Sbornik: Mathematics **197**:12 1753–1773

Matematicheskiĭ Sbornik **197**:12 43–64 DOI 10.1070/SM2006v197n12ABEH003821

Additional parameters in inverse monodromy problems

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Abstract. Several inverse problems of the analytic theory of differential equations are considered: an estimate of the number of extra singular points occurring in the construction of a Fuchsian equation for an arbitrary (for instance, reducible) monodromy representation is found; an estimate of the Poincaré rank of the unique non-Fuchsian singularity of the regular linear system constructed for an arbitrary monodromy representation is obtained; the problem of the meromorphic reduction to polynomial form of a linear system in the neighbourhood of an irregular singularity is investigated (which is related to the reduction of a linear system to a Birkhoff standard form).

Bibliography: 12 titles.

§ 1. Introduction

Consider a linear differential equation

$$\frac{d^p y}{dz^p} + b_1(z)\frac{d^{p-1}y}{dz^{p-1}} + \dots + b_p(z)y = 0 \tag{1}$$

of order p with coefficients $b_1(z), \dots, b_p(z)$ meromorphic on the Riemann sphere $\overline{\mathbb{C}}$ and holomorphic outside the set of singular points $a_1, \dots, a_n$.

By the *monodromy representation* or the *monodromy* of this equation we mean the representation

$$\chi\colon \pi_1(\overline{\mathbb{C}} \setminus \{a_1, \dots, a_n\}, z_0) \to \mathrm{GL}(p, \mathbb{C}) \tag{2}$$

of the fundamental group of the space $\overline{\mathbb{C}} \setminus \{a_1, \dots, a_n\}$ in the space of non-singular complex matrices of order p defined as follows. In the neighbourhood of a non-singular point z_0 we consider a basis $(y_1(z), \dots, y_p(z))$ in the solution space of equation (1). Analytic continuation of the functions $y_1(z), \dots, y_p(z)$ along an arbitrary loop γ outgoing from z_0 and lying in $\overline{\mathbb{C}}\setminus\{a_1, \dots, a_n\}$ transforms the basis $(y_1, \dots, y_p)$ into an (in general different) basis $(\widetilde{y}_1, \dots, \widetilde{y}_p)$. The two bases are related by means of a non-singular transition matrix G_γ corresponding to the loop γ:

$$(y_1, \dots, y_p) = (\widetilde{y}_1, \dots, \widetilde{y}_p)G_\gamma.$$

This research was carried out with the support of the Russian Foundation for Basic Research (grant no. 03-01-22000) and the Programme of Support of Leading Scientific Schools of the RF (grant no. НШ-457.2003.01).

AMS 2000 *Mathematics Subject Classification.* Primary 32G34, 34A20; Secondary 34M50, 34C20.

The map $[\gamma] \mapsto G_\gamma$ (which depends only on the homotopy class $[\gamma]$ of the loop γ) defines the representation χ. By the *monodromy matrix* of equation (1) at a singular point a_i (with respect to the basis $(y_1, \dots, y_p)$) we mean the matrix G_i corresponding to a simple loop γ_i encircling a_i, so that $G_i = \chi([\gamma_i])$.

A singular point a_i of equation (1) is said to be *Fuchsian* if the coefficient $b_j(z)$ has at this point a pole of order j or lower ($j = 1, \dots, p$). By Fuchs's theorem (see [1]) a singular point a_i is Fuchsian if and only if it is *regular* (that is, each solution has at most power growth in the neighbourhood of a_i). Equation (1) is said to be *Fuchsian* if all its singular points are Fuchsian.

The problem of the construction of Fuchsian equation (1) with prescribed singular points $a_1, \dots, a_n$ and prescribed monodromy representation (2) has a negative solution in the general case because the number of parameters determining the equation is less than the number of parameters determining the set of representations χ (see [2], [1]). Hence in the construction of a Fuchsian equation there arise extra '*apparent*' singular points (at which the coefficients of the equation are singular, but its solutions are holomorphic). If the representation (2) is irreducible, then an expression for the smallest possible number of such singular points has been obtained by Bolibrukh [2]. In the present paper we consider the case of a reducible representation.

Alongside equation (1) one can consider a linear system

$$\frac{dy}{dz} = B(z)y, \qquad y(z) \in \mathbb{C}^p, \tag{3}$$

of p equations with matrix $B(z)$ meromorphic on the Riemann sphere and holomorphic outside the points $a_1, \dots, a_n$. One defines the monodromy representation of this system in the same way as for equation (1); one merely needs to consider in place of the row $(y_1, \dots, y_p)$ a *fundamental matrix* $Y(z)$ the columns of which form a basis in the solution space of the system.

A singular point a_i of the system (3) is said to be *Fuchsian* if the matrix $B(z)$ has a simple pole at this point. A Fuchsian singular point of a linear system is always regular, although a regular singularity is not necessarily Fuchsian (see [1]). A system (3) is said to be *Fuchsian* if all its singular points are Fuchsian.

Similarly to scalar equation (1) the problem of the construction of a Fuchsian system (3) with prescribed singular points $a_1, \dots, a_n$ and prescribed monodromy representation (2) (which is called the *Riemann–Hilbert problem*) has a negative solution in the general case (see [1]–[3]). One knows various sufficient conditions for the affirmative solution of this problem (one such condition is the irreducibility of the representation (2)).

A method of the solution of problems related to the Riemann–Hilbert problem has been developed by Bolibrukh; its idea is as follows. From the representation (2) one constructs over the Riemann sphere a family $\mathscr{F}$ of holomorphic vector bundles of rank p with logarithmic connections having the prescribed singular points and the prescribed monodromy. Recall that one defines a bundle F of rank p by a set $\{U_i\}$ of neighbourhoods covering the sphere and a set $\{g_{ij}\}$ of gluing cocycles, *holomorphically invertible* matrices of order p defined on the non-empty intersections $U_i \cap U_j$ (so that $g_{ij}(z)$ is holomorphic in $U_i \cap U_j$ and $\det g_{ij}(z)$ does not vanish there),

with the following properties:
(1) $g_{ij}(z) = g_{ji}^{-1}(z)$;
(2) $g_{ij}(z)g_{jk}(z)g_{ki}(z) \equiv I$ if $U_i \cap U_j \cap U_k \neq \varnothing$.
A connection ∇ is defined by a set $\{\omega^i\}$ of local matrix-valued differential 1-forms (ω^i is defined in U_i) satisfying in the intersections $U_i \cap U_j \neq \varnothing$ the gluing conditions

$$\omega^i = (dg_{ij})g_{ij}^{-1} + g_{ij}\omega^j g_{ij}^{-1}. \tag{4}$$

The connection defines locally the system $dy = \omega^i y$. The monodromy of the connection (similarly to the monodromy of the system (3)) describes the branching pattern of solutions of these local systems after analytic continuation along closed paths encircling the singular points. A connection ∇ is said to be *logarithmic* (*Fuchsian*) if all singularities of the forms ω^i are first-order poles. The Riemann–Hilbert problem for a fixed representation (2) is solved in the affirmative if some bundle in the family $\mathscr{F}$ turns out to be holomorphically trivial (because then one can take for the cocycles g_{ij} the identity matrices and, in view of gluing condition (4), the connection ∇ defines a Fuchsian system with monodromy (2) on the entire Riemann sphere).

The Birkhoff–Grothendieck theorem states that each holomorphic vector bundle F of rank p on the Riemann sphere is equivalent to a bundle with the following description:

$$(U_0 = \mathbb{C},\ U_\infty = \overline{\mathbb{C}} \setminus \{0\},\ g_{0\infty} = z^K), \qquad K = \operatorname{diag}(k_1, \dots, k_p),$$

where $\{k_j\}$, $k_1 \geqslant \dots \geqslant k_p$, is a system of integers called the *splitting type* of the bundle F.

Let $\widetilde{\gamma}_{\min}(\chi)$ be the following quantity:

$$\widetilde{\gamma}_{\min}(\chi) = \min_{F \in \mathscr{F}} (k_1 - k_p),$$

which is defined for an arbitrary representation χ.

