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On saturating sets in projective spaces

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Abstract

Minimal saturating sets in projective spaces $PG(n,q)$ are considered. Estimates and exact values of some extremal parameters are given. In particular the greatest cardinality of a minimal 1-saturating set has been determined. A concept of saturating density is introduced. It allows to obtain new lower bounds for the smallest minimal saturating sets. A number of exhaustive results for small q are obtained. Many new small 1-saturating sets in $PG(2,q)$, $q \leqslant 587$, are constructed by computer.

Keywords: Covering codes; Projective geometry; Saturating sets

1. Introduction

We consider the n-dimensional projective space $PG(n,q)$ over the Galois field $GF(q)$. For an introduction to such spaces and geometrical objects therein, see [11–15].

Definition 1. Let ϱ be an integer with $0 \leqslant \varrho \leqslant n$. A point set S in the space $PG(n,q)$ is ϱ-saturating if ϱ is the least integer such that for any point $x \in PG(n,q)$ there exist $\varrho + 1$ points in S generating a subspace of $PG(n,q)$ in which x lies.

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One can compare it with the definitions in [2, Definition 1.1; 7, 24]. Note that the term "saturated" for points in S was applied in [24] and then was used, e.g., in [2,3,16]. But in [22] the points of $PG(n,q)\backslash S$ are said to be saturated, and as we find this definition more natural, we adopt it in [7] and here. So, the points in S are "saturating".

A q-ary linear code with codimension r has covering radius R if every r-positional q-ary column is equal to a linear combination of R columns of a parity check matrix of this code and R is the smallest value with such property. For an introduction to coverings of vector spaces over finite fields and to the concept of code covering radius, see [1].

The points of a saturating set in $PG(n,q)$ can be considered as columns of a parity check matrix of a q-ary linear code with codimension $n+1$. So, in terms of the coding theory, a ϱ-saturating l-set in $PG(n,q)$ corresponds to a parity check matrix of a q-ary linear code with length l, codimension $n+1$, and covering radius $\varrho+1$ [2,7,8,16,18]. Such code is denoted by an $[l,\ l-(n+1)]_q(\varrho+1)$ code.

Definition 2 (Ughi [24]). A ϱ-saturating set of l points is called minimal if it does not contain a ϱ-saturating set of $l-1$ points.

We denote by $l(n,q,\varrho)$ the size of the smallest possible minimal ϱ-saturating set in the space $PG(n,q)$. The corresponding *best known* value is denoted by $\bar{l}(n,q,\varrho)$. Let $\theta(n,q)=(q^{n+1}-1)/(q-1)=|PG(n,q)|$. It is clear that

$$l(n,q,0)=\theta(n,q),\quad l(n,q,n)=n+1.$$

One can use all points of $PG(n,q)$ to obtain $l(n,q,0)$ [7] and any $n+1$ independent points of $PG(n,q)$ to get $l(n,q,n)$. We have demonstrated in this paper that the greatest cardinality of a minimal 1-saturating set in $PG(n,q)$ is equal to $\theta(n-1,q)+1$ for all q. Finding values of (or good bounds on) $l(n,q,\varrho)$ is a hard problem for $0<\varrho<n$.

We introduced a concept of a saturating density for ϱ-saturating sets in the space $PG(n,q)$. The saturating density is a characteristic of quality of small saturating sets similar to the covering density of covering codes in coding theory. In particular, the saturating density allows us to see how many times in average the points of a projective space are saturated.

We obtained lower bounds on $l(n,q,\varrho)$ using the concept of saturating density and on $l(2,q,1)$ using the theory of blocking sets and complete arcs in $PG(2,q)$.

Let $t_2(2,q)$ be the size of the smallest complete arc [12] in $PG(2,q)$. The corresponding *best known* value is denoted by $\bar{t}_2(2,q)$. It can be shown that a complete arc [13, Chapters 1.2 and 1.3] in the space $PG(n,q)$ is a minimal $(n-1)$-saturating set. In particular, a complete arc in $PG(2,q)$ is a minimal 1-saturating set [24, p. 331] and it is easy to see that

$$l(2,q,1)\leqslant t_2(2,q),\quad \bar{l}(2,q,1)\leqslant \bar{t}_2(2,q).$$

The only known example with $l(2,q,1)<t_2(2,q)$ and $\bar{l}(2,q,1)<\bar{t}_2(2,q)$ was $l(2,4,1)=\bar{l}(2,4,1)=5<t_2(2,4)=\bar{t}_2(2,4)=6$ [2,7].

In this paper, by computer search in $PG(2,q)$ with $q \leqslant 587$, we obtained a number of minimal 1-saturating sets giving new values of $\bar{l}(2,q,1)$. Due to these new values in the present time $\bar{l}(2,q,1) < \bar{t}_2(2,q)$ for $q = 32, 64, 83, 97, 128, 131, 137, 103 \leqslant q \leqslant 125$, $169 \leqslant q \leqslant 587$, [5,6]. We have proved [19] that

$$l(2,q,1) = t_2(2,q) \quad \text{for } 3 \leqslant q \leqslant 16, \ q \neq 4.$$

We conjecture that

$$l(2,q,1) < 4\sqrt{q};$$

we have proved it for $q \leqslant 587$ by the new values of $\bar{l}(2,q,1)$ obtained with the help of computer [5,19].

In $PG(2,q)$ by computer we have classified all the minimal 1-saturating sets for $q \leqslant 8$ and all the smallest minimal 1-saturating sets for $q \leqslant 13$ [19].

