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On Saturating Sets in Small Projective Geometries

ALEXANDER A. DAVYDOV AND PATRIC R. J. ÖSTERGÅRD

A set of points, $S \subseteq PG(r, q)$, is said to be ϱ-saturating if, for any point $x \in PG(r, q)$, there exist $\varrho + 1$ points in S that generate a subspace in which x lies. The cardinality of a smallest possible set S with this property is denoted by $k(r, q, \varrho)$. We give a short survey of what is known about $k(r, q, 1)$ and present new results for $k(r, q, 2)$ for small values of r and q. One construction presented proves that $k(5, q, 2) \leq 3q + 1$ for $q = 2, q \geq 4$. We further give an upper bound on $k(\varrho + 1, p^m, \varrho)$.

1. INTRODUCTION

We denote the Galois field of q elements by $GF(q)$ (so q is a prime or a prime power), and let $GF(q)^* = GF(q) \setminus \{0\}$. We say that a set of points, $S \subseteq PG(r, q)$, is *ϱ-saturating* if, for any point $x \in PG(r, q)$, there exist $\varrho + 1$ points in S that generate a subspace in which x lies. The cardinality of a smallest possible set S with this property is denoted by $k(r, q, \varrho)$.

The term saturated was to our knowledge coined by Ughi in [22] and used therein for the points in S. This term has later been used, for example, in [6, 16]. In [19], however, the points in $PG(r, q) \setminus S$ are said to be saturated, and as we find this definition more natural, we adopt it here (so the points in S are saturating).

Exact values of $k(r, q, \varrho)$ are only known for the smallest parameters; in other cases, we can try to construct ϱ-saturating sets to find upper bounds on this function. If $\varrho = 0$, we clearly have to include all points of $PG(r, q)$ to get a saturating set. Hence

$$k(r, q, 0) = |PG(r, q)| = (q^{r+1} - 1)/(q - 1).$$

For $\varrho > 0$, the problem of determining values of (or good bounds on) $k(r, q, \varrho)$ is highly non-trivial. In Section 2, some known results on $k(r, q, 1)$ are surveyed. Several of these results were obtained in the context of coding theory. In fact, ϱ-saturating sets in projective geometry correspond to linear codes with covering radius $R = \varrho + 1$ in coding theory. See [6, 16] for further details regarding this correspondence.

In Section 3, we consider $k(r, q, 2)$ for $r < 5$. In Section 4, we give a construction that proves $k(5, q, 2) \leq 3q + 1$ for $q = 2, q \geq 4$. Finally, in Section 5, best known upper bounds on $k(r, q, 2)$ are tabulated for $r = 3, 4, 5$ and $q \leq 16$. Some of these bounds are obtained using a computer.

2. ON 1-SATURATING SETS

The function $k(r, q, 1)$ has been fairly intensively studied, in particular, in the framework of linear codes with covering radius 2; see, for example, [4–11].

Trivially $k(1, q, 1) = 2$ (take any two distinct points in $PG(1, q)$). The following theorem gives an upper bound on $k(2, q, 1)$ for all $q \geq 8$ [13, p. 59].

THEOREM 1. *For* $q \geq 8$, $k(2, q, 1) \leq \lfloor q/2 + 2 \rfloor$.

This bound comes from constructions of *complete caps* (or *complete arcs*), which have the additional requirement that no three of the points be collinear. Work has been done on finding

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upper bounds on the smallest size of complete caps in $PG(2,q)$ that are asymptotically better than $q/2$. Such work has, for example, led to bounds of asymptotic size $q/3$ [1, 17, 21, 23], $q/4$ [17], and $2q^{9/10}$ [21]; all these bounds, however, only hold for special values of q.

For 1-saturated sets, Ughi [22, Example B] obtained a bound of order $3q^{1/2}$, and this result was slightly improved in [6, Theorem 5.2]:

THEOREM 2. *For* $p \geq 2$, $k(2, p^2, 1) \leq 3p - 1$.

The construction in [22, Example B] can, for example, be generalized as follows to obtain families of 1-saturating sets in $PG(2,q)$ of size asymptotically $2q^{(m-1)/m}$ when $m \geq 3$.