The main results of §§ 3, 4 of the present paper are as follows.

Theorem 1. *The quantity $\widetilde{\gamma}_{\min}(\chi)$ has the following estimate*:

$$\widetilde{\gamma}_{\min}(\chi) \leqslant (p-1)(n-1).$$

The *Poincaré rank* of a singular point of the system (3) is by definition one less than the order of the pole of the matrix $B(z)$ at this point (for instance, the Poincaré rank of a Fuchsian singularity is zero).

Although we have pointed out already that the Riemann–Hilbert problem has a negative solution in the general case, by Plemelj's theorem (see [1], [3]), for a fixed representation (2) one can construct a system (3) that is Fuchsian at all points but one, at which it is regular; and we present here an estimate for the Poincaré rank of the regular singularity of the so-constructed system.

Corollary 1. *Each representation* (2) *can be realized as the monodromy representation of a system* (3) *that is Fuchsian at all points but one, at which it is regular, such that its Poincaré rank at this point is at most* $(n-1)(p-1)$.

Theorem 2. *For an arbitrary representation (2) there exists Fuchsian equation (1) with fixed monodromy such that the number m of extra apparent singular points of this equation satisfies the inequality*

$$m \leqslant \frac{(n+1)p(p-1)}{2} + 1.$$

In our § 5, which is mostly of an independent nature, we study questions related to the problem of the meromorphic transformation of a system of linear differential equations into a Birkhoff standard form.

§ 2. Proof of Deligne's lemma

If the monodromy representation of a system (3) is irreducible, then one can associate with this system a linear differential equation

$$\frac{1}{W(u_1,\dots,u_p)} \det \begin{pmatrix} u_1 & \dots & u_p & y \\ \dfrac{du_1}{dz} & \dots & \dfrac{du_p}{dz} & \dfrac{dy}{dz} \\ \dots & \dots & \dots & \dots \\ \dfrac{d^p u_1}{dz^p} & \dots & \dfrac{d^p u_p}{dz^p} & \dfrac{d^p y}{dz^p} \end{pmatrix} = 0 \tag{5}$$

of the form (1) with respect to an unknown function $y(z)$, where $u_1,\dots,u_p$ are the entries in an arbitrary row of a fundamental matrix of the system and $W(u_1,\dots,u_p)$ is their Wronskian. One easily demonstrates that the entries in an arbitrary row of a fundamental matrix of a system (3) with irreducible monodromy are linearly independent, therefore the functions $u_1,\dots,u_p$ form a basis in the solution space of the above equation and its monodromy coincides with that of the system. In this case the extra apparent singular points of equation (5) are zeros of the Wronskian $W(u_1,\dots,u_p)$, which is not identically zero under the above assumptions.

In the case when the monodromy representation of the system (3) is reducible one cannot in general proceed to an equation in the above-described fashion. For instance, if the matrix $B(z)$ of the system is diagonal, then the entries in each row of an arbitrary fundamental matrix of this system are linearly dependent. However, there exists another method of passing from the system to a scalar equation with the same monodromy, which is based on a result usually called Deligne's lemma ([4], Lemma II.1.3). We present here an analytic proof of this lemma (stated in [4] in algebraic terms).

We say that a transformation $\widetilde{y} = \Gamma(z)y$ of the system (3) is *meromorphically invertible* if its matrix $\Gamma(z)$ is meromorphic and $\det \Gamma(z) \not\equiv 0$. Such a transformation takes the system (3) to another system

$$\frac{d\widetilde{y}}{dz} = \widetilde{B}(z)\widetilde{y} \tag{6}$$

with matrix of coefficients

$$\widetilde{B}(z) = \Gamma B(z)\Gamma^{-1} + \frac{d\Gamma}{dz}\Gamma^{-1}. \tag{7}$$

Lemma 1 (Deligne [4]). *For each system* (3) *there exists a transformation* $\widetilde{y}=\Gamma(z)y$ *meromorphically invertible on the Riemann sphere that takes it to a system* (6), (7) *with matrix of coefficients* $\widetilde{B}(z)$ *of the following form*:

$$\widetilde{B}(z)=\begin{pmatrix} 0 & 1 & & 0 \\ & \ddots & \ddots & \\ 0 & & 0 & 1 \\ -b_p & \dots & \dots & -b_1 \end{pmatrix}, \tag{8}$$

where $b_1(z),\dots,b_p(z)$ *are meromorphic functions.*

Proof (based on Exercise 7 in [5], § 1.1). We consider a p-dimensional row vector $t_0(z)$ with polynomials of degree $p-1$ as components and express $t_0(z)$ in the following form:

$$t_0(z)=\alpha_0+\alpha_1(z-z_0)+\frac{1}{2}\,\alpha_2(z-z_0)^2+\dots+\frac{1}{(p-1)!}\,\alpha_{p-1}(z-z_0)^{p-1},$$

where z_0 is a point distinct from the singular points of the system (3). Next, we define the meromorphic vector-valued functions $t_1(z),\dots,t_p(z)$ on the Riemann sphere by the formulae

$$t_{j+1}=\frac{dt_j}{dz}+t_jB(z),\qquad j=0,\dots,p-1. \tag{9}$$

Consider the matrix $\Gamma(z)$ with rows $t_0(z),\dots,t_{p-1}(z)$. Let C be a constant non-singular matrix of order p with rows $\mathbf{c}_0,\dots,\mathbf{c}_{p-1}$. We claim that we can select the vector $t_0(z)$ such that $\Gamma(z_0)=C$, that is, $\det\Gamma(z)\not\equiv 0$. To this end we set

$$\begin{aligned} &t_0(z_0)=\alpha_0=\mathbf{c}_0,\\ &t_1(z_0)=\alpha_1+\alpha_0B(z_0)=\mathbf{c}_1,\qquad \text{that is,}\ \ \alpha_1=\mathbf{c}_1-\mathbf{c}_0B(z_0).\end{aligned}$$

In the general case

$$t_j(z_0)=\alpha_j+F_j(\alpha_0,\alpha_1,\dots,\alpha_{j-1})=\mathbf{c}_j,$$

where $F_j(\alpha_0,\alpha_1,\dots,\alpha_{j-1})$ is an already known vector, that is,

$$\alpha_j=\mathbf{c}_j-F_j(\alpha_0,\alpha_1,\dots,\alpha_{j-1}),\qquad j=1,\dots,p-1.$$

We see that we have selected the coefficients α_j of the polynomial $t_0(z)$ such that the corresponding matrix $\Gamma(z)$ is meromorphically invertible on the entire Riemann sphere. Consider the vector-valued function $b(z)=(b_p(z),\dots,b_1(z))$ such that

$$t_p(z)=-b(z)\Gamma(z)$$

and consider the corresponding matrix

$$\widetilde{B}(z)=\begin{pmatrix} 0 & 1 & & 0 \\ & \ddots & \ddots & \\ 0 & & 0 & 1 \\ -b_p & \dots & \dots & -b_1 \end{pmatrix}.$$

It follows by (9) that
$$\frac{d\Gamma}{dz} = \widetilde{B}\Gamma - \Gamma B,$$
therefore the transformation $\widetilde{y} = \Gamma(z)y$ takes (3) to a system with matrix of coefficients
$$\Gamma B\Gamma^{-1} + \frac{d\Gamma}{dz}\Gamma^{-1} = \Gamma B\Gamma^{-1} + \widetilde{B} - \Gamma B\Gamma^{-1} = \widetilde{B}$$
of the required form. The proof of Lemma 1 is complete.

Alongside singular points $a_1, \dots, a_n$ of the original system (3), the system (6), (7) has singularities at the zeros $z_1, \dots, z_m$ of the function $\det\Gamma(z)$ (not contributing to the monodromy). We shall assume that one singular point of the system (3) (for instance, a_n) is at infinity (otherwise we could use a linear fractional transformation $\zeta = 1/(z - a_n)$ of the Riemann sphere taking the singular points $a_1, \dots, a_{n-1}, a_n$ to singular points $b_1 = 1/(a_1 - a_n), \dots, b_{n-1} = 1/(a_{n-1} - a_n)$, $b_n = \infty$ of the same Poincaré ranks).

