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New Quaternary Linear Codes with Covering Radius 2[1]

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A new quaternary linear code of length 19, codimension 5, and covering radius 2 is found in a computer search using tabu search, a local search heuristic. Starting from this code, which has some useful partitioning properties, different lengthening constructions are applied to get an infinite family of new, record-breaking quaternary codes of covering radius 2 and odd codimension. An algebraic construction of covering codes over alphabets of even characteristic is also given. © 2000 Academic Press

1. INTRODUCTION

A linear code of length n, codimension r, and covering radius R over the finite field F_q is denoted by $[n, n-r]_q R$. The covering radius of a linear code is the smallest integer R such that all nonzero words in F_q^r (alternatively, all points in the projective space $PG(r-1, q)$) can be obtained as a linear combination of at most R columns of the $r \times n$ parity check matrix of the code. If, in addition, all nonzero words in F_q^r can be obtained as a linear

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combination of at least R' columns, we indicate this by writing $[n, n-r]_q R(R')$. A partition of the columns such that a required linear combination with the columns belonging to distinct subsets can always be obtained is called an (R, R')-partition. For a survey of covering codes, see [4].

For given values of r, R, and q, we consider the problem of finding $l(r, R; q)$, the minimum length n of an $[n, n-r]_q R$ code. For $R = 1$ we know that $l(r, 1; q) = (q^r - 1)/(q - 1)$, which follows from existence of Hamming codes, but the problem is highly nontrivial for $R \geq 2$.

Upper bounds on $l(r, R; q)$ are obtained by constructing corresponding linear codes. The first author has recently developed methods for constructing infinite series of codes with a given, small covering radius [5–8]. To apply these constructions effectively, we need good starting codes, which can be obtained, for example, by algebraic constructions or computer search.

In Section 2 a computer search for linear covering codes is discussed and a new quaternary linear code showing that $l(5, 2; 4) \leq 19$ is presented. Lengthening constructions are considered in Section 3. These can be applied to the new code to obtain other record-breaking quaternary codes with odd codimension and covering radius 2. In Section 4 an algebraic construction for linear codes over alphabets of even characteristic is presented. The paper is concluded in Section 5 by giving an updated table on $l(r, 2; 4)$ for $r \leq 25$.

2. COMPUTER SEARCH FOR LINEAR COVERING CODES

We shall briefly discuss here how linear covering codes can be found by computer. We define the problem as an optimization problem, to which we apply a so-called local search heuristic. A survey of earlier results on local search heuristics in coding theory can be found in [10].

We define the following parameters before the search: the order of the field q and the size of the parity check matrix $r \times n$. A feasible solution is such a parity check matrix with entries from F_q: $\mathbf{H} = [h_1\ h_2 \cdots h_n]$. The columns h_i are points in a projective space, so we further use the convention that the first nonzero entry of a column is 1.

When searching for an $[n, n-r]_q 2$ code, we want to find a solution with covering radius 2. We here minimize the number of points in the projective space that cannot be obtained as a linear combination of at most two columns of $\mathbf{H}$:

$$|\{x \in PG(r-1, q) \mid x \neq sh_i + th_j \quad \text{for all } s, t \in F_q,\ 1 \leq i < j \leq n\}|. \tag{1}$$

If we explicitly search for an $[n, n-r]_q 2(2)$ code, this can be accomplished by replacing F_q by $F_q^* = F_q \setminus \{0\}$ in (1). (This approach can in a straightforward manner be generalized to covering radii other than 2.)

To use local search we also need to define how the current solution can be changed to obtain the next solution; here such a neighbor is obtained by deleting any column of the current solution and adding any point in $PG(r-1, q)$.

With these definitions we applied *tabu search* [1], a local search heuristic, and found (within some 10 s) a quaternary $[19, 14]_4 2(2)$ code, which also has useful partitioning properties.