THEOREM 3. *For* $p \geq 2$ *and* $m \geq 2$, $k(2, p^m, 1) \leq 2p^{m-1} + p$.

PROOF. An element in $GF(q = p^m)$ can be expressed as

$$A = a_{m-1}\alpha^{m-1} + \cdots + a_1\alpha + a_0, \tag{1}$$

where α is a primitive element of $GF(q)$ and $a_i \in GF(p)$, $0 \leq i \leq m-1$. The 1-saturating set is given by the columns of the matrix

$$\mathbf{H} = \left[\begin{array}{ccc|ccc|ccc|ccc} 1 & 0 & 0 & 1 & \cdots & 1 & 1 & \cdots & 1 & 0 & \cdots & 0 \\ 0 & 1 & 0 & \xi_1 & \cdots & \xi_{p^{m-1}-1} & 0 & \cdots & 0 & 1 & \cdots & 1 \\ 0 & 0 & 1 & 0 & \cdots & 0 & \xi_1 & \cdots & \xi_{p^{m-1}-1} & e_1 & \cdots & e_{p-1} \end{array}\right], \tag{2}$$

where $\{e_1, e_2, \ldots, e_{p-1}\} = GF(p)^*$, and $\{\xi_1, \xi_2, \ldots, \xi_{p^{m-1}-1}\} = E \subset GF(q)^*$ consists of all non-zero elements of the form (1) with $a_{m-1} = 0$.

We shall now show that any point can be obtained as a linear combination of at most two columns of $\mathbf{H}$. For points with a 0 in a coordinate, we can use (at most) two of the first three columns of $\mathbf{H}$. So, we just need to show that a column

$$\left[\begin{array}{c} 1 \\ A = a_{m-1}\alpha^{m-1} + \cdots + a_1\alpha + a_0 \\ B = b_{m-1}\alpha^{m-1} + \cdots + b_1\alpha + b_0 \end{array}\right]$$

can be obtained as a linear combination of two columns of $\mathbf{H}$. If $a_{m-1} = 0$ or $b_{m-1} = 0$, then we take $(1, A, 0) + B(0, 0, 1)$ or $(1, 0, B) + A(0, 1, 0)$, respectively. If $a_{m-1} \neq 0$ and $b_{m-1} \neq 0$, then we have $b_{m-1} = ka_{m-1}$ where $k \in GF(p)$, and we take $(1, 0, B - kA) + A(0, 1, k)$. Note that $B - kA \in E \cup \{0\}$. □

If $m = 2$, then we obtain three independent lines in a Baer subplane of $PG(2, p^2)$ as in [22, Example B].

A further generalization of this result will be given later. For small q, better values can often be obtained by determining the exact value or by constructively finding a good upper bound (often by computer search); see [18] and [19, Table 1]. In all but one case, $k(2, 4, 1) = 5$, the exact value of or the best known upper bound on $k(2, q, 1)$ is attained by a complete cap.

For $r = 3$, we have the following result [6, Theorem 5.1], which was earlier proved in [3] for even q.

THEOREM 4. *For* $q \geq 4$, $k(3, q, 1) \leq 2q + 1$.

3. On 2-Saturating Sets

Values of $k(r, q, 2)$ for small r have previously been considered for $q = 2$ and $q = 3$; see [2, 5, 6, 12].

In this section, we consider 2-saturating sets with $r < 5$. For $r = 2$, we can take any three points that are not collinear and find $k(2, q, 2) = 3$.

Before we proceed, we present an elementary bound. This well-known bound comes from the direct sum construction in coding theory. Special cases of this result are proved in [22, (12) and Lemma 10].

THEOREM 5. $k(r + r' + 1, q, \varrho + \varrho' + 1) \le k(r, q, \varrho) + k(r', q, \varrho')$.

We can often improve on the bounds obtained using Theorem 5, but it turns out that it gives a few best known bounds for small r and q with $\varrho = 2$ (using $k(0, q, 0) = 1$, $k(1, q, 0) = q+1$, and bounds on $k(r, q, 1)$).

We shall now give a generalization of Theorem 3, which gives ϱ-saturating sets in $PG(\varrho + 1, p^m)$.