One readily sees that the first component of a solution of the system (6) with matrix of coefficients $\widetilde{B}(z)$ of the form (8) is a solution of an equation of the form (1) with singular points $a_1, \dots, a_n$ and extra apparent singularities $z_1, \dots, z_m$. Hence one can take for a basis in the solution space of this equation the components $u_1, \dots, u_p$ of the first row of a fundamental matrix $Y(z)$ of the system (6). The functions $u_1, \dots, u_p$ are linearly independent since the matrix $Y(z)$ has the following form:
$$Y(z) = \begin{pmatrix} u_1 & \dots & u_p \\ \dfrac{du_1}{dz} & \dots & \dfrac{du_p}{dz} \\ \dots & \dots & \dots \\ \dfrac{d^{p-1}u_1}{dz^{p-1}} & \dots & \dfrac{d^{p-1}u_p}{dz^{p-1}} \end{pmatrix}$$
and the linear dependence of the functions $u_1, \dots, u_p$ would imply the linear dependence of the columns of the matrix $Y(z)$. Thus, the monodromy of the equation so constructed coincides with the monodromy of the system (6) and therefore of the system (3) (recall that the monodromy matrices of the constructed equations at the additional singular points are equal to the identity). Moreover, it follows from Fuchs's theorem that if all singularities of (3) are regular, then the resulting equation is Fuchsian. We now find an estimate for the number m of its apparent singularities $z_1, \dots, z_m$.

We write the Laurent series of the matrix of coefficients $B(z)$ of the system (3) in a neighbourhood of $a_i \neq \infty$ in the following form:
$$B(z) = \frac{B^i_{-r_i-1}}{(z - a_i)^{r_i+1}} + \dots + \frac{B^i_{-1}}{z - a_i} + B^i_0 + \cdots, \qquad B^i_{-r_i-1} \neq 0$$
(as concerns the neighbourhood of $a_n = \infty$, the principal part of the Laurent series of the matrix $B(z)$ is a polynomial of degree $r_n - 1$), where r_i is the Poincaré rank of the system at the singular point a_i.

The function $\det\Gamma(z)$ (in Lemma 1) has zeros and poles at $z_1, \dots, z_m, a_1, \dots, a_n = \infty$.

It follows by formulae (9) that in the neighbourhood of the point $a_i \neq \infty$ the (vector-valued) function $t_j(z)$ has the following representation:

$$t_j(z) = \frac{1}{(z-a_i)^{j(r_i+1)}}\,\widetilde{t}_j(z), \qquad j = 0, \ldots, p-1,$$

where $\widetilde{t}_j(z)$ is a holomorphic function in the neighbourhood of a_i, therefore

$$\Gamma(z) = \operatorname{diag}\bigl(1, (z-a_i)^{-(r_i+1)}, \ldots, (z-a_i)^{-(p-1)(r_i+1)}\bigr)\widetilde{\Gamma}(z),$$

where $\widetilde{\Gamma}(z)$ is a holomorphic matrix in the neighbourhood of a_i. Hence

$$\operatorname{ord}_{a_i} \det\Gamma(z) \geqslant -\frac{p(p-1)}{2}(r_i+1), \qquad i = 1, \ldots, n-1.$$

At the same time, in the neighbourhood of infinity the matrix $B(z)$ has the following representation:

$$B(z) = z^{r_n-1}B_0(z),$$

where $B_0(z)$ is a matrix holomorphic in this neighbourhood. Hence the function $t_j(z)$ has in this neighbourhood the following expression:

$$t_j(z) = z^{p-1+j(r_n-1)}\widetilde{t}_j(z), \qquad j = 0, \ldots, p-1,$$

where $\widetilde{t}_j(z)$ is a holomorphic function in the neighbourhood of infinity. Then

$$\Gamma(z) = \operatorname{diag}\bigl(z^{p-1}, z^{p-1+(r_n-1)}, \ldots, z^{p-1+(p-1)(r_n-1)}\bigr)\widetilde{\Gamma}(z),$$

where $\widetilde{\Gamma}(z)$ is a matrix holomorphic in the neighbourhood of infinity. Consequently,

$$\operatorname{ord}_{a_n} \det\Gamma(z) \geqslant -p(p-1) - \frac{p(p-1)}{2}(r_n-1) = -\frac{p(p-1)}{2}(r_n+1).$$

Let d_i be the order of the zero of $\det\Gamma(z)$ at z_i. Then

$$0 = \sum_{i=1}^{m} d_i + \sum_{i=1}^{n} \operatorname{ord}_{a_i} \det\Gamma(z) \geqslant \sum_{i=1}^{m} d_i - \left(\sum_{i=1}^{n} r_i + n\right)\frac{p(p-1)}{2}\,,$$

so that

$$m \leqslant \sum_{i=1}^{m} d_i \leqslant \left(\sum_{i=1}^{n} r_i + n\right)\frac{p(p-1)}{2}\,.$$

We have thus established the following auxiliary result.

Lemma 2. *For an arbitrary system* (3) *one can construct a scalar equation* (1) *with the same monodromy such that the number m of its apparent singularities satisfies the inequality*

$$m \leqslant \frac{(R+n)p(p-1)}{2}\,,$$

where $R = \sum_{i=1}^{n} r_i$ is the sum of the Poincaré ranks of singular points of the system. Moreover, if the system has only regular singularities, then the resulting equation is Fuchsian.

§ 3. Holomorphic bundles and regular systems

In this section we prove Theorem 1. Consider a family of holomorphic vector bundles F^Λ on the Riemann sphere with logarithmic connections ∇^Λ having the prescribed monodromy (2). The method of constructing the family $\mathscr{F}$ has been explained in detail by Bolibrukh (see [1], [3]). We recall here only the central points of this construction.

Each bundle F^Λ is defined by a system $\Lambda = \{\Lambda_1, \dots, \Lambda_n\}$ of *admissible matrices* Λ_i (diagonal matrices with integer entries λ_i^j forming a non-increasing sequence $\lambda_i^j \geqslant \lambda_i^{j+1}$, $j = 1, \dots, p-1$) and has the following coordinate description. One covers the sphere by a system $O_1, \dots, O_n$ of small neighbourhoods of the points $a_1, \dots, a_n$ and a set $\{U_\alpha\}$ complementing $\{O_i\}$. For each non-empty intersection $O_i \cap U_\alpha$ one can express the gluing cocycle $g_{i\alpha}(z)$ in the following form:

$$g_{i\alpha}(z) = (z - a_i)^{\Lambda_i}(z - a_i)^{E_i} S_i^{-1},$$

where

$$E_i = \frac{1}{2\pi i}\ln(S_i^{-1} G_i S_i)$$

is an upper-triangular matrix with eigenvalues ρ_i^j such that

$$0 \leqslant \operatorname{Re}\rho_i^j < 1, \qquad j = 1, \dots, p, \tag{10}$$

and S_i is a non-singular constant matrix reducing the monodromy matrix G_i to upper-triangular form.

For non-empty intersections $O_i \cap U_\alpha \cap U_\beta$ the cocycle $g_{i\beta}(z)$ is an analytic continuation of the cocycle $g_{i\alpha}(z)$ and $g_{\alpha\beta}(z) \equiv \mathrm{const}$. Strictly speaking, F^Λ depends also on the system $S = \{S_1, \dots, S_n\}$ of matrices S_i reducing the monodromy matrices G_i to upper-triangular form. Hence, in view of this connection, one should denote elements of $\mathscr{F}$ by $F^{\Lambda,S}$, but we require in what follows only their dependence on the system Λ and mean by F^Λ the bundle constructed for the fixed system Λ and some system S.

