Coding-invariant Classification of Infinite Sequences

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In this talk we discuss some coding-invariant classification of infinite sequences [1].

An infinite binary sequence α is reducible to an infinite binary sequence β if $\alpha = F(\beta)$ for some computable operation F (denote this $\alpha \prec \beta$). Two sequences α and β are information equivalent (denote this $\alpha \equiv \beta$), if $\alpha \prec \beta$ and $\beta \prec \alpha$. Informally speaking, the sequences $\alpha \equiv \beta$ have the same information content (up to finite descriptions of the coding methods).

Let Ω be the set of all infinite binary sequences. A set $A \subseteq \Omega$ is called *coding invariant* if $\omega \in A$ and $\alpha \equiv \omega$ implies $\alpha \in A$ for any ω and α.

We use the concepts of semicomputable semimeasure (first introduced in [2], see also [1]) and the *a priory* semimeasure M. For any such semimeasure P the maximal measure $\bar{P}$ can be defined such that $\bar{P} \leq P$.

Let I be the Boolean algebra of all coding invariant Borel subsets of Ω, and let Υ be the factor-algebra of I with respect to the equivalence relation

$$A \sim B \Longleftrightarrow \bar{M}((A \setminus B) \cup (B \setminus A)) = 0,$$

where $A, B \in I$.

Let $\mathbf{a} = [A]$ be an element of Υ defined by a coding invariant set $A \in I$. We also define $P(\mathbf{a}) = \bar{P}(A)$ for any semicomputable semimeasure P, where $A \in \mathbf{a}$.

By ([2], Theorem 3.1) any sequence Martin-Löf random with respect to a computable measure is computable or information equivalent to a sequence random with respect to the uniform measure. Therefore, we can define two natural elements $\mathbf{r} = [\bar{R}]$ and $\mathbf{c} = [C]$ of Υ, where R be a set of all Martin-Löf sequences random with respect to the uniform measure, C be a set of all computable sequences. Evidently $\bar{M}(\mathbf{r}) > 0$ and $\bar{M}(\mathbf{c}) > 0$ and $P(\mathbf{r}) = 0$ for each computable measure P.

The zero element $\mathbf{0}$ of Υ consists of all coding invariant subsets of Ω of $\bar{M}$-measure 0. Then $\bar{M}(\mathbf{0}) = 0$. The unit element of Υ is defined $\mathbf{1} = [\Omega]$.

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An element $\mathbf{d}$ of Υ is called *atom element* if $\mathbf{d} \neq \mathbf{0}$ and it cannot be represented as a disjoint union $\mathbf{d} = \mathbf{a} \cup \mathbf{b}$ of two non-zero elements of Υ, i.e., such that $\mathbf{a} \cap \mathbf{b} = \emptyset$, $\mathbf{a} \neq \mathbf{0}$ and $\mathbf{b} \neq \mathbf{0}$. This can be interpreted as that any set of sequences generating an atom cannot be divided on two nontrivial parts by their information content.

Theorem 1 *(Levin [1]). The elements* $\mathbf{r}$ *and* $\mathbf{c}$ *are atom elements of* Υ.

The information-theoretic structure of the algebra Υ is described by the following theorem.

Theorem 2 *Let* $\mathbf{a}_1, \mathbf{a}_2, \mathbf{a}_3, \dots$ *be all atom elements of* Υ. *The following decomposition of the unit element of the algebra* Υ *is valid*

$$\mathbf{1} = \cup_{i=1}^{\infty} \mathbf{a}_i \cup \mathbf{d},$$

where

- *the atom* $\mathbf{a_1} = \mathbf{r}$ *is generated by all Martin-Löf random sequences;*
- *the atom* $\mathbf{a}_2 = \mathbf{c}$ *is generated by all computable sequences;*
- *the atoms* $\mathbf{a}_3, \mathbf{a}_4, \dots$ *are generated by infinite sequences which cannot be Martin-Löf random with respect to any computable measure, and even, they cannot be information equivalent to Martin-Löf random sequences;*
- $\mathbf{d}$ *is the non-zero infinitely divisible element of* Υ *generated by the measure-theoretic complement of all atoms.*

By definition the element $\mathbf{d} = \mathbf{1} \setminus \bigcup_{i=1}^{\infty} \mathbf{a}_i$ is infinitely divisible, i.e., for any non-zero $\mathbf{x} \subseteq \mathbf{d}$ a decomposition $\mathbf{x} = \mathbf{x}_1 \cup \mathbf{x}_2$ is valid, where $\mathbf{x}_1 \cap \mathbf{x}_2 = \mathbf{0}$, $\mathbf{x}_1 \neq \mathbf{0}$ and $\mathbf{x}_2 \neq \mathbf{0}$.

References

[1] Levin L.A., V'yugin V.V. Invariant properties of informational bulks, Springer Lecture Notes on Computer Science, 1977, v.53, p.359-364.

[2] A.K. Zvonkin and L.A. Levin. The complexity of finite objects and the algorithmic concepts of information and randomness, Russ. Math. Surv. 1970, 25, 83-124.