THEOREM 6. *For $p \ge 2$ and $m \ge \varrho + 1$, $k(\varrho + 1, p^m, \varrho) \le (p - 1)\binom{\varrho + 1}{2} + p^{m-\varrho}(\varrho + 1) + 1$.*

PROOF. We consider points in $PG(\varrho+1, p^m)$ as $(\varrho+2)$-tuples over $GF(p^m)$ with homogeneous coordinates, and express an element in $GF(q = p^m)$ as $A = a_{m-1}\alpha^{m-1} + \cdots + a_1\alpha + a_0$ (α is a primitive element in $GF(p^m)$). The set $E \subset GF(p^m)^*$ consists of all non-zero elements with $a_{m-1}, a_{m-2}, \ldots, a_{m-\varrho} = 0$.

We shall now prove that the following points make up a ϱ-saturating set: all points with one non-zero coordinate (that is, with weight one), and all points with two non-zero coordinates where the second non-zero coordinate is in $GF(p)$ if the first coordinate is zero and in E otherwise (that is, points of the form $(0, \ldots, 0, 1, 0, \ldots, 0, e \in GF(p)^*, 0, \ldots, 0)$ and $(1, 0, \ldots, 0, \xi \in E, 0, \ldots, 0)$). The total number of such points clearly coincides with the upper bound in the theorem.

It is not difficult to see that the requirement of being ϱ-saturating is fulfilled if we consider a point which has zero coordinates or which has, in any but the first coordinate, coordinate values in E. (We then take a linear combination of points of weight one and possibly—with coefficient 1—a point with a one in the first coordinate and an element from E in some other coordinate.) Hence, we need only consider points

$$\begin{bmatrix} 1 \\ A_1 = a_{1,m-1}\alpha^{m-1} + \cdots + a_{1,1}\alpha + a_{1,0} \\ \vdots \\ A_{\varrho+1} = a_{\varrho+1,m-1}\alpha^{m-1} + \cdots + a_{\varrho+1,1}\alpha + a_{\varrho+1,0} \end{bmatrix}$$

with $a_{i,j} \in GF(p)$ where, for all i, at least one of $a_{i,m-1}, a_{i,m-2}, \ldots, a_{i,m-\varrho}$ is non-zero.

We now write $A_i = B_i\alpha^{m-\varrho} + a_{i,m-\varrho-1}\alpha^{m-\varrho-1} + \cdots + a_{i,0}$. The polynomials B_i are of the form $B_i = b_{i,\varrho-1}\alpha^{\varrho-1} + \cdots + b_{i,0}$, where $b_{i,j} = a_{i,j+m-\varrho} \in GF(p)$. We then have $\varrho + 1$ polynomials B_i, which we can consider to be in a vector space with ϱ coordinates in $GF(p)$. These polynomials must then be linearly dependent with coefficients from $GF(p)$. Then there exists a B_i that can be expressed as a linear combination of the other polynomials. Without loss of generality, due to symmetry of the points in our saturating set, we may assume that we can write

$$B_{\varrho+1} = \sum_{i=1}^{\varrho} k_i B_i$$

with $k_i \in GF(p)$. Note that the elements $A_i - B_i\alpha^{m-\varrho} \in E \cup \{0\}$. Our proof is now completed by the fact that $(1, A_1, A_2, \dots, A_{\varrho+1}) = (1, 0, \dots, 0, A_{\varrho+1} - k_1A_1 - k_2A_2 - \cdots - k_\varrho A_\varrho) + A_1(0, 1, 0, \dots, 0, k_1) + A_2(0, 0, 1, 0, \dots, 0, k_2) + \cdots + A_\varrho(0, \dots, 0, 1, k_\varrho)$. □

If $m = \varrho + 1$, then the ϱ-saturating set in the construction in Theorem 6 consists of $(\varrho + 1)(\varrho + 2)/2$ lines in the subgeometry $PG(\varrho + 1, p)$. A further generalization of this approach, using planes, etc. in subgeometries seems possible.

