \geq l_0+1, \quad n \geq r+l_0. \tag{45}$$

Construction Z_9

We use $\mathbf{Z}'_0$ of equation (39) and add the last row to the $(1+l_0+s_0+1)$-th row from the bottom. Similarly to Construction Z_8 we remove one column and obtain $\mathbf{Z}'$. By equations (11),(36), the size k of the complete cap $\mathbf{H}\subset \mathrm{PG}(n,2)$ obtained with the help of $\mathbf{Z}'$ of Construction Z_9 is

$$k=2^r+2^l+2^{r_0}+2\Delta_0-1=2^r+2^{n+1-r}+2^{r_0}+2\Delta_0-1,$$
$$r\geq r_0+1,\quad l\geq l_0+3,\quad n\geq r+l_0+2. \tag{46}$$

4.5. Using the Smallest Known Complete Caps

In Constructions Z_{10}–Z_{12} as the starting complete caps $\mathbf{Z}'_0$ we use the smallest known complete $f(n)$-caps in $\mathrm{PG}(n,2)$, $n\geq 7$, with $f(n)$ of equation (2), see [9]. In formulas of [9] we choose convenient parameters e_i, e_u, and so on, see below.

Construction Z_{10}

As the starting complete cap $\mathbf{Z}'_0$ with $s=m=4$ we take the complete 28-cap in $\mathrm{PG}(7,2)$ of [9, formula (51)]. The 28-th column of $\mathbf{Z}'_0$ contains s zeroes in the s-region. We add the sum of two last rows of $\mathbf{Z}'_0$ to the 4-th row and obtain a complete 28-cap $\mathbf{Z}'$ for which all conditions $\mathscr{A}',\mathscr{B}',\mathscr{C}'$ and the properties $U^*_{4,0}$, $L^*_{4,0}$ hold. The reader can easy check this. To satisfy the condition $\mathscr{D}'$ we must take $r\geq s+1=5$, $l\geq m+1=5$. By equation (11), the size k of the complete cap $\mathbf{H}\subset \mathrm{PG}(n,2)$ obtained with the help of $\mathbf{Z}'$ of Construction Z_{10} is

$$k=2^r+2^l-4=2^r+2^{n+1-r}-4,\quad r\geq 5,\quad l\geq 5,\quad n\geq r+4. \tag{47}$$

Construction Z_{11}

Here $s=m=v\geq 5$, $\mathbf{Z}'=\mathbf{Z}'_0=U^{2v}$, where U^{2v} is the matrix of [9, formulas (31),(39)–(42)] with $e_i\neq 0$ in [9, formula (31)]. By formulas mentioned one can see that U^{2v} gives a complete $(15\cdot 2^{v-3}-3)$-cap in $\mathrm{PG}(t,2)=\mathrm{PG}(2v-1,2)$ for which all conditions $\mathscr{A}',\mathscr{B}',\mathscr{C}'$ and the properties $U^*_{v,0}$, $L^*_{v,0}$ hold. To satisfy the condition $\mathscr{D}'$ we must take $r\geq s+1=v+1\geq 6, l\geq m+1=v+1\geq 6$. By equation (11), the size k of the complete cap $\mathbf{H}\subset \mathrm{PG}(n,2)$ obtained with the help of $\mathbf{Z}'$ of Construction Z_{11} is

$$\begin{aligned} k&=2^r+2^l-2^{v-3}-3\\ &=2^r+2^{n+1-r}-2^{v-3}-3,\quad v\geq 5,\ r\geq v+1,\ l\geq v+1,\ n\geq r+v. \end{aligned} \tag{48}$$

