On Finiteness of Trajectories for One Mapping Associated with Quasi-Inversion of Rotation Mapping on Integer Planar Lattice

V. Kozyakin
Institute of Information Transmission Problems
Russian Academy of Sciences
Bolshoi Karetny lane 19, Moscow 101447, Russia
e-mail: kozyakin@ippi.ac.msk.su

Keywords: Spatial discretizations, Rotation mapping, Finiteness of trajectories

ABSTRACT

Investigation of spatial discretizations, even in simplest cases, is collided with essential difficulties and requires to develop rather non-standard methods, as indicated in [1, 2, 3, 4, 5]. Typical in this respect is the problem of analysis of discretization of the planar rotation mapping which demonstrates that often the behaviour of initial continuous mapping may differ dramatically from that of its discrete counterpart. In the paper it is shown that, while the analysis of discrete analogs of the planar rotation mapping is quite difficult, the pair "discretization of an original mapping + discretization of the inverse mapping" may have strongly correlated properties.

INTRODUCTION

As was noted by many authors [1, 2], the research of spatial discretizations even of simplest mappings is conjugated with significant theoretical difficulties, and often behaviour of initial continuous mappings differs dramatically from that of their discrete analogs. The task to research the phase portrait for discretization of the planar linear rotation mapping is typical in this respect. Some approaches to investigation of originating problems, requiring developments of rather non-standard methods, are indicated in works [3, 4, 5]. However complete clearness concerning connection of properties of discretized mappings and their continuous counterparts is not until now present.

The present work continues investigation of properties of discretizations of the planar rotation mapping. It is shown, in particular that, though an analysis of discrete analogs of the planar rotation mapping faces serious difficulties, the pair of mappings "discretization T^z_θ of the rotation mapping T_θ on an angle θ" plus "discretization $T^z_{-\theta}$ of the inverse mapping $T_{-\theta}$" has a strongly correlated behaviour, that can be expressed in a rather simple dynamics of the mapping $T^z_{-\theta} \circ T^z_\theta$.

Structure of work is as follows. In Section 1 the problem is stated and the main Theorem 1 about a finiteness of trajectories of the mapping $T^z_{-\theta} \circ T^z_\theta$ is formulated. In Section 2 the approach reducing research of the mapping $T^z_{-\theta} \circ T^z_\theta$ on the lattice $\mathbb{Z}^2$ to some auxiliary piecewise continuous mapping J_θ on a square with geometrically clear properties is described. It is necessary to note, that the initial research of properties of mapping $T^z_{-\theta} \circ T^z_\theta$ has been based on geometric constructions "distributed throughout the whole space $\mathbb{R}^2$". The approach presented below essentially utilizes ideas of I. Vladimirov, indicated by him during discussion of the mentioned "distributed" constructions. In Section 3 the analysis of trajectories of the mapping J_θ is carried out. At last, in Section 4 the proof of the basic Theorem 1 is presented.

1 STATEMENT OF THE PROBLEM

Let $\mathbb{Z} = \{m : \ m = 0, \pm 1, \pm 2, \ldots\}$ be the lattice of points with integer coordinates in $\mathbb{R}^1$, and let $\mathbb{Z}^2 := \mathbb{Z} \times \mathbb{Z} = \{(m,n) : \ m,n = 0, \pm 1, \pm 2, \ldots\}$ be the lattice of points with integer coordinates in $\mathbb{R}^2$. Denote by $[\![x]\!]$ *the operator of rounding off up to nearest integer* on the straight line $\mathbb{R}^1$ defined in the usual manner: $[\![x]\!] = i$ where $i \in \mathbb{Z}$ is such that $x - \frac{1}{2} \le i < x + \frac{1}{2}$. Similarly, define coordinate-wise the operator of rounding off up to nearest integer on the plane $\mathbb{R}^2$:

$$[\![x]\!] = ([\![x_1]\!], [\![x_2]\!]) \in \mathbb{Z}^2, \qquad x = (x_1, x_2) \in \mathbb{R}^2.$$

We shall consider the mapping of a rotation of the plane $\mathbb{R}^2$ on an angle θ, $0 \le \theta \le \frac{\pi}{2}$, around the origin

$$x \mapsto T_\theta x, \qquad \text{where} \qquad T_\theta = \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}.$$