Given a field $GF(p^r)$, where p is a prime, we get the best bound by finding the smallest factor in r that is greater than or equal to $\varrho + 1$ and letting this be the value of m when Theorem 6 is applied. For example, $k(3, p^6, 2) = k(3, (p^2)^3, 2) \le 6p^2 - 2$.

4. An Infinite Family with $r = 5$

In this section we shall give a construction that shows that $k(5, q, 2) \le 3q + 1$ for $q \ne 3$. The construction can be seen as taking, with slight modifications, two ovals and one line in this projective space. It can further be seen as a generalization of the (oval plus line) construction giving Theorem 4. The points of the constructions are columns of the following matrix (of size $(3q + 1) \times 6$):

$$\mathbf{H} = \left[\begin{array}{ccc|c|ccc|ccc|c|c} 1 & \cdots & 1 & 0 & 0 & \cdots & 0 & 0 & \cdots & 0 & 0 & 0 \\ a_1 & \cdots & a_q & 1 & 0 & \cdots & 0 & 0 & \cdots & 0 & 0 & 0 \\ a_1^2 & \cdots & a_q^2 & 0 & 1 & \cdots & 1 & 0 & \cdots & 0 & 0 & 0 \\ 0 & \cdots & 0 & 0 & a_2 & \cdots & a_q & 1 & \cdots & 1 & 0 & 0 \\ 0 & \cdots & 0 & 0 & 0 & \cdots & 0 & a_2 & \cdots & a_q & 0 & 1 \\ 0 & \cdots & 0 & 0 & 0 & \cdots & 0 & a_2^2 & \cdots & a_q^2 & 1 & 0 \end{array}\right] = [\mathbf{h}_1 \ \cdots \ \mathbf{h}_{3q+1}], \tag{3}$$

where $\{a_1 = 0, \dots, a_q\} = GF(q)$. We can also present the points in the following isomorphic way, thereby observing a symmetry that will later be useful (coordinates of each point $\mathbf{h}_{2q+2}, \dots, \mathbf{h}_{3q-1}$ are divided by a_i^2):

$$\mathbf{H}' = \left[\begin{array}{ccc|c|ccc|ccc|c} 1 & \cdots & 1 & 0 & 0 & \cdots & 0 & 0 & \cdots & 0 & 0 \\ a_1 & \cdots & a_q & 1 & 0 & \cdots & 0 & 0 & \cdots & 0 & 0 \\ a_1^2 & \cdots & a_q^2 & 0 & 1 & \cdots & 1 & 0 & \cdots & 0 & 0 \\ 0 & \cdots & 0 & 0 & a_2 & \cdots & a_q & a_1^2 & \cdots & a_q^2 & 0 \\ 0 & \cdots & 0 & 0 & 0 & \cdots & 0 & a_1 & \cdots & a_2 & 1 \\ 0 & \cdots & 0 & 0 & 0 & \cdots & 0 & 1 & \cdots & 1 & 0 \end{array}\right] = [\mathbf{h}'_1 \ \cdots \ \mathbf{h}'_{3q+1}]. \tag{4}$$

As the points $\mathbf{h}_{q+2}, \dots, \mathbf{h}_{2q}$ further can be given with a 1 in the fourth position, we clearly have a symmetry given by the permutation

$$(1\,6)(2\,5)(3\,4) \tag{5}$$

on the coordinates. We will now prove that every point of $PG(5, q)$ is a linear combination of at most three columns of $\mathbf{H}$ and hence that the set is 2-saturating.

We use the following notations for the points: $C^* = \{\mathbf{h}_1, \dots, \mathbf{h}_{q+1}\}$, $D^* = \{\mathbf{h}_1, \dots, \mathbf{h}_q\}$, $L^* = \{\mathbf{h}_{q+2}, \dots, \mathbf{h}_{2q}\}$.

The points $(0, 0, 1, 0, 0, 0)$ and $(0, 0, 0, 1, 0, 0)$ are called A_1 and A_2, respectively. We let $C = C^* \cup \{A_1\}$ and $D = D^* \cup \{A_1\}$, so C is a hyperoval if q is even, and D is an oval if q is odd [14, Ch. 8].