We have described constructions of a minimal 1-saturating $(\theta(n-1,q)+1)$-set and a minimal 1-saturating $\theta(n-1,q)$-set in $PG(n,q)$. We use the following notations for the space $PG(n,q)$: $m(n,q,\varrho)$ is the size of the largest minimal ϱ-saturating sets, $m'(n,q,\varrho)$ and $m''(n,q,\varrho)$ are the sizes, respectively, of the second and third largest minimal ϱ-saturating sets. We have proved that $m(n,q,1) = \theta(n-1,q)+1$ for all q and n, $m'(n,q,1) = \theta(n-1,q)$ for $q \geqslant 3$, $n \geqslant 2$, and $m''(2,q,1) = q$ for $7 \leqslant q \leqslant 25$ (see Tables 2 and 3).

Note that a ϱ-saturating set in $PG(n,q)$, $\varrho + 1 \leqslant n$, can generate an infinite family of ϱ-saturating sets in $PG(N,q)$ with $N = n + (\varrho+1)m$, $m = 1,2,3,\ldots$ (see [1, Chapter 5.4; 2; 3, Example 6]) and references therein where such infinite families are considered as linear codes with covering radius $\varrho + 1$. In this work we present many 1-saturating sets in $PG(2,q)$.

In Section 2 we obtain the values of $m(n,q,1)$, $m'(n,q,1)$, and $m''(2,q,1)$. In Section 3 a concept of saturating density is introduced for the space $PG(n,q)$. In Section 4 we give lower bounds on $l(n,q,\varrho)$ and $l(2,q,1)$. Section 5 gives results in $PG(2,q)$ for small q. In particular the spectrum of the sizes of the minimal 1-saturating sets in $PG(2,q)$, for $q \leqslant 16$ has been obtained. Section 6 contains a list of small 1-saturating sets in $PG(2,q)$, $q \leqslant 587$, obtained by computer.

2. Values of $m(n,q,1)$, $m'(n,q,1)$, and $m''(2,q,1)$

Construction A. In the space $PG(n,q)$ let us consider a $(\theta(n-1,q)+1)$-set S_A of the following form: a whole hyperplane V of $\theta(n-1,q)$ points plus one point P not belonging to V.

Theorem 1. *The point set S_A of Construction* A *is a minimal* 1-*saturating* $(\theta(n-1,q)+1)$-*set in the space $PG(n,q)$ for all q and n.*

Proof. Let us consider the $\theta(n-1,q)$ lines containing the point P and one point of the hyperplane V. Every line contains two points of S and therefore is 1-saturated.

All the lines mentioned cover the whole space $PG(n,q)$. It is easy to see that $S_A \backslash G$ is not a saturating set where G is an arbitrary point of S_A. Hence S_A is a minimal 1-saturating set. $\square$

Remark 1. Construction A and Theorem 1 can be considered as an example of using [24, Lemma 10]. This lemma is treated as the "direct sum" construction in covering codes theory [1, Section 3.2].

Theorem 2. *Any $\theta(n-1,q)+1$ points in the space $PG(n,q)$ are a 1-saturating set.*

Proof. Let S be a $(\theta(n-1,q)+1)$-set in a space $PG(n,q)$ and let P be an arbitrary point of the space not belonging to S. Let us consider the lines containing the point P and at least one point of the set S. The number of lines through a point of $PG(n,q)$ is at most $\theta(n-1,q)$. Hence at least one line contains two or more points of S. So, the point P is 1-saturated. $\square$

Corollary 1. *The greatest cardinality of a minimal 1-saturating set in a space $PG(n,q)$ is equal to $\theta(n-1,q)+1$, i.e., $m(n,q,1)=\theta(n-1,q)+1$ for all q and n.*

Proof. By Theorem 2, $m(n,q,1) \leqslant \theta(n-1,q)+1$. On the other hand, a minimal 1-saturating $(\theta(n-1,q)+1)$-set exists, see Theorem 1. $\square$

Example 1. In the plane $PG(2,q)$ we have $m(2,q,1)=q+2$ and a $(q+2)$-set containing a whole line l of $q+1$ points and one point $P \notin l$ is a largest minimal 1-saturating set.

Example 2. For q even in the plane $PG(2,q)$ a hyperoval of $q+2$ points is another interesting example of a largest minimal 1-saturating set. Of course, a hyperoval is not connected with Construction A.

Construction B. Let $V=\{V_1, V_2, \dots, V_{\theta(n-1,q)}\}$ be a hyperplane in the space $PG(n,q)$ consisting of the points V_i. Denote by P an external point for V. Let T be a point on the line through the points V_1 and P and $P \neq T \neq V_1$. Let us consider a $\theta(n-1,q)$-set $S_B=\{V_3, V_4, \dots, V_{\theta(n-1,q)}, P, T\}$.

Theorem 3. *The point set S_B of Construction* B *is a minimal 1-saturating $\theta(n-1,q)$-set in a space $PG(n,q), q \geq 3,\ n \geqslant 2$.*

Proof. Denote by v_i a line through points V_i and P, $i=1,2,\dots,\theta(n-1,q)$. All points on the lines $v_3, v_4, \dots, v_{\theta(n-1,q)}$ are 1-saturated since $\{V_3, V_4, \dots, V_{\theta(n-1,q)}, P\} \subset S_B$. All points on the line v_1 are 1-saturated since $P, T \in v_1$.