Associate with the mapping T_θ its discretization[1] on the lattice $\mathbb{Z}^2 \subset \mathbb{R}^2$ defined as

$$T_\theta^z x := [\![T_\theta x]\!], \qquad x \in \mathbb{Z}^2.$$

The problem on correspondence of properties of the mapping $T_\theta x$ and its discrete analog $T_\theta^z x$ is far from being simple. Thus properties of mapping

$$I_\theta(x) = [\![T_{-\theta}[\![T_\theta x]\!]]\!], \qquad x \in \mathbb{Z}^2$$

yield to a more detailed research. Taking into account that $I_\theta \equiv T_{-\theta}^z \circ T_\theta^z$, and that the mapping $T_{-\theta}^z$ is possible to treat as the natural applicant to be a "mapping converse to T_θ^z", then the properties of mapping I_θ are clearly important for understanding of a "degree of reversibility or irreversibility" of the mapping T_θ^z on $\mathbb{Z}^2$. The main result of the paper is then as follows.

Theorem 1 *If $\theta = \frac{k\pi}{2}$ then $I_\theta = I$. If $0 < \theta < \frac{\pi}{2}$ then for each trajectory $\{x(n)\}_{n=0}^\infty$ of the mapping J_θ a non-negative number*

$$N_x \le \left[\frac{\left|\cos\frac{\theta}{2} - \sin\frac{\theta}{2}\right|}{2\cos\frac{\theta}{2}\sin\theta}\right],$$

can be found such that $x(n) = x(N_x)$ for $n \ge N_x$. If, in addition, $N_x > 0$ then

$$x(1) - x(0) = x(2) - x(1) = \ldots = x(N_x) - x(N_x - 1). \tag{1}$$

Geometrically, relations (1) mean, that all the elements $\{x(n)\}_{n=0}^\infty$ lay on the intersection of some straight line, parallel to one of the coordinate axes, with the lattice $\mathbb{Z}^2$.

2 PASSAGE TO A MAPPING ON A SQUARE

In this Section a construction reducing the analysis the mappings I_θ on $\mathbb{Z}^2$ to analysis of an auxiliary piecewise continuous mapping defined on the square

$$\Pi = \left[-\frac{1}{2}, \frac{1}{2}\right) \times \left[-\frac{1}{2}, \frac{1}{2}\right) \subset \mathbb{R}^2$$

[1] In view of obvious "symmetry" of properties of mapping T_θ^z at a modification of an angle θ on magnitude multiple to $\frac{\pi}{2}$, we restrict ourselves by reviewing only the case $0 \le \theta \le \frac{\pi}{2}$.

is described.

Denote by $\{x\} := x - [\![x]\!]$ the operator of taking the fractional part of a vector; clearly, $\{x\} \in \Pi$ for any vector $x \in \mathbb{R}^2$. Consider also the set

$$\Pi_\theta = \left\{ x \in \Pi : \quad x = \{T_\theta z\}, \ z \in \mathbb{Z}^2 \right\} \subset \Pi,$$

associated with the mapping I_θ (or T_θ), and define on it the mapping

$$J_\theta(x) = \left\{ x - T_\theta [\![T_\theta^{-1} x]\!] \right\}, \qquad x \in \Pi.$$

Lemma 1 *The relations* $\left(J_\theta|_{\Pi_\theta} \circ S_\theta\right)(x) = \left(S_\theta \circ I_\theta\right)(x)$ *and* $[\![T_{-\theta} S_\theta(x)]\!] = x - I_\theta(x)$*, where* $S_\theta(x) := \{T_\theta x\}$*, are valid for any* $x \in \mathbb{Z}^2$*.*

According to Lemma 1 the mapping S_θ establishes the relation of semi-conjugacy between the mappings $J_\theta|_{\Pi_\theta}$ and I_θ. Conditions, under which the mapping S_θ is one-to-one are formulated in the following Lemma 2.

Lemma 2 *The mapping S_θ establishes a one-to-one correspondence between the lattice $\mathbb{Z}^2$ and the set Π_θ only in the case when at least one of the values $\cos\theta$ or $\sin\theta$ is irrational.*

From Lemma 1 it follows that the analysis of dynamics of the mapping I_θ on $\mathbb{Z}^2$ can be reduced to that of the mapping J_θ on its invariant set Π_θ. In turn, for the analysis of properties of the mapping J_θ on $\Pi_\theta \subset \Pi$ it is enough to study properties of this mapping on area Π, as will be done in the present Section.