The following notations are used for spaces. The projective plane $(a, b, c, 0, 0, 0)$ is called π, the line $(0, 0, a, b, 0, 0)$ is called L, and the three-dimensional space $(a, b, c, d, 0, 0)$ is called V.

TABLE 1.
Upper bounds on $k(r, q, 2)$ for $3 \leq r \leq 5$ and $q \leq 16$.

$r \backslash q$	2	3	4	5	7	8	9	11	13	16
3	5.[d]	5.[e]	5.[b]	6.[b]	7.[a]	7.[a]	7.[a]	8[a]	8[c]	9[c]
4	6.[d]	8.[e]	9[c]	10[c]	13[c]	15[a]	16[a]	19[a]	22[a]	26[a]
5	7.[d]	11[e]	13[f]	16[f]	22[f]	25[f]	28[f]	34[f]	40[f]	49[f]

[a] Theorem 5 applied to [19, Table I]; [b] Complete arc; [c] This paper (computer search); [d] [12]; [e] [6]; [f] Theorem 7.

TABLE 2.
Explicitly listed saturating sets.

$k(3, 13, 2) \leq 8$: {(1,0,0,0), (0,1,0,0), (0,0,1,0), (0,0,0,1), (1,9,8,2), (1,3,10,11), (1,1,10,5), (1,2,1,0)}
$k(3, 16, 2) \leq 9$: {(1,0,0,0), (0,1,0,0), (0,0,1,0), (0,0,0,1), (1,0,5,5), (1,4,4,15), (1,9,9,8), (1,13,12,14), (1,12,1,3)}
$k(4, 4, 2) \leq 9$: {(1,0,0,0,0), (0,1,0,0,0), (0,0,1,0,0), (0,0,0,1,0), (0,0,0,0,1), (1,3,2,2,0), (1,1,0,3,2), (1,1,2,1,2), (0,1,3,0,3)}
$k(4, 5, 2) \leq 10$: {(1,0,0,0,0), (0,1,0,0,0), (0,0,1,0,0), (0,0,0,1,0), (0,0,0,0,1), (1,1,0,1,3), (1,0,3,1,4), (1,4,2,4,3), (0,1,1,0,3), (1,1,3,3,2)}
$k(4, 7, 2) \leq 13$: {(1,0,0,0,0), (0,1,0,0,0), (0,0,1,0,0), (0,0,0,1,0), (0,0,0,0,1), (0,1,1,1,6), (1,1,1,1,1), (1,0,1,4,4), (0,1,3,0,2), (1,5,6,4,5), (1,1,5,2,6), (1,3,3,6,2), (1,3,4,5,6)}

THEOREM 7. *The sets given by* (3) *and* (4) *are* 2*-saturating for* $q \geq 4$.

PROOF. We shall show that any point $Z = (z_1, z_2, z_3, z_4, z_5, z_6)$ can be expressed as a linear combination of at most three columns of (3) or (4). Taking the symmetry (5) into account, we need to consider four cases.

Case 1. $Z \in L$: $z_1 = z_2 = z_5 = z_6 = 0$. These points are on the line L and such a point can be expressed as a linear combination of any two distinct points on the line, that is, using at most two columns in L^*.

Case 2. $Z \in \pi$: $z_4 = z_5 = z_6 = 0$. We consider the cases q even and q odd separately. Let q be even. Then the hyperoval C has no unisecants and every point of $\pi \setminus C$ lies on $q/2 + 1$ bisecants of this hyperoval [14, Section 8.1]. Hence every point of $\pi \setminus C$ lies on $q/2$ bisecants of the point set C^*. Such bisecants exist when $q/2 \geq 1$, that is, $q \geq 2$.

Now let q be odd. Then every point of the oval D has one unisecant and every point of $\pi \setminus D$ lies on at least $(q-1)/2$ bisecants of this oval [14, Table 8.2]. Hence every point of $\pi \setminus D$ lies on at least $(q-3)/2$ bisecants of the point set D^*. Such bisecants exist when $(q-3)/2 \geq 1$, that is, $q \geq 5$.

Since, according to Case *1*, the point A_1 can also be expressed as a linear combination of two points in L^*, every point of π is a linear combination of at most two points of H.