For simplicity we suppose that the line v through V_1 and V_2 contains points $V_3, V_4, \dots, V_{q+1}$. Denote by v_i' a line through points V_i and T, $i=3,4,\dots,q+1$. The lines v, v_2, and v_i', $i=3,4,\dots,q+1$, lie in the same plane π. All points on the lines $v'_3, v'_4, \ldots, v'_{q+1}$ are 1-saturated and all these lines intersect the line v_2 in $q-1$ distinct points others than P and V_2. Finally, the point V_2 is 1-saturated since $q+1-2 \geqslant 2$ and the points V_3, V_4 always exist. So, the line v_2 is 1-saturated.

All lines v_i, $i = 1, 2, \ldots, \theta(n-1, q)$, cover the whole space $PG(n, q)$. So, S_B is a 1-saturating set. Let G be an arbitrary point of S_B. It is easy to see that $S_B \backslash G$ is not a saturating set. For example, if $G = V_3$ then one point of the line v_2 is not saturated taking into account that $V \cap \pi = v$. Hence S_B is a minimal 1-saturating set. $\square$

Corollary 2. *The cardinality of the second largest minimal* 1-*saturating set in* $PG(n, q)$, $q \geqslant 3$, $n \geqslant 2$, *is equal to* $\theta(n-1, q)$, *i.e.*, $m'(n, q, 1) = \theta(n-1, q)$ *for* $q \geqslant 3$, $n \geqslant 2$.

Remark 2. In the plane $PG(2, q)$ we have $m'(2, q, 1) = q + 1$. For q odd in $PG(2, q)$ an oval of $q+1$ points is an example of minimal 1-saturating $(q+1)$-set not connected with Construction B.

Now we consider the values of $m''(2, q, 1)$. Using computer, we got [19] minimal 1-saturating q-sets for $7 \leqslant q \leqslant 16$, see Section 5.

Example 3. We put $q = 7$. A minimal 1-saturating 7-set S_7 in $PG(2, 7)$ has the form [19]

$$S_7 = \begin{bmatrix} 1 & 0 & 0 & 1 & 1 & 1 & 1 \\ 0 & 1 & 0 & 1 & 0 & 1 & 3 \\ 0 & 0 & 1 & 1 & 5 & 6 & 5 \end{bmatrix}.$$

The following construction allows us to obtain minimal 1-saturating sets with a relatively great cardinality. We use it to get values (or lower bound on) $m''(2, q, 1)$ in this section and Section 5 by computer.

Construction C. Let an l-set S_C in $PG(2, q)$ consists of three special points $C_1 = (1, 0, 0)$, $C_2 = (1, 0, a)$, $a \in GF^*(q)$, and $C_3 = (1, 1, 0)$, and $l - 3$ points placed on a line d. The points of the line d belonging to S_C have form $(0, 0, 1)$ and $(0, 1, d_j)$ where $d_j \in GF(q)$, $j = 1, 2, \ldots, l-4$, $d_1 = 0$. The point set S_C has the form

$$S_C = \begin{bmatrix} 1 & 1 & 1 & 0 & 0 & 0 & \ldots & 0 \\ 0 & 0 & 1 & 0 & 1 & 1 & \ldots & 1 \\ 0 & a & 0 & 1 & d_1 = 0 & d_2 & \ldots & d_{l-4} \end{bmatrix}. \tag{1}$$

Using Construction C by computer we found [4] minimal 1-saturating q-sets in $PG(2, q)$ with $9 \leqslant q \leqslant 25$. For $q = 27, 29$ this construction gives minimal 1-saturating sets of size $q - 1$.

Example 4. We put $q = 17$, $a = 2$. A minimal 1-saturating 17-set S_{17} in $PG(2,17)$ has the form

$$S_{17} = \begin{bmatrix} 1 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\ 0 & 2 & 0 & 1 & 0 & 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 & 11 & 12 & 15 \end{bmatrix}.$$

Now we have

Theorem 4. *Let $7 \leqslant q \leqslant 25$. Then the cardinality of the third largest minimal 1-saturating set in $PG(2,q)$ is equal to q, i.e., $m''(2,q,1) = q$ for $7 \leqslant q \leqslant 25$.*

Remark 3. As in the plane $PG(2,q)$ a q-arc is always incomplete [12], the minimal 1-saturating q-sets cannot be arcs.

3. Saturating density in $PG(n,q)$

Definition 3. Let S be a ϱ-saturating set in the geometry $PG(n,q)$ and let x be a point of $PG(n,q)$. A generating linear combination for the point x is a linear combination of points from S having the form

$$x = \sum_{j=1}^{i} c_j a_j, \quad c_j \in GF^*(q), \ \ a_j \in S, \ \ j = 1,2,\ldots,i, \ \ 1 \leqslant i \leqslant \varrho + 1, \tag{2}$$

where we may put one of the coefficients c_k, $k = 1,\ldots,i$, equal to 1.

By Definitions 1 and 3, if S is a ϱ-saturating set in $PG(n,q)$ then there exist at least one generating linear combination for every point x of $PG(n,q)$.

Definition 4. Let S be a ϱ-saturating set in the geometry $PG(n,q)$. The saturating density $\varphi_\varrho(n,q)$ of S is the average number of generating linear combinations for the points of $PG(n,q)$.

Let S be a ϱ-saturating l-set in the space $PG(n,q)$. By Definition 4, the saturating density $\varphi_\varrho(n,q)$ of S can be calculated as follows:

$$\varphi_\varrho(n,q) = \frac{\sum_{i=1}^{\varrho+1} (q-1)^{i-1}\binom{l}{i}}{|PG(n,q)|} = \frac{\sum_{i=1}^{\varrho+1} (q-1)^{i}\binom{l}{i}}{q^{n+1}-1}. \tag{3}$$

In relation (3), $\binom{l}{i}$ is the number of subsets consisting of i points of S and $(q-1)^{i-1}$ is the number of generating linear combinations which can be obtained from the given subset of i points.