The connection ∇^Λ is defined by the forms ω^α trivial in U_α and by the forms ω^{Λ_i} that have the following expressions in the neighbourhoods O_i:

$$\omega^{\Lambda_i} = \bigl(\Lambda_i + (z - a_i)^{\Lambda_i} E_i (z - a_i)^{-\Lambda_i}\bigr)\frac{dz}{z - a_i} \tag{11}$$

(it follows from the definition of an admissible matrix Λ_i and the upper-triangular matrix E_i that $z = a_i$ is a first-order pole of the form ω^{Λ_i}). One easily verifies that the forms $\omega^{\Lambda_i}, \omega^\alpha$ and $\omega^\alpha, \omega^\beta$ satisfy gluing conditions (4) on the non-empty intersections $O_i \cap U_\alpha$ and $U_\alpha \cap U_\beta$.

Definition 1. One calls the eigenvalues $\beta_i^j = \lambda_i^j + \rho_i^j$ of the matrix $\Lambda_i + E_i$ *exponents of the connection* ∇^Λ at the point $z = a_i$.

It follows from the expression (11) for the connection form ω^{Λ_i} that at a Fuchsian point $z = a_i$ exponents are eigenvalues of the residue matrix $\operatorname{res}_{a_i}\omega^{\Lambda_i}$.

As already pointed out in the introduction, a logarithmic connection on a trivial bundle defines a Fuchsian system (3). On the other hand, in the case of the holomorphic bundle F^Λ of splitting type $(k_1,\dots,k_p)$, for which we consider the equivalent coordinate description

$$\big(U_1=\mathbb{C},\ U_\infty=\overline{\mathbb{C}}\setminus\{a_1\},\ g_{1\infty}=(z-a_1)^K\big),\qquad K=\operatorname{diag}(k_1,\dots,k_p),$$

the logarithmic connection ∇^Λ defines a system (3) that is Fuchsian at all points but one ($z=a_1$), at which it is regular, and the fundamental matrix $Y_1(z)$ has the following form in the neighbourhood of this point:

$$Y_1(z)=(z-a_1)^{-K}V_1(z)(z-a_1)^{\Lambda_1}(z-a_1)^{E_1}, \tag{12}$$

where $V_1(z)$ is a holomorphically invertible matrix in a neighbourhood of a_1. As regards the neighbourhoods of the other (Fuchsian) singularities a_i, there exist in these neighbourhoods fundamental matrices $Y_i(z)$ of the following form:

$$Y_i(z)=V_i(z)(z-a_i)^{\Lambda_i}(z-a_i)^{E_i}, \tag{13}$$

where $V_i(z)$ is holomorphically invertible in a neighbourhood of a_i (see [1], [3] for details).

Definition 2. The *degree* $\deg F^\Lambda$ of a bundle F^Λ is the quantity

$$\deg F^\Lambda=\sum_{i=1}^n \operatorname{res}_{a_i}\operatorname{tr}\omega^{\Lambda_i}=\sum_{i=1}^n\sum_{j=1}^p\beta_i^j.$$

The degree of a bundle is an integer equal to the sum of the coefficients k_i of the splitting type of this bundle. This can be demonstrated as follows. Consider a system (3) with form $\omega=B(z)\,dz$ corresponding to the connection ∇^Λ. One sees from the expressions (12), (13) for the fundamental matrix $Y_1(z)$ and the other fundamental matrices $Y_i(z)$ of this system and also from Liouville's formula $d\ln\det Y_i(z)=\operatorname{tr}\omega$ that

$$\begin{aligned}\operatorname{res}_{a_1}\operatorname{tr}\omega&=\operatorname{res}_{a_1}\left(-\frac{\operatorname{tr}K}{z-a_1}\,dz+\frac{d\det V_1}{\det V_1}+\frac{\operatorname{tr}(\Lambda_1+E_1)}{z-a_1}\,dz\right)\\&=-\operatorname{tr}K+\operatorname{tr}(\Lambda_1+E_1),\\ \operatorname{res}_{a_i}\operatorname{tr}\omega&=\operatorname{res}_{a_i}\left(\frac{d\det V_i}{\det V_i}+\frac{\operatorname{tr}(\Lambda_i+E_i)}{z-a_i}\,dz\right)=\operatorname{tr}(\Lambda_i+E_i)\end{aligned}$$

(the differential forms $d\det V_i/\det V_i$ are holomorphic at the corresponding points a_i). By the theorem on the sum of the residues

$$\sum_{i=1}^n\operatorname{res}_{a_i}\operatorname{tr}\omega=0,$$

that is,

$$\sum_{i=1}^n\operatorname{tr}(\Lambda_i+E_i)-\operatorname{tr}K=0$$

and

$$\deg F^{\Lambda} = \sum_{i=1}^{n} \operatorname{tr}(\Lambda_i + E_i) = \operatorname{tr} K.$$

Bolibrukh showed that for the splitting type of the bundle $F^{\Lambda} \in \mathscr{F}$ constructed from the irreducible representation (2) one has the inequalities

$$k_i - k_{i+1} \leqslant n - 2, \qquad i = 1, \dots, p-1$$

(see [1], [3]). On this basis we prove the following result, one consequence of which is Theorem 1.

Proposition 1. *Consider a bundle $F^{\Lambda} \in \mathscr{F}$ with logarithmic connection ∇^{Λ} the exponents of which satisfy the condition $0 \leqslant \operatorname{Re} \beta_i^j < M$, $M \in \mathbb{N}$. Then the following inequalities hold for the splitting type of this bundle*:

$$k_i - k_{i+1} \leqslant nM - 1, \qquad i = 1, \dots, p-1.$$

Proof. We consider two separate cases.

Case 1. For the splitting type of the bundle F^{Λ} one has the inequalities

$$k_i - k_{i+1} \leqslant n - 2, \qquad i = 1, \dots, p-1.$$

Since $M \in \mathbb{N}$, the required result is in this case a direct consequence of these relations.

Case 2. For some l one has $k_l - k_{l+1} \geqslant n-1$. In this case we claim that $k_l - k_{l+1} \leqslant nM - 1$.

We show first that the bundle F^{Λ} has a subbundle F^1 of rank l *stabilized* by the connection ∇^{Λ} and of splitting type $(k_1, \dots, k_l)$. In terms of the coordinate description of the $(F^{\Lambda}, \nabla^{\Lambda})$ the existence of such a subbundle means that the cocycles g_{ij} and the forms ω_i have the block upper-triangular form:

$$g_{ij} = \begin{pmatrix} g_{ij}^1 & * \\ 0 & g_{ij}^2 \end{pmatrix}, \qquad \omega_i = \begin{pmatrix} \omega_i^1 & * \\ 0 & \omega_i^2 \end{pmatrix},$$

with all the blocks g_{ij}^1 and ω_i^1 of size $l \times l$. In that case the forms ω_i^1 define a restriction ∇^1 of the connection ∇^{Λ} to the subbundle F^1.

Consider the system (3) with regular singular point a_1 and Fuchsian singularities $a_2, \dots, a_n$ corresponding to the connection ∇^{Λ}. The form $\omega' = B(z)\,dz$ of the coefficients of this system has simple poles in $O_\infty = \overline{\mathbb{C}} \setminus \{a_1\}$, and in the neighbourhood O_1 of the point a_1 it has the following form:

$$\omega' = (dY_1)Y_1^{-1} = -\frac{K}{z - a_1}\,dz + (z-a_1)^{-K}\omega(z-a_1)^{K}, \tag{14}$$

where the form ω has a logarithmic singularity at the point a_1. This follows from the form (12) of the fundamental matrix of this system in the neighbourhood of the point $z = a_1$:

$$Y_1(z) = (z-a_1)^{-K} V_1(z)(z-a_1)^{\Lambda_1}(z-a_1)^{E_1}.$$

Recall that the matrix $V_1(z)$ is holomorphically invertible at the point a_1 and the set of diagonal elements of the matrix K coincides with the splitting type of the bundle F^Λ.