Construction Z_{12}

We put $s=4, m=5$. As starting complete cap $\mathbf{Z}'_0$ we take the complete 43-cap in $\mathrm{PG}(8,2)$ of [9, Theorem 5, Remark 2]. For $\mathbf{Z}'_0$ in [9, formulas(31),(39)–(42),(50)] we take $\beta=(001)$, $\gamma=(010)$, $\delta=(011)$, $w_1=w_2=w_3=1$, $e_i=(0001)$, $e_u=(0001)$. To get $\mathbf{Z}'$ we change $\mathbf{Z}'_0$ writing the 1st row as the last one in [9, formula (50)]. We obtain

$$\mathbf{Z}'=\begin{bmatrix} 1\,45\,67 & 32\,3\,2\,32\,2 & 32323\,2\,2 & 89\mathrm{ABCDEF} & 11111111111111111\\ -\,- & -\,-\,-\,- & -\,-\,-\,- & -\,-\,-\,- & -\,-\,-\,-\,-\,-\,-\,-\\ 3\,55\,66 & 82\mathrm{A}4\mathrm{C}6\mathrm{E} & \mathrm{B}197\mathrm{F}5\mathrm{D} & 11111111 & 0123456789\mathrm{ABCDEF}\\ 0\,00\,00 & 0000000 & 0000000 & 00000000 & 1111111111111111 \end{bmatrix}. \tag{49}$$

where hexadecimal notation is used. As it is said in [9, Remark 2], we examined by computer that $\mathbf{Z}'$ is a complete cap. By equation (49), the matrix has the properties $U^*_{4,0}$ and $L^*_{5,0}$. So, the conditions $\mathscr{A}', \mathscr{B}', \mathscr{C}'$ hold. To satisfy the condition $\mathscr{D}'$ one must take $r \geq s+1=5$, $l \geq m+1=6$, see Remark 2 of this work. By equation (11), the size k of the complete cap $\mathbf{H} \subset \mathrm{PG}(n,2)$ obtained with the help of $\mathbf{Z}'$ of Construction Z_{12} is

$$k = 2^r + 2^l - 5 = 2^r + 2^{n+1-r} - 5, \quad r \geq 5, \ l \geq 6, \ n \geq r+5. \tag{50}$$

4.6. *Computer Search for Caps* Z'

We consider the situation when an infinity family of complete caps $\mathbf{H}$ is produced by Construction S, see Section 4.1, and the only "starting" cap $\mathbf{Z}'$ is found by computer. We use the term "Construction $\mathrm{Z}_{13,i}$" when for given parameters s, m we have found by computer a cap $\mathbf{Z}'_{13,i}$ for which all conditions $\mathscr{A}', \mathscr{B}', \mathscr{C}'$ and the properties $U^*_{s,0}$, $L^*_{m,0}$ hold. Here i is the ordinal number. For all Constructions $\mathrm{Z}_{13,i}$ we put $r \geq s+1$ and $l \geq m+1$, see Remark 2. Therefore the condition $\mathscr{D}'$ holds. We give caps $\mathbf{Z}'_{13,i}$ in hexadecimal notation.

Constructions $\mathrm{Z}_{13,1}$ and $\mathrm{Z}_{13,2}$

We put $s=m=3$. We found by computer a 15-cap $\mathbf{Z}'_{13,1}$ and a 16-cap $\mathbf{Z}'_{13,2}$.

$$\mathbf{Z}'_{13,1} = \left[\frac{11}{15}\ \frac{22}{26}\ \frac{3}{1}\ \frac{444}{134}\ \frac{55}{37}\ \frac{6}{4}\ \frac{7777}{3467}\right], \quad \mathbf{Z}'_{13,2} = \left[\frac{1111}{1567}\ \frac{22}{25}\ \frac{33}{16}\ \frac{44}{35}\ \frac{55}{17}\ \frac{66}{45}\ \frac{77}{67}\right].$$

By (11), the size k of the complete cap $\mathbf{H} \subset \mathrm{PG}(n,2)$ obtained with the help of $\mathbf{Z}'_{13,j}$ is

$$k = 2^r + 2^l + j - 2 = 2^r + 2^{n+1-r} + j - 2, \quad r \geq 4, \ l \geq 4, \ n \geq r+3, \ j = 1,2. \tag{51}$$