By above, for any ϱ-saturating set S we have

$$\varphi_\varrho(n,q) \geqslant 1. \tag{4}$$

The saturating density of the smallest known ϱ-saturating sets in the space $PG(n,q)$ is denoted by $\bar{\varphi}_\varrho(n,q)$.

For comparison with coding theory note that the covering density $\mu_{\varrho+1}$ of a covering code with length l, codimension $n+1$, and covering radius $\varrho+1$ is calculated in a form close to (3) [1]:

$$\mu_{\varrho+1} = \frac{\sum_{i=0}^{\varrho+1} (q-1)^i \binom{l}{i}}{q^{n+1}}, \tag{5}$$

where $\sum_{i=0}^{\varrho+1} (q-1)^i \binom{l}{i}$ is the cardinality of the Hamming sphere of radius $\varrho+1$ in the space of q-ary vectors of length l.

4. Lower bounds on $l(n,q,\varrho)$ and $l(2,q,1)$

By (3) and (4), we have the following natural bounds on $l(n,q,\varrho)$:

$$\sum_{i=1}^{\varrho+1} (q-1)^{i-1} \binom{l(n,q,\varrho)}{i} \geqslant \frac{q^{n+1}-1}{q-1} = |PG(n,q)| \tag{6}$$

or

$$\sum_{i=0}^{\varrho+1} (q-1)^{i} \binom{l(n,q,\varrho)}{i} \geqslant q^{n+1}. \tag{7}$$

We can call bounds (6) and (7) by "saturating density bounds". The corresponding lower bound in covering code theory obtained from (5) is called the "sphere bound" [1].

For $PG(2,q)$ and $\varrho = 1$, by (6),

$$l(2,q,1) + (q-1)l(2,q,1)(l(2,q,1)-1)/2 \geqslant q^2+q+1. \tag{8}$$

In fact, bound (8) is a natural lower bound for complete arcs. There exist other lower bounds for complete arcs (say, A-bounds), e.g., [12, Theorems 9.12 and 9.13]. We can slightly improve these bounds obtaining *bounds for* 1-*saturating sets that are not complete arcs* (say, NA-bounds), i.e., bounds for 1-saturating sets that contain at least one subset of three points on the same line. Let a lower NA-bound give $l(2,q,1) \geqslant l_{NA}$. Then

$$l(2,q,1) \geqslant \min\{t_2(2,q), \lceil l_{NA} \rceil\}. \tag{9}$$

If $\lceil l_{NA} \rceil \geqslant t_2(2,q)$ we can take a $t_2(2,q)$-arc as a 1-saturating set of the smallest size. Moreover, if $\lceil l_{NA} \rceil > t_2(2,q)$ then only $t_2(2,q)$-arcs are 1-saturating sets of the smallest size. Note that for $q \leqslant 29$ the exact values of $t_2(2,q)$ are known [9,10,20].

To obtain NA-bounds we can use approaches for A-bounds taking into account that there exists at least one subset of three points on the same line. For example, we use the approach of [12, Theorem 9.12].

Theorem 5. *Let S be a* 1*-saturating l-set in $PG(2,q)$ containing three points on the same line and let $l<q+2$. Then*

$$l \geqslant \sqrt{2q+4.25}+1.5. \tag{10}$$

Moreover,

$$l \geqslant \sqrt{2q+4.25}+2.5 \quad \text{if } A_q \geqslant 6 \text{ is an even integer} \tag{11}$$

where $A_q = \sqrt{2q+4.25}+1.5$.

Proof. Let P_i be a point of $S=\{P_1,P_2,\dots,P_l\}$ and let points P_1,P_2,P_3 lie on the same line t. Since a line is not a 1-saturating set there exists a point $P_j \notin t$ with $j \geqslant 4$. As $l<q+2$, there is a unisecant to S at every point of S. Denote by u_i a unisecant to S at the point P_i. Since S is a 1-saturating set, all q points of $u_j \setminus \{P_j\}$ belong to d-secants of $S \setminus \{P_j\}$ with $d \geqslant 2$. The number N_l of such d-secants is at most $\frac{1}{2}(l-1)(l-2)-2$. So, $\frac{1}{2}(l-1)(l-2)-2 \geqslant N_l \geqslant q$. It gives (10).

Let $A_q \geqslant 6$ be an even integer. Then we can put $l=A_q$ and obtain $\frac{1}{2}(l-1)(l-2)-2=q$. We suppose also that $\{P_1,P_2,P_3\}$ is the only subset of three collinear points of S. Else $N_l<\frac{1}{2}(l-1)(l-2)-2$ and $N_{A_q}<q$, i.e., we immediately obtain $l \geqslant A_q+1$. So, $N_l=\frac{1}{2}(l-1)(l-2)-2$ and $N_{A_q}=q$. We will show that for $l=A_q$ there exists a unisecant u_{j^*}, $j^* \geqslant 4$, such that some its point is the intersectional point I of two d-secants of S, say d_1 and d_2, with $d \geqslant 2$. So, at least one point of the unisecant u_j is 1-saturated with the help of two d-secants of S, $d \geqslant 2$. It will imply the requirement $N_l \geqslant q+1$ and we will get the desired bound $l \geqslant A_q+1$.