By (14) the entries ω'_{mj} and ω_{mj} of the matrix-valued differential 1-forms ω' and ω are connected for $m \neq j$ by the equality

$$\omega'_{mj} = (z-a_1)^{-k_m+k_j}\omega_{mj};$$

$\operatorname{ord}_{a_1}\omega_{mj} \geqslant -1$. By assumption $k_l - k_{l+1} > n-2$ for some l, therefore we have $k_j - k_m > n-2$ for $j \leqslant l$, $m > l$. Hence the orders $\operatorname{ord}_{a_1}\omega'_{mj}$ at the point a_1 of the differential forms ω'_{mj} with indicated indices are greater than $n-3$, whereas the sum of the orders $\operatorname{ord}_{a_i}\omega'_{mj}$ at the singular points distinct from a_1 is at least $-n+1$ (since the form ω' has logarithmic singularities at these points). We thus obtain for meromorphic forms ω'_{mj} with indicated indices on the Riemann sphere that the sum of their orders over all singularities and zeros is greater than -2, although this sum is known to be -2 for a non-trivial differential form (the degree of the canonical divisor; see [6], § 17). Hence these forms (as well as the ω_{mj}) are identically equal to zero, so that the forms ω' and ω are block upper triangular:

$$\omega' = \begin{pmatrix} \omega^1 & * \\ 0 & \omega^2 \end{pmatrix}, \qquad \omega = \begin{pmatrix} \omega_0^1 & * \\ 0 & \omega_0^2 \end{pmatrix}, \tag{15}$$

where the matrix-valued forms ω^1 and ω_0^1 have size $l \times l$ and satisfy (in view of (14)) the gluing condition

$$\omega^1 = -\frac{K^1}{z-a_1}\,dz + (z-a_1)^{-K^1}\omega_0^1(z-a_1)^{K^1}, \tag{16}$$

where $K^1 = \operatorname{diag}(k_1,\dots,k_l)$.

Thus the vector bundle F^Λ has an equivalent coordinate description

$$\bigl(O_1,\ O_\infty = \overline{\mathbb{C}} \setminus \{a_1\},\ g_{1\infty} = (z-a_1)^K\bigr), \qquad K = \operatorname{diag}(k_1,\dots,k_p),$$

and the logarithmic connection ∇^Λ is defined in the neighbourhoods O_1 and O_∞ by forms ω, ω' of the structure (15) satisfying gluing condition (14). Hence the bundle F^Λ has a subbundle F^1 of rank l with connection ∇^1 defined in the neighbourhoods O_1 and O_∞ by the forms ω_0^1, ω^1 satisfying gluing condition (16). By construction the connection ∇^Λ stabilizes the subbundle F^1 and coincides on it with the connection ∇^1, and the splitting type of the subbundle F^1 is equal to $(k_1,\dots,k_l)$.

Assume that $k_l - k_{l+1} \geqslant nM$. Then for the mean value of the exponents ${}^1\beta_i^j$ of the connection ∇^1 (which are the eigenvalues of the matrices $\operatorname{res}_{a_1}\omega_0^1$ and $\operatorname{res}_{a_i}\omega^1$) on the subbundle F^1 we have the lower bound

$$\frac{1}{ln}\sum_{i=1}^{n}\sum_{j=1}^{l}{}^1\beta_i^j = \frac{\deg F^1}{ln} = \frac{k_1+\dots+k_l}{ln} \geqslant \frac{k_{l+1}}{n} + M,$$

while for the mean value of the other exponents ${}^2\beta_i^j$ (the eigenvalues of the matrices $\operatorname{res}_{a_1}\omega_0^2$ and $\operatorname{res}_{a_i}\omega^2$) we have the upper bound

$$\frac{1}{(p-l)n}\sum_{i=1}^{n}\sum_{j=1}^{p-l}{}^2\beta_i^j = \frac{\deg F - \deg F^1}{(p-l)n} = \frac{k_{l+1}+\dots+k_p}{(p-l)n} \leqslant \frac{k_{l+1}}{n}\,.$$

Hence the mean value of the exponents ${}^1\beta_i^j$ is larger by M at least than the mean value of the exponents ${}^2\beta_i^j$, while by the hypothesis the real parts of all the exponents of the connection ∇^Λ are strictly less than M. We arrive at a contradiction, therefore $k_l - k_{l+1} \leqslant nM - 1$ for each l. The proof of Proposition 1 is complete.

Proof of Theorem 1. It is sufficient to consider the canonical bundle F^0 corresponding to the system $\Lambda = \{0, \dots, 0\}$ of zero matrices. In that case the exponents β_i^j of the connection ∇^0 satisfy, in view of (10), the condition

$$0 \leqslant \operatorname{Re} \beta_i^j = \operatorname{Re} \rho_i^j < 1.$$

Then it follows by Proposition 1 that for the coefficients k_i^0 of the splitting type of the bundle F^0 we have the inequalities

$$k_i^0 - k_{i+1}^0 \leqslant n - 1, \qquad i = 1, \dots, p-1.$$

Hence

$$\widetilde{\gamma}_{\min}(\chi) = \min_{F^\Lambda}(k_1 - k_p) \leqslant k_1^0 - k_p^0 = \sum_{i=1}^{p-1}(k_i^0 - k_{i+1}^0) \leqslant (n-1)(p-1).$$

The proof of Theorem 1 is complete.

Proof of Corollary 1. Consider the canonical bundle F^0 with logarithmic connection ∇^0 constructed from the representation (2). It has splitting type $(k_1^0, \dots, k_p^0)$ with $k_1^0 - k_p^0 \leqslant (n-1)(p-1)$ (see the proof of Theorem 1). Corresponding to the connection ∇^0 is a system (3) with regular singularity at a_1 and Fuchsian singularities at $a_2, \dots, a_n$ that has the prescribed monodromy (2). For an estimate of the Poincaré rank r_1 at the singular point a_1 recall that the form $\omega' = B(z)\,dz$ of the coefficients of this system has the structure (14) in the neighbourhood of a_1:

$$\omega' = -\frac{K^0}{z - a_1}\,dz + (z - a_1)^{-K^0}\omega(z - a_1)^{K^0},$$

where $K^0 = \operatorname{diag}(k_1^0, \dots, k_p^0)$ and the form ω has a simple pole at a_1. Hence the order of the pole of the matrix elements of the form ω' at this point is at most $k_1^0 - k_p^0 + 1$ and therefore $r_1 \leqslant k_1^0 - k_p^0 \leqslant (n-1)(p-1)$. The proof of Corollary 1 is complete.

We point out that for the dimension $p = 3$ (the lowest dimension in which there exists a counterexample in the Riemann–Hilbert problem) one has a better estimate of the Poincaré rank r_1 of the regular singularity of the system constructed from the representation (2):

$$r_1 \leqslant \left[\frac{n}{2}\right] - 1,$$

where $[x]$ is the integer part of the quantity x ([2], Corollary 2.3.3). Moreover, for $p = 3$ there exist representations for which one cannot construct a similar system of lower Poincaré rank r_1 (see [2], Proposition 2.2.4, the proof of Theorem 2.3.3, and Corollary 2.3.2), so that the non-sharp estimate of Corollary 1 is not completely pointless (there exist representations (2) from which one cannot construct a system (3) with regular singularities and of low Poincaré rank at the unique non-Fuchsian singular point).

§ 4. Additional singularities of a Fuchsian equation

Consider the family $\mathscr{F}$ of holomorphic vector bundles F^Λ with logarithmic connections ∇^Λ constructed from the representation (2). By the *Fuchsian weight* of the bundle $F^\Lambda \in \mathscr{F}$ we mean the quantity

$$\gamma(F^\Lambda) = \sum_{i=1}^{p}(k_1 - k_i),$$

where $(k_1, \dots, k_p)$ is the splitting type of the bundle F^Λ.

The function $\gamma\colon \mathscr{F} \to \mathbb{N} \cup \{0\}$ is bounded if and only if the representation (2) is irreducible ([2], Theorem 4.2.1). Moreover, for the splitting type of an arbitrary bundle $F^\Lambda \in \mathscr{F}$ constructed from an irreducible representation one has the inequalities

$$k_i - k_{i+1} \leqslant n - 2, \qquad i = 1, \dots, p-1,$$

and for such a representation one can define the quantity

$$\gamma_{\max}(\chi) = \max_{F^\Lambda \in \mathscr{F}} \gamma(F^\Lambda) \leqslant \frac{(n-2)p(p-1)}{2}\,,$$

which is called the *maximum Fuchsian weight* of the irreducible representation χ.