THEOREM 1. *There exists a* $[19, 14]_4 2(2)$ *code having a* $(2, 2)$*-partition into* 16 *subsets.*

Proof. We denote $F_4 = \{0, 1 = \alpha^0, 2 = \alpha^1, 3 = \alpha^2\}$, where α is a generator of the field. The code with the following parity check matrix is a $[19, 14]_4 2(2)$ code, which is easily checked by computer:

$$[h_1\ h_2\ \cdots\ h_{19}]$$
$$= \begin{bmatrix} 0 & 0 & 0 & 0 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\ 0 & 1 & 1 & 1 & 0 & 0 & 1 & 1 & 1 & 1 & 1 & 1 & 2 & 2 & 2 & 3 & 3 & 3 & 3 \\ 1 & 0 & 2 & 3 & 2 & 3 & 1 & 1 & 1 & 2 & 2 & 3 & 0 & 2 & 3 & 2 & 2 & 3 & 3 \\ 3 & 2 & 0 & 2 & 1 & 3 & 0 & 2 & 3 & 1 & 3 & 3 & 2 & 3 & 2 & 0 & 0 & 0 & 3 \\ 2 & 0 & 3 & 3 & 3 & 0 & 1 & 0 & 1 & 2 & 1 & 3 & 1 & 1 & 0 & 0 & 1 & 2 & 2 \end{bmatrix}.$$

One possible $(2, 2)$-partition of the columns into 16 subsets is $\{h_1, h_{19}\} \cup \{h_9, h_{17}\} \cup \{h_{10}, h_{16}\} \cup \{h_2\} \cup (h_3\} \cup \{h_4\} \cup \{h_5\} \cup \{h_6\} \cup \{h_7\} \cup \{h_8\} \cup \{h_{11}\} \cup \{h_{12}\} \cup \{h_{13}\} \cup \{h_{14}\} \cup \{h_{15}\} \cup \{h_{18}\}$. ■

We have not tried to minimize the number of subsets in the $(2, 2)$-partition, since minor further improvements on it would be insignificant for the subsequent constructions.

3. SOME RECURSIVE CONSTRUCTIONS

We shall see here how the new code obtained in Theorem 1 can be used to get other improved codes. Two constructions, which have been considered in depth in [8], will be presented.

The first construction, which is called the $M^{(4)}$ construction in [8], is as follows. The parity check matrix of an $[n, n-r]_q 2(2)$ starting code is denoted by $\mathbf{H} = [h_1\, h_2\ \cdots\ h_n]$. The parity check matrix of the resulting code is then

$$\mathbf{H}_1 = \left[\begin{array}{cccc|c|cccc|c} h_1 & h_1 & \cdots & h_1 & \cdots & h_n & h_n & \cdots & h_n & h_1 \cdots h_1 \\ 0 & \alpha^0 & \cdots & \alpha^{q^m-2} & \cdots & 0 & \alpha^0 & \cdots & \alpha^{q^m-2} & \mathbf{0} \\ 0 & \beta_1\alpha^0 & \cdots & \beta_1\alpha^{q^m-2} & \cdots & 0 & \beta_n\alpha^0 & \cdots & \beta_n\alpha^{q^m-2} & \mathbf{H}'_m \end{array}\right].$$

Here α is a primitive element in F_{q^m}, the values β_i are elements in F_{q^m} on which some further restrictions will be imposed, $\mathbf{H}'_m$ is the parity check matrix of the $[n_m = (q^m - 1)/(q - 1), n_m - m]_q 1$ Hamming code, and $\mathbf{0}$ is a zero matrix of the same size as $\mathbf{H}'_m$. The elements of the last two rows are in F_{q^m}; to get a matrix over F_q, these are mapped to m-element columns over F_q. The value of m is chosen so that the following theorems can be applied.

THEOREM 2. *If* $\mathbf{H}$ *is the parity check matrix of an* $[n, n - r]_q 2(2)$ *code having a* $(2, 2)$*-partition into* P *subsets,* $P \leq q^m$*, and* $\beta_i \neq \beta_j$ *when* h_i *and* h_j *belong to distinct subsets in the partition, then* $\mathbf{H}_1$ *is a parity check matrix for an* $[n' = nq^m + (q^m - 1)/(q - 1), n' - (2m + r)]_q 2(2)$ *code having a* $(2, 2)$*-partition into* $P + 1$ *subsets.*

Proof. Since $\mathbf{H}$ is the parity check matrix of an $[n, n - r]_q 2(2)$ code, every word $a \in F_{q^r}^*$ can be represented as $a = sh_i + th_j$ with h_i and h_j belonging to distinct subsets in the given $(2, 2)$-partition and $s, t \in F_q^*$. We now want to show that every nonzero $x = (a, b, c) \in F_{q^r} F_{q^m} F_{q^m}$ can be obtained as a linear combination of exactly two columns of $\mathbf{H}_1$.