Let $b_{i,v}$ be a 2-secant of S through points P_i and P_v. We put $l=A_q$, $d_1=t$, $d_2=b_{4,5}$. Let us consider the lines through the intersectional point I and points P_i, $i \geqslant 6$. As A_q-5 is odd, at least one line mentioned is an unisecant u_{j^*}, $j^* \geqslant 6$. □

We will obtain once more an NA-bound. We apply the approach connected with using lower bounds on blocking sets for obtaining A-bounds, see, for example, [23]. We slightly paraphrase [12, Lemma 13.9].

Lemma 1. *Let S be a* 1*-saturating l-set in $PG(2,q)$ containing three points on the same line and let $l<q+2$, $q \geqslant 3$. The dual in $PG(2,q)$ of the set of d-secants of S with $d \geqslant 2$ is a blocking set of size at most $\frac{1}{2}l(l-1)-2$.*

Proof. The proof is the same as the proof of [12, Lemma 13.9]. □

Corollary 3. *Let S be a* 1*-saturating l-set in $PG(2,q)$ containing three points on the same line with $l<q+2$, $q \geqslant 3$. Let B_q be a lower bound on the size of a blocking set in $PG(2,q)$. Then*

$$l \geqslant \sqrt{2B_q+4.25}+0.5. \tag{12}$$

Proof. By Lemma 1, $\frac{1}{2}l(l-1)-2 \geqslant B_q$. □

Corollary 4. *Let S be a* 1*-saturating l-set in $PG(2,q)$ containing three points on the same line with $l<q+2$, $q\geqslant 3$. Then*

$$l\geqslant\sqrt{3q+7.25}+0.5 \quad \text{if } q \text{ is prime}, \tag{13}$$

$$l\geqslant\sqrt{2q+2\sqrt{q}+6.25}+0.5 \quad \text{if } q \text{ is square}, \tag{14}$$

$$l\geqslant\sqrt{2q+2\sqrt{pq}+6.25}+0.5 \quad \text{if } q=p^h,\ h\geqslant 3 \text{ is odd}, \tag{15}$$

$$l\geqslant\sqrt{2q+\sqrt[3]{4q^2}+6.25}+0.5 \quad \text{if } q=p^h,\ h\geqslant 3 \text{ is odd},\ p=2,3, \tag{16}$$

$$l\geqslant\sqrt{2q+2\sqrt[3]{q^2}+6.25}+0.5 \quad \text{if } q=p^h,\ h\geqslant 3 \text{ is odd},\ p>3. \tag{17}$$

Proof. By [12, Theorem 13.18], $B_q\geqslant\frac{3}{2}(q+1)$ if q is prime, $B_q\geqslant q+\sqrt{q}+1$ if q is square, $B_q\geqslant q+\sqrt{pq}+1$ if $q=p^h$, $h\geqslant 3$ is odd. Besides, by [14, Table 6.1], $B_q\geqslant q+\sqrt[3]{q^2/2}+1$ *if* $q=p^h$, $h\geqslant 3$ is odd, $p=2,3$, and $B_q\geqslant q+\sqrt[3]{q^2}+1$ if $q=p^h$, $h\geqslant 3$ is odd, $p>3$. $\square$

We denote by l_q and l'_q, respectively, the lower NA-bound on the size l of 1-saturating l-sets in $PG(2,q)$ given in Theorem 5 and Corollaries 3 and 4. By (9)

$$l(2,q,1)\geqslant\min\{t_2(2,q),\lceil\max\{l_q,l'_q\}\rceil\} \tag{18}$$

The theoretical lower bounds on $l(2,q,1)$ for $3\leqslant q\leqslant 29$ are written in the 6th column of Table 1 where $*$ notes the situation when only $t_2(2,q)$-arcs can be saturating sets of the smallest size, $T_q=\lceil\max\{l_q,l'_q\}\rceil$, $C_q=\lfloor 4\sqrt{q}-\bar{l}(2,q,1)\rfloor$, "Ref." means "References", "comp." means "computer", and $\bar{\varphi}_1$ denotes $\bar{\varphi}_1(2,q)$.

5. Computer search in $PG(2,q)$ for small q

This section contains the results of a computer search in $PG(2,q)$ for small q. For $3\leqslant q\leqslant 16$ some researches are exhaustive.

We have proved that for $3\leqslant q\leqslant 16$, $q\neq 4$, there is the equality $l(2,q,1)=t_2(2,q)$ [19]. For $q=3$, $8\leqslant q\leqslant 13$, all the smallest minimal 1-saturating sets are complete arcs while for $q=5,7$ there exist examples of size $l(2,q,1)$ that are not arcs.

The results about the values of $\bar{l}(2,q,1)$ and the spectrum of values for which a minimal 1-saturating set exists have been found using the randomized greedy algorithm described in the next section.

The computer results about the values of $l(2,q,1)$, $3\leqslant q\leqslant 16$, have been found using an exhaustive algorithm that exploits the equivalence properties among sets of points of $PG(2,q)$ to reduce the search space. Using the same algorithm all the minimal 1-saturating sets have been classified for $q\leqslant 8$.