The smallest possible number m_0 of extra apparent singular points arising in the construction of Fuchsian equation (1) from the irreducible representation (2) can be expressed by the following formula ([2], Theorem 4.4.1):

$$m_0 = \frac{(n-2)p(p-1)}{2} - \gamma_{\max}(\chi). \tag{17}$$

This question was earlier considered also in [7], where one can find an upper estimate of the quantity m_0 in the case when the monodromy representation is irreducible and one of the monodromy matrices G_i is diagonalizable (in that paper the authors considered equations on a compact Riemann surface of arbitrary genus).

Proof of Theorem 2. It follows from Plemelj's theorem (see also [1]) that each representation (2) can be realized by a Fuchsian system with an extra apparent singularity. We now discuss this in greater detail.

Consider a holomorphic vector bundle F^Λ with logarithmic connection ∇^Λ constructed from the representation χ^* obtained from (2) by the addition of an extra singular point a_{n+1} with identity monodromy matrix.

Corresponding to the connection ∇^Λ is a system (3) with Fuchsian singularities $a_1, \dots, a_n$ and regular singularity a_{n+1} that has the prescribed monodromy χ^*. In the neighbourhood of $z = a_{n+1}$ the fundamental matrix $Y(z)$ of this system has the form (12):

$$Y(z) = (z - a_{n+1})^{-K} V(z) (z - a_{n+1})^{\Lambda_{n+1}},$$

where K is an integer diagonal matrix and $V(z)$ a holomorphically invertible matrix in this neighbourhood ($E_{n+1} = 0$ because $G_{n+1} = I$).

By Bolibrukh's rearrangement lemma ([2], Lemma 4.1.3) there exists a matrix $\Gamma(z)$ holomorphically invertible outside the point a_{n+1} and a matrix $U(z)$ holomorphically invertible in a neighbourhood of a_{n+1} such that

$$\Gamma(z)(z-a_{n+1})^{-K}V(z)=U(z)(z-a_{n+1})^{\widetilde{K}},$$

where $\widetilde{K}$ is an integer diagonal matrix with entries that are a rearrangement of the diagonal entries of the matrix $-K$. Then the transformation $y'=\Gamma(z)y$ takes the above system to another system for which $a_1,\dots,a_n$ remain Fuchsian singularities (the matrix $\Gamma(z)$ is holomorphically invertible in the neighbourhood of these points), while in the neighbourhood of $z=a_{n+1}$ its fundamental matrix $Y'(z)=\Gamma(z)Y(z)$ has the following form:

$$Y'(z)=U(z)(z-a_{n+1})^{\widetilde{K}}(z-a_{n+1})^{\Lambda_{n+1}}=U(z)(z-a_{n+1})^{\widetilde{K}+\Lambda_{n+1}}.$$

Thus, the point a_{n+1} is also Fuchsian for the transformed system because the coefficient matrix

$$B'(z)=\frac{dY'}{dz}Y'^{-1}=\frac{dU}{dz}U^{-1}+U\frac{\widetilde{K}+\Lambda_{n+1}}{z-a_{n+1}}U^{-1}$$

has a simple pole at this point. Furthermore, all the solutions of the so-constructed system are single-valued meromorphic functions in the neighbourhood of $z=a_{n+1}$ and the sum R of the Poincaré ranks of the singular points is equal to zero.

Using Lemma 2 we now construct a Fuchsian differential equation with monodromy χ^* such that the number m of apparent singular points satisfies the inequality

$$m\leqslant\frac{(n+1)p(p-1)}{2}.$$

Bearing in mind that $z=a_{n+1}$ is also an apparent singularity of the so-constructed equation with respect to originally prescribed singular points $a_1,\dots,a_n$ we obtain the required estimate. (It is assumed in the definition of an apparent singularity that solutions of the equation are single-valued holomorphic in its neighbourhood, and we only know so far that they are meromorphic. However, after the transformation $y'=(z-a_{n+1})^N y$ of the unknown function $y(z)$, where N is the highest order of the pole at a_{n+1} of the solutions of the constructed equation, we obtain a Fuchsian equation with the same singularities and monodromy that now has holomorphic solutions in the neighbourhood of $z=a_{n+1}$.) The proof of Theorem 2 is complete.

In the case of a representation (2) for which the Riemann–Hilbert problem has an affirmative solution, the estimate of above-proved Theorem 2 can be refined in a natural way to $m\leqslant np(p-1)/2$ (because the representation is realized by a Fuchsian system with prescribed singularities $a_1,\dots,a_n$). In particular, one can obtain this estimate for an irreducible representation χ, which is weaker than (17). Hence one cannot earnestly call Theorem 2 a generalization of relation (17) to the case of an arbitrary representation; it is rather a supplement to this relation.

§ 5. Meromorphic reduction of a linear system

The problem of the transformation of a system of linear differential equations in the neighbourhood of an *irregular* (that is, not a regular) singularity to a Birkhoff standard form reads as follows (the singular point is normally put at infinity).

Consider a system

$$z\,\frac{dy}{dz} = C(z)y, \qquad C(z) = \sum_{n=-\infty}^{r} C_n z^n, \tag{18}$$

of p linear differential equations in a neighbourhood $O_\infty = \{z \in \overline{\mathbb{C}} : |z| > R\}$ of an irregular singularity ∞ of Poincaré rank r ($C_r \neq 0$).

A linear transformation

$$\widetilde{y} = \Gamma(z)y \tag{19}$$

takes (18) to the system

$$z\,\frac{d\widetilde{y}}{dz} = \widetilde{C}(z)\widetilde{y}, \qquad \widetilde{C}(z) = z\,\frac{d\Gamma}{dz}\,\Gamma^{-1} + \Gamma C(z)\Gamma^{-1}. \tag{20}$$

One chooses (19) to be either *analytic* (with $\Gamma(z)$ holomorphically invertible in O_∞), so that one speaks about the *analytic equivalence* of the systems (18) and (20), or *meromorphic* (the matrix $\Gamma(z)$ is meromorphically invertible in O_∞), when one speaks about the *meromorphic equivalence* of these systems. An analytic transformation does not change the Poincaré rank of the original system, whereas a meromorphic one can increase or decrease the Poincaré rank.

If the matrix $\widetilde{C}(z)$ of the transformed system (20) has the polynomial form

$$\widetilde{C}(z) = \widetilde{C}_{r'} z^{r'} + \dots + \widetilde{C}_0, \qquad \widetilde{C}_{r'} \neq 0, \tag{21}$$

and $r' \leqslant r$, then one says that the system (20), (21) is a *Birkhoff standard form* of the original system (18).

An analytic transformation of a linear system to a Birkhoff standard form is not always possible: a counterexample was discovered by Gantmakher (see [8]).

We say that a system (18) is *reducible* if it can be reduced by a transformation (19) to a system (20) with block upper-triangular matrix of coefficients $\widetilde{C}(z)$:

$$\widetilde{C}(z) = \begin{pmatrix} C' & * \\ 0 & C'' \end{pmatrix} \tag{22}$$

(irrespective of whether the transformation is analytic or meromorphic: one can show that if a system (18) is reduced to the form (22) by a meromorphic transformation, then it can also be reduced to a similar block upper-triangular form by a holomorphic transformation). Otherwise we say that the system (18) is *irreducible*.

One sufficient condition for the reduction of a linear system to a Birkhoff standard form by an analytic transformation is due to Bolibrukh: *if a system* (18) *is irreducible*, *then it can be analytically transformed into a Birkhoff standard form* (see [2], [1], [3]).

The question of the existence of a meromorphic transformation of a linear system to the Birkhoff standard form is not yet resolved. As is known, this question has an affirmative answer in dimensions $p = 2$ and $p = 3$; one also knows various conditions for an affirmative solution in an arbitrary dimension p (one can learn details from Balser's survey [9]).