Introduce auxiliary notations. Consider the set $T_\theta^{-1}\Pi$ (see Fig. 1); it consists of four non-overlapping triangles $\Pi_{(-1,0)}$, $\Pi_{(1,0)}$, $\Pi_{(0,-1)}$, $\Pi_{(0,1)}$, not belonging to square Π, and also of the octagon $\Pi \cap T_\theta^{-1}\Pi$, entirely lying in the square Π. Therefore the set $[\![T_\theta^{-1}\Pi]\!]$ consists of five elements of the lattice $\mathbb{Z}^2$, points with coordinates $(-1,0)$, $(1,0)$, $(0,-1)$, $(0,1)$ and $(0,0)$.

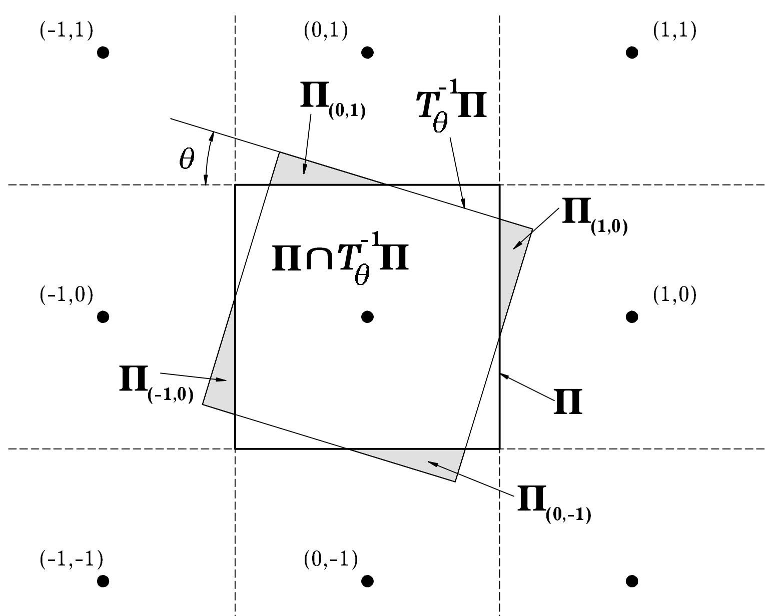

Figure 1: *Mutual disposition of the sets* Π, $\Pi \cap T_\theta^{-1}\Pi$ *and* $\Pi_{(i,j)}$.

The pre-images of points $(-1,0)$, $(1,0)$, $(0,-1)$, $(0,1)$ at mapping $x \mapsto [\![T_\theta^{-1}x]\!]$ are the triangles (see Fig. 2) $\Pi^{(nw)}, \Pi^{(ne)}, \Pi^{(sw)}, \Pi^{(se)} \subset \Pi$, and the pre-image of the point $(0,0)$ is the octagon

$$\Pi^{(c)} = \Pi \backslash \left\{\Pi^{(nw)} \cup \Pi^{(ne)} \cup \Pi^{(sw)} \cup \Pi^{(se)}\right\}.$$

Hereinafter some properties of defined above sets will be important for us.

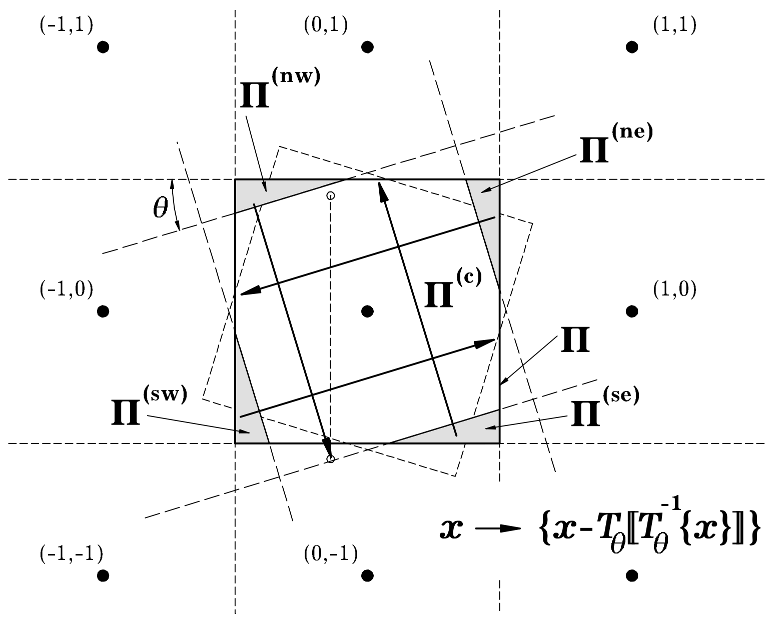

Figure 2: *Action of the mapping J_θ on the square* Π.