Table 1
Bounds on $l(2,q,1)$, $3 \leqslant q \leqslant 29$, $T_q = \lceil \max\{l_q, l'_q\} \rceil$, $C_q = \lfloor 4\sqrt{q} - \bar{l}(2,q,1) \rfloor$, $\bar{\varphi}_1 = \bar{\varphi}_1(2,q)$

q	$t_2(2,q)$	l_q	l'_q	T_q	$l(2,q,1)$ theory	$l(2,q,1)$ comp.	$\bar{l}(2,q,1)$	$4\sqrt{q}$	C_q	$\bar{\varphi}_1$	Ref.
3	4	4.70	4.53	5	4^*		4.	6.9	2	1.23.	[19]
4	6	5	4.77	5	5–6	5	5.	8	3	1.67.	[2]
5	6	5.28	5.22	6	6		6.	8.9	2	2.13.	[19]
7	6	5.77	5.82	6	6		6.	10.4	4	1.68.	[19]
8	6	7	6	7	6^*		6.	11.3	5	1.52.	[19]
9	6	6.22	6	7	6^*		6.	12	6	1.38.	[19]
11	7	6.62	6.84	7	7	7^*	7.	13.3	6	1.63.	[19]
13	8	7	7.30	8	8	8^*	8.	14.4	6	1.88.	[19]
16	9	7.52	7.30	8	8–9	9	9.	16	7	2.01.	[19]
17	10	7.68	8.13	9	9–10		10	16.5	6	2.38	[4]
19	10	9	8.52	9	9–10		10	17.4	7	2.15	[4]
23	10	8.59	9.23	10	10		10.	19.2	9	1.81.	[4]
25	12	8.86	8.64	9	9–12		12	20	8	2.45	[4]
27	12	9.13	9.35	10	10–12		12	20.8	8	2.28	[4]
29	13	9.39	10.21	11	11–13		13	21.5	8	2.52	[4]

Table 2
All sizes of minimal 1-saturating l-sets in $PG(2,q)$, $3 \leqslant q \leqslant 13$

q	$l(2,q,1)$	Sizes l of minimal 1-saturating sets with $l(2,q,1) < l \leqslant q$	$m'(2,q,1) = q+1$	$m(2,q,1) = q+2$	Ref.
3	4_1^*		4_1	5_1	[19]
4	5_1		5_1	6_3	[19]
5	6_6		6_6	7_1	[19]
7	6_3	7_7	8_{31}	9_3	[19]
8	6_1^*	$7_2, 8_{60}$	9_{18}	10_5	[19]
9	6_1^*	$7 \leqslant l \leqslant 9 = q$	10	11	[19]
11	7_1^*	$8 \leqslant l \leqslant 11 = q$	12	13	[19]
13	8_2^*	$9 \leqslant l \leqslant 13 = q$	14	15	[19]

In the following tables we use these notations: a point indicates the cases $\bar{l}(2,q,1) = l(2,q,1)$ and $\bar{\varphi}_1(2,q) = \varphi_1(2,q)$, $C_q = \lfloor 4\sqrt{q} - \bar{l}(2,q,1) \rfloor$, the asterisk $*$ notes that only $t_2(2,q)$-arcs are the 1-saturating sets of the smallest size, "Ref." means "References".

In Table 1 new lower bounds on $l(2,q,1)$, obtained by computer, are written in the 7th column. In this table values of $\bar{l}(2,q,1)$ and $\bar{\varphi}_1(2,q)$ are given as well. By Table 1, for $q \leqslant 29$, $q \neq 4$, we have $\bar{l}(2,q,1) = t_2(2,q)$.

In Table 2 all sizes of minimal 1-saturating sets in $PG(2,q)$, for small q , are given. The subscripts indicate the number of nonequivalent minimal 1-saturating sets.

In the examples below $|S_i| = i$ and, besides, similarly to [7,21] we represent elements of Galois fields as follows. If q is prime, the elements are $GF(q) = \{0, 1, \ldots, q-1\}$ and we operate on these modulo q. If q is a degree of a prime, we denote $GF(q) = \{0, 1 = \alpha^0, 2 = \alpha^1, \ldots, q-1 = \alpha^{q-2}\}$ where α is a primitive element. This defines multiplication. For addition we use a primitive polynomial generating the field. For example, we can design the table of Zech logarithms [7,17,21]. In this work the primitive polynomials are [17] $x^2 + x + 2$ for $q = 25$ and $x^3 + 2x^2 + x + 1$ for $q = 27$.

Example 5. For the case $l(2, q, 1) = t_2(2, q)$ we give the examples of the smallest 1-saturating sets S_i that are not complete caps.

$q = 5$, $l(2, 5, 1) = t_2(2, 5) = 6, S_6 = \{(1, 1, 0), (1, 2, 0), (1, 3, 0), (1, 4, 0), (1, 0, 1), (1, 1, 1)\}$.

$q = 7$, $l(2, 7, 1) = t_2(2, 7) = 6$, $S_6 = \{(1, 0, 0), (0, 1, 0), (0, 0, 1), (1, 1, 1), (1, 1, 6), (1, 6, 4)\}$.

For $q = 5$ we used Construction B with $L_i = (1, i-2, 0), P = (1, 0, 1), T = (1, 1, 1)$.

In Table 3 all known sizes of minimal 1-saturating sets in $PG(2, q)$, for $16 \leqslant q \leqslant 29$, are given.

By Tables 2 and 3 one can see that for $3 \leqslant q \leqslant 25$, $q \neq 23$, there exist minimal 1-saturating sets of all the sizes in the interval $[\bar{l}(2, q, 1),\ q+2]$. Besides, $m''(2, q, 1) \geqslant q-1$ for $q = 27, 29$.

Example 6. For the case $\bar{l}(2, q, 1) = \bar{t}_2(2, q) = t_2(2, q)$ we give examples when a value of $\bar{l}(2, q, 1)$ is achieved by a 1-saturating set that is not a complete cap.

$q = 17$, $\bar{l}(2, 17, 1) = \bar{t}_2(2, 17) = t_2(2, 17) = 10$, $S_{10} = \{(1, 0, 0), (1, 1, 0), (0, 1, 0), (1, 1, 9), (1, 10, 3), (1, 10, 2), (1, 6, 1), (1, 9, 4), (1, 2, 13), (1, 2, 3)\}$.