Using meromorphic transformations (19) one can always reduce the system (18) to the polynomial form (20), (21) (of higher Poincaré rank r' though). In this section we obtain an estimate for r'. We shall consider only reducible systems (since one can always transform analytically an irreducible system to a Birkhoff standard form).

We can assume that the matrix $C(z)$ of the coefficients of the system (18) has a block upper-triangular form:

$$C(z) = \begin{pmatrix} C^1(z) & * & * \\ 0 & \ddots & * \\ 0 & 0 & C^m(z) \end{pmatrix}, \tag{23}$$

where $C^1, \dots, C^m$ are irreducible blocks of sizes $p_1, \dots, p_m$, respectively, $2 \leqslant m \leqslant p$. By [10] this system has a *formal fundamental matrix* $\widehat{Y}(z)$ of the following form:

$$\widehat{Y}(z) = \widehat{F}(z) z^L e^{Q(z)}, \tag{24}$$

where $\widehat{F}(z)$ is a formal (matrix) Laurent series (in $1/z$) of block upper-triangular form (23) with finite principal part such that $\det \widehat{F}(z)$ is distinct from the zero series; L is a constant block upper-triangular matrix of the form (23) such that the real parts of the eigenvalues of L lie in the half-open interval $[0, 1)$; $Q(z)$ is a diagonal matrix with polynomials of $z^{1/s}$ on the diagonal (for some positive integer s) of degree at most r (with respect to z).

The formal substitution of the matrix $\widehat{Y}(z)$ in the system (18) makes it a correct identity (although the series $\widehat{F}(z)$ can have an empty convergence annulus).

For a further discussion we require the following technical result.

Lemma 3. *One can transform the system* (18), (23) *by means of a meromorphic transformation* (19) *into a system* (20) *of a similar block upper-triangular form of Poincaré rank at most r and with formal fundamental matrix* (24), *where the formal Laurent series $\widehat{F}(z)$ is replaced by*

$$\widehat{W}(z) = \sum_{n=0}^{\infty} W_n z^{-n},$$

which is an invertible (*in the sense that* $\det W_0 \neq 0$) *formal Taylor series in* $1/z$.

Proof. By an analogue of Sauvage's lemma (see [11]) for formal series, one obtains

$$U(z)\widehat{F}(z) = z^D \widehat{W}(z),$$

where $U(z)$ is an upper-triangular (matrix) polynomial of z and $1/z$ ($\det U(z) \equiv 1$), D is a diagonal integer matrix, $\widehat{W}(z)$ is a formal invertible (matrix) Taylor series in $1/z$ (of the block upper-triangular form (23) because $\widehat{W}(z) = z^{-D} U(z) \widehat{F}(z)$).

The required meromorphic transformation is defined by the upper-triangular matrix $\Gamma(z) = z^{-D}U(z)$.

The Poincaré rank of the transformed system is equal to the Poincaré rank of the system with fundamental matrix $Y(z) = z^L e^{Q(z)}$ (because the formal analytic transformation $\widetilde{y} = \widehat{W}^{-1}(z)y$ does not change the Poincaré rank) and the coefficient matrix

$$C'(z) = z\frac{dY}{dz}Y^{-1} = z\frac{d}{dz}(z^L e^{Q(z)})e^{-Q(z)}z^{-L} = L + z^L\biggl(z\frac{dQ}{dz}\biggr)z^{-L}.$$

The (generally speaking, fractional) degree of the polynomial $z\,dQ/dz$ is at most r, and the real parts of the eigenvalues of L lie in the half-open interval $[0,1)$, therefore the leading power in the expansion of the matrix $C'(z)$ is strictly less than $r+1$. However, $C'(z)$ contains only integer powers of z, so that this degree is at most r. The proof of Lemma 3 is complete.

One can study the problem of the reduction of a linear system to the Birkhoff standard form, similarly to the Riemann–Hilbert problem, by means of the theory of holomorphic bundles with connections (see [1], [3]). We shall use the main methods of these papers for the proof of the following result.

Theorem 3. *Using a meromorphic transformation* (19) *one can transform the system* (18), (23) *into a system* (20) *with matrix of coefficients* $\widetilde{C}(z)$ *having the polynomial form* (21), *where*

$$r' \leqslant 1 + r\max_{1\leqslant j\leqslant m} p_j.$$

Proof. Consider the fundamental matrix $Y(z)$ of the system (18) having the same block upper-triangular form as the matrix (23) of the coefficients of the system. Then

$$Y(z) = T(z)z^E,$$

where

$$T(z) = \begin{pmatrix} T^1(z) & * & * \\ 0 & \ddots & * \\ 0 & 0 & T^m(z) \end{pmatrix}, \qquad E = \frac{1}{2\pi i}\ln G = \begin{pmatrix} E^1 & * & * \\ 0 & \ddots & * \\ 0 & 0 & E^m \end{pmatrix}. \tag{25}$$

The matrix $T(z)$ is single-valued and holomorphically invertible in $O_\infty \setminus \{\infty\}$, G is the monodromy matrix of the system (18) (with respect to the basis of the columns of $Y(z)$), and the eigenvalues ρ^j of the matrix E satisfy the condition

$$0 \leqslant \operatorname{Re}\rho^j < 1, \qquad j = 1,\dots,p.$$

From the system (18) we construct over the Riemann sphere a holomorphic vector bundle F with coordinate description $F = (O_\infty,\ O_0 = \mathbb{C},\ g_\infty 0 = T(z))$. The matrix-valued differential 1-forms

$$\omega_0 = \frac{E}{z}\,dz, \qquad \omega_\infty = \frac{C(z)}{z}\,dz,$$

defined in the neighbourhoods O_0 and O_∞, respectively, define a connection ∇ in the bundle F because in the intersection $O_0 \cap O_\infty = O_\infty \setminus \{\infty\}$ they satisfy gluing condition (4):

$$\omega_\infty = \frac{C(z)}{z}\,dz = (dY)Y^{-1} = (dT)T^{-1} + T\biggl(\frac{E}{z}\,dz\biggr)T^{-1} = (dg_{\infty 0})g_{\infty 0}^{-1} + g_{\infty 0}\omega_0 g_{\infty 0}^{-1}.$$

It follows from the form (25) of the cocycle $g_{\infty 0}(z) = T(z)$ that the bundle F has a family of subbundles $0 = F^0 \subset F^1 \subset \dots \subset F^m = F$,

$$F^j/F^{j-1} = \bigl(O_\infty,\ O_0 = \mathbb{C},\ g^j_{\infty 0} = T^j(z)\bigr), \qquad j = 1,\dots,m.$$

The matrix-valued differential 1-forms $\omega^j_0 = (E^j/z)\,dz$ and $\omega^j_\infty = (C^j(z)/z)\,dz$, which are defined in O_0 and O_∞, respectively, define a connection ∇^j in the quotient bundle F^j/F^{j-1} constructed from the irreducible system $z\,dy/dz = C^j(z)y$ of size p_j. For the splitting type $(k^j_1,\dots,k^j_{p_j})$ of this bundle we have the inequalities

$$0 \leqslant k^j_i - k^j_{i+1} \leqslant r, \qquad i = 1,\dots,p_j - 1 \tag{26}$$

([2], Proposition 4.5.1, see also [1], [3]).

In view of Lemma 3, we can assume that the formal fundamental matrix $\widehat{Y}(z)$ of the system (18), (23) has the form

$$\widehat{Y}(z) = \widehat{W}(z) z^L e^{Q(z)},$$

where $\widehat{W}(z)$ is an invertible (matrix-valued) formal Taylor series (in $1/z$) of the block upper-triangular form (23), and the matrices L and $Q(z) = \operatorname{diag}(Q^1(z),\dots, Q^m(z))$ are the same as in (24). Hence the formal fundamental matrices $\widehat{Y}^j(z)$ of the systems $z\,dy/dz = C^j(z)y$ have the following representations:

$$\widehat{Y}^j(z) = \widehat{W}^j(z) z^{L^j} e^{Q^j(z)}, \qquad j = 1,\dots,m$$

(we denote by $\widehat{W}^j(z)$ and L^j the diagonal blocks of the matrices $\widehat{W}(z)$ and L, respectively).