We get three cases. If $a = 0$ and $\beta_1 b - c \neq 0$, we take $p(h_1, b/p, \beta_1 b/p) - p(h_1, 0, w)$, where w is a column of $\mathbf{H}'_m$ such that $wp = \beta_1 b - c$. If $a = 0$ and $\beta_1 b - c = 0$, then $b \neq 0$ since (a, b, c) is nonzero, and we take $(h_1, b, \beta_1 b) - (h_1, 0, 0)$. Finally, if $a \neq 0$, we solve the following system of equations for e_1 and e_2:

$$sh_i + th_j = a$$

$$se_1 + te_2 = b$$

$$s\beta_i e_1 + t\beta_j e_2 = c.$$

Since the determinant from the last two of these equations is $st(\beta_j - \beta_i)$ and $\beta_i \neq \beta_j$ when h_i and h_j belong to distinct subsets in the given $(2, 2)$-partition, there is a solution.

The new code has a $(2, 2)$-partition into $P + 1$ subsets. Namely, if h_i and h_j belong to the same subset in the partition of the original code, then we can have $[h_i \cdots]^T$ and $[h_j \cdots]^T$ in the same partition of the new code. Exceptions to this rule are the last $(q^m - 1)/(q - 1)$ columns, which are put in the same partition as columns of type $[h_t \cdots]^T$, where h_t is not in the same partition as h_1 in the original code, and $[h_1\, 0\, 0]^T$, which makes up a subset on its own. ∎

The condition on the values of β_i gives that for the number of subsets in the original $(2, 2)$-partition, $P \leq q^m$ is a necessary condition for the construction to work. With a slightly different matrix $\mathbf{H}_1$, this condition can be somewhat relaxed to get $P \leq q^m + 1$. However, this minor improvement (which is carried out for a similar construction in Theorem 3) is ineffectual for the results in this work. See [8] for details.

The second construction to be presented in this section is called the $M^{(5)}$ construction in [8]. Now we start from an $[n, n-r]_q 2(2)$ code with a parity check matrix $\mathbf{H} = [h_1\, h_2\, \cdots\, h_n]$. The parity check matrix of the new code is $\mathbf{H}_2 = [\mathbf{H}_{2,1}\, \mathbf{H}_{2,2}]$, where $n' < n$ and

$$\mathbf{H}_{2,1} = \left[\begin{array}{cccc|c|cccc} h_1 & h_1 & \cdots & h_1 & \cdots & h_{n'-1} & h_{n'-1} & \cdots & h_{n'-1} \\ 0 & \alpha^0 & \cdots & \alpha^{q^m-2} & \cdots & 0 & \alpha^0 & \cdots & \alpha^{q^m-2} \\ 0 & \beta_1\alpha^0 & \cdots & \beta_1\alpha^{q^m-2} & \cdots & 0 & \beta_{n'-1}\alpha^0 & \cdots & \beta_{n'-1}\alpha^{q^m-2} \end{array}\right],$$

$$\mathbf{H}_{2,2} = \left[\begin{array}{cccc|c|cccc} h_{n'} & h_{n'} & \cdots & h_{n'} & \cdots & h_n & h_n & \cdots & h_n \\ 0 & 0 & \cdots & 0 & \cdots & 0 & 0 & \cdots & 0 \\ 0 & \alpha^0 & \cdots & \alpha^{q^m-2} & \cdots & 0 & \alpha^0 & \cdots & \alpha^{q^m-2} \end{array}\right].$$

It is required that $h_{n'}, \ldots, h_n$ all belong to the same subset in the $(2,2)$-partition. This construction has other, stronger conditions than the one in Theorem 2.

THEOREM 3. *If* $\mathbf{H}$ *is the parity check matrix of an* $[n, n-r]_q 2(2)$ *code having a* $(2,2)$*-partition into* P *subsets,* $P \le q^m + 1 \le n$, *if* $\beta_i \ne \beta_j$ *when* h_i *and* h_j *belong to distinct subsets in this* $(2,2)$*-partition, and if* $\bigcup_{i=1}^{n'-1} \beta_i = F_{q^m}$, *then* $\mathbf{H}_2$ *is a parity check matrix for an* $[n'' = nq^m, n'' - (2m+r)]_q 2(2)$ *code having a* $(2,2)$*-partition into* $2P$ *subsets.*

Proof. Since $\mathbf{H}$ is the parity check matrix of an $[n, n-r]_q 2(2)$ code, every word $a \in F_{q^r}^*$ can be represented as $a = sh_i + th_j$ with h_i and h_j belonging to distinct subsets in the $(2,2)$-partition and $s, t \in F_q^*$. We shall show that every nonzero $x = (a, b, c) \in F_{q^r} F_{q^m} F_{q^m}$ can be obtained as a linear combination of exactly two columns of $\mathbf{H}_2$.