$q = 19$, $\bar{l}(2, 19, 1) = \bar{t}_2(2, 19) = t_2(2, 19) = 10$, $S_{10} = \{(1, 0, 0), (1, 10, 9), (1, 0, 14), (1, 18, 10), (1, 6, 7), (1, 3, 5), (1, 3, 0), (1, 16, 13), (0, 1, 14), (1, 9, 2)\}$.

$q = 25$, $\bar{l}(2, 25, 1) = \bar{t}_2(2, 25) = t_2(2, 25) = 12$, $S_{12} = \{(1, 0, 0), (1, 2, 12), (1, 3, 8), (1, 24, 7), (1, 3, 15), (1, 7, 10), (1, 8, 14), (1, 0, 7), (1, 13, 2), (1, 14, 14), (1, 16, 8), (1, 2, 1)\}$.

Table 3
The sizes of the known minimal 1-saturating l-sets in $PG(2, q)$, $16 \leqslant q \leqslant 29$.

q	$l(2, q, 1)$	Sizes l of the known minimal 1-saturating sets with $l(2, q, 1) \leqslant l \leqslant q$	$m'(2, q, 1) = q+1$	$m(2, q, 1) = q+2$	Ref.
16	9	$9 \leqslant l \leqslant 16 = q$	17	18	[4,19]
17	$\geqslant 9$	$10 \leqslant l \leqslant 17 = q$,	18	19	[4]
19	$\geqslant 9$	$10 \leqslant l \leqslant 19 = q$,	20	21	[4]
23	10	$10 \leqslant l \leqslant 23 = q$, $l \neq 11$	24	25	[4]
25	$\geqslant 9$	$12 \leqslant l \leqslant 25 = q$,	26	27	[4]
27	$\geqslant 10$	$12 \leqslant l \leqslant 26 = q-1$	28	29	[4]
29	$\geqslant 11$	$13 \leqslant l \leqslant 28 = q-1$	30	31	[4]

$q=27$, $\bar{l}(2,27,1)=\bar{t}_2(2,27)=t_2(2,27)=12$, $S_{12}=\{(1,0,0),(1,8,17),(1,7,10),(1,23,23),\ (1,14,25),\ (0,9,5),\ (1,20,2),\ (1,7,12),\ (1,22,0),\ (1,12,1),\ (1,19,17),(1,14,6)\}$.

$q=29$, $\bar{l}(2,29,1)=\bar{t}_2(2,29)=t_2(2,29)=13$, $S_{13}=\{(1,0,0),(1,2,16),(0,1,7),(1,0,15),\ (1,15,19),\ (1,14,0),\ (1,17,12),\ (1,8,22),\ (1,15,9),\ (1,19,2),(1,21,12),(1,28,4),(1,2,12)\}$.

6. Small 1-saturating sets in $PG(2,q)$

In this section we use a randomized greedy algorithm to construct examples of small 1-saturating sets. On every step an algorithm minimizes an objective function f but some steps are executed in a random manner. The number of these steps and their ordinal numbers have been taken intuitively. Besides, if the same extremum of f can be get in distinct ways, a way is chosen randomly.

We begin to construct a saturating set by computer using a starting set of points S_0. On every step one point is added to the set. As value of the objective function f we consider the number of points in the projective space that are ϱ-saturated by the set obtained. As S_0 we can use a subset of points of a complete arc (for example, from [6]) or of a minimal ϱ-saturating set obtained in previous stages of the computer search. A generator of random numbers is used for a random choice.

The smallest known sizes $\bar{l}(2,q,1)$ of minimal 1-saturating sets in planes $PG(2,q)$ and saturating density $\bar{\varphi}_1(2,q)$ for $31\leqslant q\leqslant 587$ are given in Table 4 where $C_q=\lfloor 4\sqrt{q}-\bar{l}(2,q,1)\rfloor$, $\bar{\varphi}_1$ denotes $\bar{\varphi}_1(2,q)$.

Table 4
The minimal known sizes $\bar{l}(2,q,1)$ and saturating density $\bar{\varphi}_1=\bar{\varphi}_1(2,q)$ of 1-saturating sets in planes $PG(2,q)$. $C_q=\lfloor 4\sqrt{q}-\bar{l}(2,q,1)\rfloor$

q	$\bar{l}(2,q,1)$	$4\sqrt{q}$	C_q	$\bar{\varphi}_1$	Ref.	q	$\bar{l}(2,q,1)$	$4\sqrt{q}$	C_q	$\bar{\varphi}_1$	Ref.
31	14	22.3	8	2.76	[5,22]	131	35	45.8	10	4.48	[5]
32	13	22.6	9	2.30	[5]	137	36	46.8	10	4.53	[5]
37	16	24.3	8	3.08	[5,22]	139	37	47.2	10	4.72	[5,6]
41	16′	25.6	9	2.80	[6]	149	39	48.8	9	4.91	[5,6]
43	16′	26.2	10	2.67	[6]	151	39	49.2	10	4.84	[5,6]
47	18	27.4	9	3.13	[5,6]	157	40	50.1	10	4.91	[5,6]
49	18′	28	10	3.00	[6]	163	41	51.1	10	4.97	[5,6]
53	18′	29.1	11	2.78	[6]	167	42	51.7	9	5.10	[5,6]
59	20′	30.7	10	3.12	[6]	169	38	52	14	4.11	[2]
61	22	31.2	9	3.67	[5,22]	173	42	52.6	10	4.92	[5]
64	19	32	13	2.59	[5]	179	43	53.5	10	4.99	[5]
67	23	32.7	9	3.67	[5,21]	181	43	53.8	10	4.94	[5]
71	24	33.7	9	3.78	[5,6]	191	45	55.3	10	5.13	[5]
73	24	34.2	10	3.68	[5]	193	45	55.6	10	5.08	[5]
79	26	35.6	9	4.02	[5,6]	197	46	56.1	10	5.20	[5]
81	26	36	10	3.92	[5,6]	199	46	56.4	10	5.15	[5]
83	26	36.4	10	3.83	[5]	211	48	58.1	10	5.30	[5]