Constructions $\mathrm{Z}_{13,3}, \mathrm{Z}_{13,4}$, and $\mathrm{Z}_{13,5}$

Let $s=4$, $m=3$. We found by computer a 27-cap $\mathbf{Z}'_{13,3}$, a 28-cap $\mathbf{Z}'_{13,4}$, and a 29-cap $\mathbf{Z}'_{13,5}$.

$$\mathbf{Z}'_{13,3} = \left[\frac{11}{24}\ \frac{2}{6}\ \frac{33}{13}\ \frac{4}{4}\ \frac{555}{123}\ \frac{66}{67}\ \frac{77}{16}\ \frac{8888}{1467}\ \frac{99}{17}\ \frac{A}{3}\ \frac{B}{3}\ \frac{C}{1}\ \frac{D}{1}\ \frac{EE}{45}\ \frac{FF}{34}\right],$$

$$\mathbf{Z}'_{13,4} = \left[\frac{111}{126}\ \frac{2}{3}\ \frac{33}{46}\ \frac{44}{24}\ \frac{5}{7}\ \frac{66}{25}\ \frac{77}{15}\ \frac{88}{16}\ \frac{99}{26}\ \frac{AAA}{137}\ \frac{B}{4}\ \frac{C}{7}\ \frac{DD}{37}\ \frac{EE}{26}\ \frac{FF}{25}\right],$$

$$\mathbf{Z}'_{13,5} = \left[\frac{11}{45}\ \frac{2}{7}\ \frac{33}{16}\ \frac{44}{45}\ \frac{555}{457}\ \frac{6}{7}\ \frac{77}{16}\ \frac{8}{1}\ \frac{9}{7}\ \frac{AA}{23}\ \frac{BBB}{123}\ \frac{CC}{16}\ \frac{DD}{17}\ \frac{EE}{23}\ \frac{FFF}{123}\right].$$

By equation (11), the size k of the complete cap $\mathbf{H} \subset PG(n,2)$ obtained with the help of $\mathbf{Z}'_{13,t}$ is

$$k = 2^r + 2^l + t = 2^r + 2^{n+1-r} + t, \quad r \geq 5, \ l \geq 4, \ n \geq r+4, \ t = 3,4,5. \tag{52}$$

Table 2. The sizes $k < 2^{n-1}$ of the small complete caps in PG$(n, 2)$ obtained by distinct constructions.