Lemma 3 *In each pair of the triangles* $\left\{\Pi_{(-1,0)}, \Pi_{(1,0)}\right\}$, $\left\{\Pi_{(0,-1)}, \Pi_{(0,1)}\right\}$, $\left\{\Pi^{(nw)}, \Pi^{(se)}\right\}$ *and* $\left\{\Pi^{(ne)}, \Pi^{(sw)}\right\}$ *(See Fig. 1 and 2) the hypotenuse of only one of the triangles in the pair belongs to the corresponding triangle.*

From definition of the sets $\Pi^{(nw)}$, $\Pi^{(ne)}$, $\Pi^{(sw)}$, $\Pi^{(se)}$ and $\Pi^{(c)}$ it is easy to see, that on each of these sets the mapping $[\![T_\theta^{-1}x]\!]$ (and the mapping $T_\theta[\![T_\theta^{-1}x]\!]$ together with it) takes a constant value, i.e. there can be found vectors $s^{(nw)}$, $s^{(ne)}$, $s^{(sw)}$ $s^{(se)}$ and $s^{()}$, such that

$$T_\theta[\![T_\theta^{-1}x]\!] = \begin{cases} s^{(nw)} & \text{for} \quad x \in \Pi^{(nw)}, \\ s^{(ne)} & \text{for} \quad x \in \Pi^{(ne)}, \\ s^{(sw)} & \text{for} \quad x \in \Pi^{(sw)}, \\ s^{(se)} & \text{for} \quad x \in \Pi^{(se)}, \\ s^{(c)} & \text{for} \quad x \in \Pi^{(c)}. \end{cases}$$

Remark also, that clearly $s^{(c)} = 0$. At the same time, Euclidean norms of the vectors $s^{(nw)}$, $s^{(ne)}$, $s^{(sw)}$, $s^{(se)}$ are equal to 1, and each of these vectors is orthogonal to the hypotenuse of the corresponding triangle $\Pi^{(nw)}$, $\Pi^{(ne)}$, $\Pi^{(sw)}$, $\Pi^{(se)}$ and is oriented in direction of the center of the square Π. The proof immediately follows from that fact, that the mapping $[\![T_\theta^{-1}x]\!]$ is constant on each of triangles $\Pi^{(nw)}$, $\Pi^{(ne)}$, $\Pi^{(sw)}$ and $\Pi^{(se)}$ and takes one of the values $(-1,0)$, $(1,0)$, $(0,-1)$, $(0,1)$, while the linear mapping T_θ is orthogonal, i.e., preserves angles and distance.

Thus, the following Lemma is proved.

Lemma 4 *The mapping J_θ can be represented on the square Π as $J_\theta(x) = \{x - s(x)\}$ where $s(x) \equiv \mathrm{const}$ on each of the sets $\Pi^{(nw)}$, $\Pi^{(ne)}$, $\Pi^{(sw)}$, $\Pi^{(se)}$ and $\Pi^{(c)}$. In addition*

(i) $s(x) \equiv 0$ for $x \in \Pi^{(c)}$;

(ii) $\|s(x)\| = 1$ for $x \in \Pi^{(nw)} \cup \Pi^{(ne)} \cup \Pi^{(sw)} \cup \Pi^{(se)}$;

(iii) on each of the triangles $\Pi^{(nw)}$, $\Pi^{(ne)}$, $\Pi^{(sw)}$, $\Pi^{(se)}$ the vector $s(x)$ is orthogonal to the diagonal of the corresponding triangle and translates a neighborhood of the right angle of the corresponding triangle inside of the square Π.