Table 4 (*continued*)

q	$\bar{l}(2,q,1)$	$4\sqrt{q}$	C_q	$\bar{\varphi}_1$	Ref.	q	$\bar{l}(2,q,1)$	$4\sqrt{q}$	C_q	$\bar{\varphi}_1$	Ref.
89	28	37.7	9	4.16	[5,6]	223	49	59.7	10	5.23	[5]
97	29	39.4	10	4.10	[5]	227	50	60.3	10	5.35	[5]
101	30	40.2	10	4.22	[5,6]	229	50	60.5	10	5.30	[5]
103	30	40.6	10	4.14	[5]	233	51	61.1	10	5.43	[5]
107	31	41.4	10	4.27	[5]	239	51	61.8	10	5.29	[5]
109	31	41.8	10	4.20	[5]	241	52	62.1	10	5.46	[5]
113	32	42.5	10	4.32	[5]	243	52	62.4	10	5.41	[5]
121	32	44	12	4.03	[2]	251	53	63.4	10	5.45	[5]
125	34	44.7	10	4.42	[5]	256	47	64	17	4.19	[2]
127	35	45.1	10	4.61	[5,6]	257	54	64.1	10	5.53	[5]
128	34	45.3	11	4.32	[5]	263	55	64.9	9	5.60	[5]
269	56	65.6	9	5.68	[5]	421	73	82.1	9	6.21	[5]
271	56	65.8	9	5.64	[5]	431	75	83.04	8	6.41	[5]
277	57	66.6	9	5.72	[5]	433	75	83.2	8	6.38	[5]
281	57	67.1	10	5.64	[5]	439	75	83.8	8	6.29	[5]
283	58	67.3	9	5.80	[5]	443	76	84.2	8	6.40	[5]
289	50	68	18	4.21	[5]	449	76	84.8	8	6.32	[5]
293	59	68.5	9	5.80	[5]	457	77	85.5	8	6.38	[5]
307	60	70.1	10	5.73	[5]	461	77	85.9	8	6.32	[5]
311	61	70.5	9	5.85	[5]	463	77	86.1	9	6.29	[5]
313	61	70.8	9	5.81	[5]	467	78	86.4	8	6.40	[5]
317	62	71.2	9	5.93	[5]	479	79	87.5	8	6.41	[5]
331	63	72.8	9	5.86	[5]	487	80	88.3	8	6.46	[5]
337	64	73.4	9	5.95	[5]	491	81	88.6	7	6.57	[5]
343	64	74.1	10	5.84	[5]	499	81	89.4	8	6.47	[5]
347	65	74.5	9	5.96	[5]	503	82	89.7	7	6.58	[5]
349	65	74.7	9	5.93	[5]	509	82	90.2	8	6.50	[5]
353	66	75.2	9	6.04	[5]	512	82	90.5	8	6.46	[5]
359	66	75.8	9	5.94	[5]	521	84	91.3	7	6.67	[5]
361	56	76	20	4.24	[2]	523	83	91.5	8	6.48	[5]
367	67	76.6	9	5.99	[5]	529	68	92	24	4.29	[2]
373	68	77.3	9	6.08	[5]	541	85	93.04	8	6.58	[5]
379	69	77.9	8	6.16	[5]	547	86	93.6	7	6.66	[5]
383	69	78.3	9	6.09	[5]	557	87	94.4	7	6.69	[5]
389	70	78.9	8	6.12	[5]	563	87	94.9	7	6.62	[5]
397	71	79.7	8	6.23	[5]	569	88	95.4	7	6.70	[5]
401	71	80.1	9	6.17	[5]	571	88	95.5	7	6.68	[5]
409	72	80.9	8	6.22	[5]	577	89	96.1	7	6.76	[5]
419	73	81.9	8	6.24	[5]	587	90	96.9	6	6.79	[5]

In column $\bar{l}(2,q,1)$ of Table 4 and a prime notes that all the known examples of minimal 1-saturating sets of size $\bar{l}(2,q,1)$ are complete arcs. Of course, it is more interesting when the value of $\bar{l}(2,q,1)$ is achieved by a 1-saturating set that is not a complete arc. For $q \geqslant 121$, q is square, in Table 4 we use the result of [2, Theorem 5.2] that gives $l(2,p^2,1) \leqslant 3p-1$.

Since $l(2,q,1) \leqslant \bar{l}(2,q,1)$, by Tables 1 and 4, we have

Theorem 6. *For the size $l(2,q,1)$ of the smallest minimal 1-saturating sets in the plane $PG(2,q)$ it holds that*

$$4\sqrt{q} - l(2,q,1) \geqslant 2 \quad \text{for } 3 \leqslant q \leqslant 587, \tag{19}$$

$$4\sqrt{q} - l(2,q,1) \geqslant 8 \quad \text{for } 23 \leqslant q \leqslant 487. \tag{20}$$

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