n	k
5	13_1
6	$21_1\; \mathbf{22}_2\; 24_W \cdots 29_W\; 31_W$
7	$28_0\; 29_1\; \mathbf{30}_2\; \mathbf{31}_{13,1}\; \mathbf{32}_{13,2}\; \mathbf{33}_4\; 35_W\; 37_1\; \mathbf{38}_2\; 39_W \cdots 63_W$
8	$43_0\; 45_1\; \mathbf{46}_2\; \mathbf{47}_{13,1}\; \mathbf{48}_{13,2}\; \mathbf{49}_4\; \mathbf{50}_2\; \mathbf{51}_{13,3}\; \mathbf{52}_{13,4}\; \mathbf{53}_{13,5}\; 56^D_0\; 57_W\; 58^D_1$ $\mathbf{60}^D_2\; \mathbf{62}^D_{13,1}\; 63_W\; \mathbf{64}^D_{13,2}\; \mathbf{66}^D_4\; 67_W\; 69_1\; 70^D_W\; 71_W \cdots 127_W$
9	$57_0\; \mathbf{60}_{10}\; 61_1\; \mathbf{62}_2\; \mathbf{63}_{13,1}\; \mathbf{64}_{13,2}\; \mathbf{65}_4\; \mathbf{66}_2\; \mathbf{67}_{13,3}\; \mathbf{68}_{7+1}\; \mathbf{69}_{13,5}\; \mathbf{72}_3\; \mathbf{73}_4$ $\mathbf{74}_{5+1}\; 77_1\; \mathbf{78}_2\; \mathbf{79}_{13,1}\; \mathbf{80}_{13,2}\; \mathbf{81}_4\; \mathbf{82}_2\; 83_W\; \mathbf{84}_{13,4}\; \mathbf{85}_{13,5}\; 86^D_0\; \mathbf{88}_3\; 89_W$ $90^D_1\; \mathbf{92}^D_2\; \mathbf{94}^D_{13,1}\; 95_W\; \mathbf{96}^D_{13,2}\; \mathbf{97}_4\; \mathbf{98}^D_4\; \mathbf{100}^D_2\; 101_W\; \mathbf{102}^D_{13,3}\; 103_W\; \mathbf{104}^D_{13,4}$ $\mathbf{105}_4\; \mathbf{106}^D_{13,5}\; 107_W\; \mathbf{108}_3\; 112^D_0\; 113_W\; 114^D_W\; 116^D_1\; 117_W\; 119_W\; \mathbf{120}^D_2$ $\mathbf{124}^D_{13,1}\; 125_W\; 126^D_W\; \mathbf{128}^D_{13,2}\; 131_W\; \mathbf{132}^D_4\; 133_1\; 134^D_W\; 135_W \cdots 255_W$
10	$89_0\; \mathbf{91}_{12}\; \mathbf{92}_{10}\; 93_1\; \mathbf{94}_2\; \mathbf{95}_{13,1}\; \mathbf{96}_{13,2}\; \mathbf{97}_4\; \mathbf{98}_2\; \mathbf{99}_{13,3}\; \mathbf{100}_{7+1}\; \mathbf{101}_{13,5}\; \mathbf{104}_3$ $\mathbf{105}_4\; \mathbf{106}_2\; \mathbf{107}_{8+2}\; \mathbf{108}_{7+1}\; 114^D_0\; \mathbf{120}^D_{10}\; 121_W\; 122^D_1\; \mathbf{124}^D_2\; \mathbf{126}^D_{13,1}\; \mathbf{128}^D_{13,2}$ $\mathbf{129}_4\; \mathbf{130}^D_4\; \mathbf{132}^D_2\; \mathbf{134}^D_{13,3}\; 135_W\; \mathbf{136}^D_{7+1}\; \mathbf{138}^D_{13,5}\; 141_1\; \mathbf{142}_2\; \mathbf{143}_{13,1}\; \mathbf{144}^D_3$ $\mathbf{145}_{8+4}\; \mathbf{146}^D_4\; 147_W\; \mathbf{148}^D_{5+1}\; 149_W\; \mathbf{152}_3\; 153_W\; 154^D_1\; \mathbf{156}^D_2\; \mathbf{158}^D_{13,1}\; 159_W$ $\mathbf{160}^D_{13,2}\; \mathbf{162}^D_4\; 163_W\; \mathbf{164}^D_2\; 165_W\; 166^D_W\; \mathbf{168}^D_{13,4}\; \mathbf{169}_4\; \mathbf{170}_2\; 171_W\; 172^D_0$ $\mathbf{176}^D_3\; 177_W\; 178^D_W\; 180^D_1\; 183_W\; \mathbf{184}^D_2\; \mathbf{188}^D_{13,1}\; 189_W\; 190^D_W\; 191_W\; \mathbf{192}^D_{13,2}$ $\mathbf{194}^D_4\; 195_W\; \mathbf{196}^D_4\; \mathbf{200}^D_2\; 201_W\; 202^D_W\; \mathbf{204}^D_{13,3}\; 205_W\; 206^D_W\; 207_W\; \mathbf{208}^D_{13,4}$ $\mathbf{210}^D_4\; \mathbf{212}_3\; 213_W\; 214^D_W\; \mathbf{216}^D_3\; 219_W\; 224^D_0\; 225_W\; 226^D_W\; 228^D_W\; 231_W$ $232^D_1\; 233_W\; 234^D_W\; 237_W\; 238^D_W\; \mathbf{240}^D_2\; 243_W\; 247_W\; \mathbf{248}^D_{13,1}\; 249_W$ $250^D_W\; 252^D_W\; 255_W\; \mathbf{256}^D_{13,2}\; 259_W\; 261_1\; 262^D_W\; 263_W \cdots 511_W$
11	$117_0\; \mathbf{121}_{11}\; \mathbf{123}_{12}\; \mathbf{124}_{10}\; 125_1\; \mathbf{126}_2\; \mathbf{127}_{13,1}\; \mathbf{128}_{13,2}\; \mathbf{129}_4\; \mathbf{130}_2\; \mathbf{131}_{13,3}$ $\mathbf{132}_{7+1}\; \mathbf{133}_{13,5}\; \mathbf{136}_3\; \mathbf{137}_4\; \mathbf{138}_2\; \mathbf{139}_{8+2}\; \mathbf{140}_{5+2}$
12	$181_0\; \mathbf{185}_{11}\; \mathbf{187}_{12}\; \mathbf{188}_{10}\; 189_1\; \mathbf{190}_2\; \mathbf{191}_{13,1}\; \mathbf{192}_{13,2}\; \mathbf{193}_4\; \mathbf{194}_2\; \mathbf{195}_{13,3}$ $\mathbf{196}_{7+1}\; \mathbf{197}_{13,5}\; \mathbf{200}_3\; \mathbf{201}_4\; \mathbf{202}_2\; \mathbf{203}_{8+2}\; \mathbf{204}_{5+2}$