3 FINITENESS OF TRAJECTORIES OF THE MAPPING J_θ

Let now $\{x(n)\}$ be some trajectory of the mapping J_θ, i.e. the sequence of elements of the set Π defined by relations:

$$x(n+1) = J_\theta(x(n)), \qquad n = 0, 1, 2, \ldots, \quad x(0) \in \Pi.$$

Lemma 5 *For each trajectory $\{x(n)\}_{n=0}^{\infty}$ of the mapping J_θ a number N_x can be found such that all the elements $x(n)$ for $n < N_x$ belong to the same set $\Pi^{(nw)}$, $\Pi^{(ne)}$, $\Pi^{(sw)}$ or $\Pi^{(se)}$, and for $n \geq N_x$ the relations $x(n) \equiv x(N_x) \in \Pi^{(c)}$ hold. In addition, for numbers N_x the following universal estimate is valid:*

$$N_x \leq \left[\frac{\left|\cos\frac{\theta}{2} - \sin\frac{\theta}{2}\right|}{2\cos\frac{\theta}{2}\sin\theta} \right].$$

Proof. If $x(0) \in \Pi^{(c)}$ then, by Statement (i) of Lemma 4, $x(1) = J_\theta(x(0)) \in \Pi^{(c)}$, and similarly, $x(n) \in \Pi^{(c)}$ for all $n \geq 0$. Therefore we shall consider only the case, when $x(0) \in \Pi^{(c)} \backslash \Pi$. By definition of the set $\Pi^{(c)}$, the point $x(0)$ lays in one of the sets $\Pi^{(nw)}$, $\Pi^{(ne)}$, $\Pi^{(sw)}$ or $\Pi^{(se)}$. Let, for definiteness, $x(0) \in \Pi^{(nw)}$ (see Fig. 3).

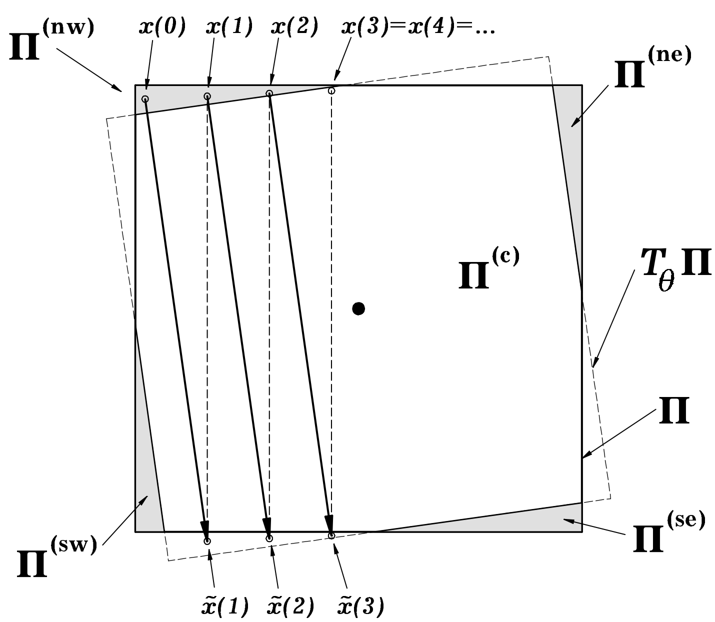

Figure 3: *Behavior of a trajectory of the mapping J_θ.*

Form an auxiliary sequence of vectors $\widetilde{x}(n) = x(n-1) - s(x(n-1))$ for $n \geq 1$. Then by Lemma 4 $x(n) = \{\widetilde{x}(n)\}$. As follows from Statements (ii) and (iii) of Lemma 4 and from Fig. 3

the element $\widetilde{x}(1)$ is obtained from $x(0)$ by shifting on a vector of unit length along one of sides of the square $T_\theta\Pi$ which is also of unit length. Since, in addition, by Lemma 3 the hypotenuse of only one of the triangles $\Pi^{(nw)}$ and $\Pi^{(se)}$ belongs to the corresponding triangle, then the element $\widetilde{x}(1)$ can either hit in the square Π (and to belong thus to the set $\Pi^{()}$) or not hit in this square. In the first case, obviously, $x(1)=\widetilde{x}(1)$, and then $x(1)=x(2)=\ldots\in\Pi^{()}$.

Consider the case when $\widetilde{x}(1)\notin\Pi^{()}$. Here either $x(1)\in\Pi^{()}$ and then, as in the previous case we obtain relations $x(1)=x(2)=\ldots\in\Pi^{()}$, or $x(1)=\{\widetilde{x}(1)\}\notin\Pi^{()}$. In the latter case the only possible situation is when $x(1)=\{\widetilde{x}(1)\}\in\Pi^{(nw)}$.

By repeating above reasonings we obtain that either such a number N_x can be found for which all the elements $x(n)$, $n<N_x$, belong to the set $\Pi^{(nw)}$ and for $n\geq N_x$ the relations $x(n)\equiv x(N_x)\in\Pi^{(c)}$ are valid, or $x(n)\in\Pi^{(nw)}$ for all values of n. We shall show that the second situation is impossible.