Case 3. $Z \in V$: $z_5 = z_6 = 0$. All points in V can be expressed as a linear combination of at most two points in $C^* \cup L^* \cup \{A_2\}$ [6, Theorem 5.1]. Moreover, from Case *1* it follows that A_2 can be obtained as a linear combination of two points in L^*, so every point in V is a linear combination of at most three columns of **H**.

Case 4. $Z \notin V$ and $Z \notin V$ after applying (5): $z_1 = 1$ or $z_1 = 0, z_2 = 1$, and not both $z_5 = z_6 = 0$. Then there is a column $\mathbf{h}_i$ in C^* of the form $(z_1, z_2, x, 0, 0, 0)$. If $x = z_3$, then we use the result from Case *2* to express the point $(0, 0, 0, z_4, z_5, z_6)$ as a linear combination of at most two columns of **H**.

TABLE 3.
Zech logarithms for extension fields.

$q\backslash k$	0	1	2	3	4	5	6	7	8	9	10	11	12	13	14
4	—	2	1												
16	—	12	9	4	3	10	8	13	6	2	5	14	1	7	11

Also, due to the symmetry, there is a column $\mathbf{h}_j$ with $2q+1 \le j \le 3q+1$ of the form $a^{-1}(0, 0, 0, y, z_5, z_6)$, and the case is settled if $y = z_4$.

If $x \neq z_3$ and $y \neq z_4$, then we get $Z = \mathbf{h}_i + a\mathbf{h}_j + (z_3 - x)(0, 0, 1, (z_4 - y)/(z_3 - x), 0, 0)$. This completes the final case and the whole proof. □

It is interesting to see that the construction also works for $q = 2$ (with a slightly altered proof), and then gives a set corresponding to $k(5, 2, 2) = 7$. In coding theoretic terms, this is a so-called perfect code.

By $k(1, q, 1) = q + 1$ and Theorems 4 and 5, we find that $k(5, q, 2) \le 3q + 2$. The current construction gives a slight improvement (by 1) on this size.

Attempts were made to find smaller 2-saturating sets than those given by Theorem 7 using a computer, but without success. This indicates that, at least for small values of q, the construction is effective.

5. A TABLE

The best known bounds on $k(r, q, 2)$ for $q \le 16$ are displayed in Table 1. A period indicates an exact value. All these follow from the so-called sphere covering bound in coding theory, except for the cases $k(3, 5, 2) = 6$, $k(3, 7, 2) = 7$, $k(3, 8, 2) = 7$, $k(3, 9, 2) = 7$, and $k(4, 3, 2) = 8$, which have been proved by Penttila [20]. Bounds can sometimes be obtained in several ways, but we have restricted the keys to one construction or reference.

Several of the bounds in Table 1 were found by computer search. The saturating sets found in this way are explicitly listed in Table 2. We used a stochastic search method called *tabu search*, and applied this to our problem as described in [15].

In listing the sets, we use the following convention for the field elements. If q is a prime field, the elements are $GF(q) = \{0, 1, \ldots, q-1\}$ and we operate on these modulo q. In the case of an extension field, we denote $GF(q) = \{0, 1 = \alpha^0, 2 = \alpha^1, \ldots, q-1 = \alpha^{q-2}\}$, where α is a primitive element. This defines multiplication. Addition is defined using Zech logarithms. The Zech logarithm is the function $Z(k)$ for which $\alpha^{Z(k)} = 1 + \alpha^k$. The Zech logarithms for the extension fields used can be found in Table 3. A dash denotes $1 + \alpha^k = 0$. The primitive polynomials used to generate the fields are $x^2 + x + 1$ for $q = 4$ and $x^4 + x^3 + 1$ for $q = 16$.

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ALEXANDER A. DAVYDOV

Institute for Information Transmission Problems,
Russian Academy of Sciences,
Bol'shoi Karetnyi per. 19, GSP-4,
Moscow, 101447,
Russia
AND

PATRIC R. J. ÖSTERGÅRD
Department of Computer Science and Engineering,
Helsinki University of Technology,
P.O. Box 5400, 02015 HUT, Finland