4.7. Tables of Sizes of Small Complete Caps

We give Table 2 with examples of sizes of caps obtained by known and new constructions. The subscripts $i \in \{1, 2, 3, 4, 10, 11, 12\}$ and $13, j \in \{13, 1 \ldots 13, 5\}$ indicate Construction Z_i and $Z_{13,j}$, respectively. Sizes of equation (1) have the subscript "1" as they can be generated by Construction Z_1. The subscripts "0" and "W" indicate, respectively, the known constructions of [9], see equation (2), and [14], see equation (3). Finally, the subscript of the form $u + i$, $u \in \{5, 6, 7, 8, 9\}$, $i \in \{1, 2, 3, 4, 10, 11, 12\}$, denotes Construction Z_u using a complete cap $\mathbf{H}_0$ obtained with the help of Construction Z_i. The superscript "D" indicates the doubling construction used for the results defined by the subscript. Boldface notes sizes obtained by new constructions and doubling of these new sizes.

For $n \leq 10$ Table 2 is filled in the following order. First, all sizes of equations (1)–(3) and applying DC to them are written. We denote $A_W \cdots B_W$ a region of sizes described in [14], see equation (3). Some sizes into such regions can be obtained also by DC. Then we consider the dimensions n in increasing order and

Table 3. The updated table of sizes of the known small complete caps in PG$(n, 2)$.

n	Sizes k of the known complete caps with $k \leq 2^{n-1}$	References
10	$91 \leq k \leq 511$, $k = 89$	[6,9,14],⋆
11	$123 \leq k \leq 1023$, $k = 117, 121$	[6,9,14],⋆
12	$187 \leq k \leq 2047$, $k = 181, 185$	[6,9,14],⋆

⋆ - results of this work

for fixed n we list sizes generated by Constructions Z_2–Z_{12} and $Z_{13,i}$. Every new size obtained is written in Table 2 together with applying DC to it. If the same new size can be obtained by several Constructions Z_i we note only one construction. For $n = 11, 12$ we give in Table 2 only relatively small sizes.

Using results written in Table 2 we can update Table 1 for $n = 10, 11, 12$, see Table 3.

Acknowledgment

The authors thank the anonymous referees for their details and useful comments.

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