As is seen from Fig. 3, $|x_n(0)-x_{n+1}(0)|=\sin\theta$ and the length of the horizontal leg of the triangle $\Pi^{(nw)}$ is equal to

$$\left|\frac{1}{2}-\left(\frac{1}{2\sin\theta}-\frac{1}{2\tan\theta}\right)\right|.$$

So, not more than

$$\left[\frac{\left|\frac{1}{2}-\left(\frac{1}{2\sin\theta}-\frac{1}{2\tan\theta}\right)\right|}{\sin\theta}\right]=\left[\frac{\left|\cos\frac{\theta}{2}-\sin\frac{\theta}{2}\right|}{2\cos\frac{\theta}{2}\sin\theta}\right]$$

pairs of points $x(n),x(n+1)$ can be contained in the set $\Pi^{(nw)}$ whence the estimate required in Lemma for the number N_x follows. Lemma is proved. □

4 PROOF OF THEOREM 1

The validity of statement of Theorem for $\theta=\frac{k\pi}{2}$ is obvious; therefore we shall consider only the case when $0<\theta<\frac{\pi}{2}$.

Associate with the trajectory $\{x(n)\}$ the sequence of vectors $u(n)=S_\theta(x(n))$, $n=0,1,\ldots$. Then from relations $x(n+1)=I_\theta(x(n))$ valid for $n\geq 0$ we obtain by Lemma 1 that

$$J_\theta(u(n))=(J_\theta\circ S_\theta)(x(n))=((S_\theta\circ I_\theta)(x(n))=S_\theta(x(n+1))=u(n+1).$$

Therefore, the sequence $\{u(n)\}_{n=0}^{\infty}$ is a trajectory of the mapping J_θ, and by Lemma 5 a number N_u can be found such that all the elements $u(n)$, $n<N_u$, belong to the same set $\Pi^{(nw)}$, $\Pi^{(ne)}$, $\Pi^{(sw)}$ or $\Pi^{(se)}$, and for $n\geq N_u$ relations $u(n)\equiv u(N_u)\in\Pi^{(c)}$ hold. Then by Lemma 1 $x(n)-x(n+1)=x(n)-I_\theta(x(n))=[\![T_{-\theta}u(n)]\!]$ for $n\geq 0$ and so, the elements $x(n)-x(n+1)$ for $n<N_u$ belong to one (and the same) of the following one-element sets

$$[\![T_{-\theta}\Pi^{(nw)}]\!]\quad\text{or}\quad[\![T_{-\theta}\Pi^{(ne)}]\!]\quad\text{or}\quad[\![T_{-\theta}\Pi^{(sw)}]\!]\quad\text{or}\quad[\![T_{-\theta}\Pi^{(se)}]\!],\tag{2}$$

and for $n\geq N_u$ relations $x(n)-x(n+1)\in[\![T_{-\theta}\Pi^{(c)}]\!]$ hold. For completion of the proof of Theorem it suffices to note that, by definition, the set of elements (2) coincides with the set of elements $(-1,0)$, $(1,0)$, $(0,-1)$, $(0,1)$, while the set $[\![T_{-\theta}\Pi^{(c)}]\!]$ is exactly the point $(0,0)$. □

REFERENCES

[1] N. Kuznetsov, P. Kloeden, *The Problem of Information Stability in Computer Studies of Continuous Systems*, Technical Report M95/01, Deakin University, Australia, p. 1 (1995).

[2] M. Blank, *Pathologies Generated by Round-off in Dynamical Systems*, Phisica D 78, p. 93 (1994).

[3] P. Diamond, P. Kloeden, V. Kozyakin, A. Pokrovskii, *Boundedness and Dissipativity of Truncated Rotations on Uniform Lattices*, Technical Report M95/06, Deakin University, Australia, p. 1 (1995).

[4] V. Kozyakin, N. Kuznetsov, A. Pokrovskii, I. Vladimirov, *Some Problems in Analysis of Discretizations of Continuous Dynamical Systems*, Deakin University, Geelong, Australia, CADSEM Report 96-019, p. 1 (1996) (accepted for publication in "Proceedings of the World Congress of Nonlinear Analysts WCNA-96").

[5] I. Vladimirov, *Quantized Linear Systems on Integer Lattices: Frequency-based Approach. Part I*, Deakin University, Geelong, Australia, CADSEM Report 96-032, p. 1 (1996).