It follows now by Liouville's formula $d\ln\det\widehat{Y}^j(z) = \operatorname{tr}\omega^j_\infty$ that

$$\operatorname{res}_\infty \operatorname{tr}\omega^j_\infty = \operatorname{res}_\infty\biggl(\frac{d\det\widehat{W}^j}{\det\widehat{W}^j} + \frac{\operatorname{tr}L^j}{z}\,dz + d\operatorname{tr}Q^j\biggr) = -\operatorname{tr}L^j.$$

Hence

$$\sum_{i=1}^{p_j} k^j_i = \deg F^j/F^{j-1} = \operatorname{res}_0 \operatorname{tr}\omega^j_0 + \operatorname{res}_\infty \operatorname{tr}\omega^j_\infty = \operatorname{tr}E^j - \operatorname{tr}L^j,$$

and since the real parts of the eigenvalues of the matrices E^j and L^j lie in $[0,1)$, it follows that

$$-p_j < \sum_{i=1}^{p_j} k^j_i < p_j, \qquad j = 1,\dots,m. \tag{27}$$

Inequalities (26) and (27) yield

$$-\frac{p_j-1}{2}r-1<k^j_{p_j}\leqslant\cdots\leqslant k^j_1<\frac{p_j-1}{2}r+1,\qquad j=1,\dots,m.$$

We have thus obtained for the coefficients k^j_i of the splitting type of each bundle F^j/F^{j-1} the common estimate

$$|k^j_i|<1+\frac{r}{2}\bigl(\max_{1\leqslant j\leqslant m}p_j-1\bigr). \tag{28}$$

By the definition of equivalent vector bundles, for the cocycle $g^j_{\infty 0}(z)$ of the bundle F^j/F^{j-1} we have the matrix relation

$$H^j_\infty(z)g^j_{\infty 0}(z)=z^{-K_j}H^j_0(z),\qquad j=1,\dots,m,$$

where $K_j=\operatorname{diag}(k^j_1,\dots,k^j_{p_j})$ and H^j_∞ and H^j_0 are matrices holomorphically invertible in O_∞ and O_0, respectively. Denoting by $H_\infty(z)$ the block diagonal matrix $H_\infty=\operatorname{diag}(H^1_\infty,\dots,H^m_\infty)$ we obtain

$$H_\infty g_{\infty 0}=\begin{pmatrix} z^{-K_1}H^1_0 & * & * \\ 0 & \ddots & * \\ 0 & 0 & z^{-K_m}H^m_0\end{pmatrix}=z^{-K}g'_{\infty 0}, \tag{29}$$

where $K=\operatorname{diag}(K_1,\dots,K_m)$, $g'_{\infty 0}(z)$ is a cocycle holomorphically invertible in $O_\infty\setminus\{\infty\}$ and equivalent to the identity cocycle, which means that one can transform the matrix $g'_{\infty 0}$ into the identity matrix by multiplication by a holomorphically invertible matrix in O_∞ on the left and by a holomorphically invertible matrix in O_0 on the right. We can demonstrate this as follows.

Let $H_0(z)$ be the block diagonal matrix

$$H_0=\operatorname{diag}(H^1_0,\dots,H^m_0).$$

Then

$$g'_{\infty 0}H_0^{-1}=\begin{pmatrix} 1 & * & * \\ 0 & \ddots & * \\ 0 & 0 & 1\end{pmatrix}.$$

We now explain by the example of a matrix of size $p=2$ that the upper-triangular cocycle

$$\widetilde{g}_{\infty 0}(z)=\begin{pmatrix} 1 & a \\ 0 & 1\end{pmatrix}$$

is equivalent to the identity cocycle. A holomorphic function $a(z)$ in $O_\infty\cap O_0$ can be represented as a sum $a(z)=a^-(z)+a^+(z)$ of a holomorphic function $a^-(z)$ in O_∞ and a holomorphic function $a^+(z)$ in O_0. Hence

$$\begin{pmatrix} 1 & -a^- \\ 0 & 1\end{pmatrix}\begin{pmatrix} 1 & a \\ 0 & 1\end{pmatrix}\begin{pmatrix} 1 & -a^+ \\ 0 & 1\end{pmatrix}=\begin{pmatrix} 1 & a^+ \\ 0 & 1\end{pmatrix}\begin{pmatrix} 1 & -a^+ \\ 0 & 1\end{pmatrix}=\begin{pmatrix} 1 & 0 \\ 0 & 1\end{pmatrix}.$$

As regards the case of an upper-triangular cocycle $\widetilde{g}_{\infty 0}(z)$ of arbitrary size p with ones on the main diagonal, one acts in a similar fashion, multiplying on the left by upper-triangular matrices with ones on the main diagonal that are holomorphic in O_∞, multiplying on the right by upper-triangular matrices with ones on the diagonal holomorphic in O_0, and transforming the columns of $\widetilde{g}_{\infty 0}$ one after another into the corresponding columns of the identity matrix.

Thus, we obtain for the cocycle $g'_{\infty 0}(z)$ the factorization

$$g'_{\infty 0}(z) = G_\infty(z) G_0(z),$$

where G_∞ and G_0 are block upper-triangular matrices (23) holomorphically invertible in O_∞ and O_0, respectively. Hence by (29),

$$G_\infty^{-1} z^K H_\infty g_{\infty 0} = G_0.$$

This matrix relation means that there exists a linear system

$$z \frac{dy}{dz} = C'(z) y, \tag{30}$$

on the entire Riemann sphere with just two singular points, 0 and ∞, that is analytically equivalent in $O_0 = \mathbb{C}$ to the system $dy = \omega_0 y$ (so that 0 is a Fuchsian singularity of (30), and therefore the matrix $C'(z)$ is holomorphic in $\mathbb{C}$) and meromorphically equivalent in O_∞ to the system $dy = \omega_\infty y$, that is, to the original system (18). The meromorphic equivalence in O_∞ is defined by means of a linear transformation with matrix $\Gamma(z) = G_\infty^{-1} z^K H_\infty$, which is therefore a composite of three transformations, the first and the third of which do not change the Poincaré rank at ∞. Thus, the Poincaré rank r' of the system (30) at ∞ is equal to the Poincaré rank of the system obtained from (18) by the linear transformation with matrix $\Gamma_2(z) = z^K$. The matrix $C''(z)$ of coefficients of such a system has the form

$$C''(z) = K + z^K C(z) z^{-K},$$

therefore $r' = r + \max_{i,j} |k_i - k_j|$ (where the k_i are diagonal entries of the matrix K). By the estimate (28),

$$|k_i - k_j| \leqslant 1 + r\bigl(\max_{1 \leqslant j \leqslant m} p_j - 1\bigr),$$

therefore $C'(z)$ is a polynomial of degree

$$r' \leqslant 1 + r \max_{1 \leqslant j \leqslant m} p_j.$$

The proof of Theorem 3 is complete.

Corollary 2 (Bruno [12]). *The system* (18) *with upper-triangular matrix of coefficients can be reduced by a meromorphic transformation* (19) *to a Birkhoff standard form, which is also upper triangular.*

Proof. Since the matrix $C(z)$ of the coefficients of (18) is upper triangular, it follows that $\max_{1 \leqslant j \leqslant m} p_j = 1$ and it follows by the estimate (28) that K (from the previous theorem) is the zero matrix. Thus, one reduces the upper-triangular system (18) to the polynomial form (21) by means of a meromorphic transformation (19) with upper-triangular matrix $\Gamma(z) = G_\infty^{-1} H_\infty \Gamma_1$ not increasing the Poincaré rank (Γ_1 is the upper-triangular matrix from Lemma 3, H_∞ is a diagonal matrix, and G_∞ is the upper-triangular matrix from the previous theorem). The proof of Corollary 2 is complete.

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Received 27/SEP/05
Translated by N. KRUZHILIN

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