We get the following cases. If $a = 0$ and $b = 0$, we take $(h_n, 0, c) - (h_n, 0, 0)$. If $a = 0$ and $b \ne 0$, we first find an index k such that $c/b = \beta_k$ (this is always possible since $\bigcup_{i=1}^{n'-1} \beta_i = F_{q^m}$). Then we take $(h_k, b, \beta_k b) - (h_k, 0, 0)$. If $a \ne 0$, we get two subcases depending on whether one of the indices i and j belongs to $\{n', \ldots, n\}$ or not. If not, we solve the following system of equations for e_1 and e_2 (which always has a solution as argued in the proof of Theorem 2):

$$sh_i + th_j = a$$

$$se_1 + te_2 = b$$

$$s\beta_i e_1 + t\beta_j e_2 = c.$$

In the last case, we assume without loss of generality that $j \geq n'$ (still $a \neq 0$), and solve the following system of equations for e_1 and e_2:

$$sh_i + th_j = a$$

$$se_1 + 0 = b$$

$$s\beta_i e_1 + te_2 = c.$$

The determinant from the last two equations is $st \neq 0$, so there is always a solution.

A $(2,2)$-partition of the new code into $2P$ subsets is as follows. We let $[h_i\,0\,0]^T$ and $[h_j\,0\,0]^T$ belong to the same subset in the partition if and only if h_i and h_j belong to the same subset in the partition of the original code. In addition to these P subsets, we get another P subsets by letting $[h_i\,a\,b]^T$ and $[h_j\,c\,d]^T$, where neither a and b nor c and d are both 0, belong to the same subset in the partition if and only if h_i and h_j belong to the same subset in the partition of the original code. ■

By applying the construction in Theorem 3 repeatedly, we get an infinite family of good codes. We first prove a lemma.

LEMMA 1. *Let $q \geq 3$. If Theorem 3 can be applied to an $[n, n-r]_q 2(2)$ code to get an $[n' = nq^m,\ n' - (r+2m)]_q 2(2)$ code, then it can be applied again to the new code to get an $[n'' = n'q^{m'},\ n'' - (r + 2m + 2m')]_q 2(2)$ code, where $m + 1 \leq m' \leq 2m$.*

Proof. When Theorem 3 is applied to an $[n, n-r]_q 2(2)$ code with a $(2,2)$-partition into P subsets, we get an $[n' = nq^m,\ n' - (r+2m)]_q 2(2)$ code with a (2,2)-partition into $2P$ subsets. In this construction we must have that $P \leq q^m + 1 \leq n$. We want to apply the construction again to get an $[n'' = n'q^{m'},\ n'' - (r + 2m + 2m')]_q 2(2)$ code. For the parameter m', we can then use $m' \geq m + 1$, as $q^{m+1} + 1 > 3 \cdot q^m > 2(q^m + 1) \geq 2P$ (as $q \geq 3$). Furthermore, for m' we can choose values up to $m' \leq 2m$ as $q^{2m} + 1 < q^m(q^m + 1) \leq q^m n = n'$. ■

If $q = 2$, we can in the same way get a slightly smaller range for the possible values of m': $m + 2 \leq m' \leq 2m$.

THEOREM 4. *Let $q \geq 3$. If Theorem 3 can be applied to an $[n_0, n_0 - r_0]_q 2(2)$ code, with a parameter $m_0 \geq 2$ (resp. $m_0 = 1$), then $l(r_0 + 2m, 2; q) \leq n_0 q^m$ for all $m \geq 3m_0 + 3$ (resp. $m \geq 10$).*

Proof. When the construction is applied repeatedly, we get a sequence of values $m_0, m_1, \ldots, m_{s-1}$ where the final value of m in the expression $n_0 q^m$ is

$$m = \sum_{i=0}^{s-1} m_i. \tag{2}$$

Moreover, Lemma 1 says that we must have

$$m_i + 1 \leq m_{i+1} \leq 2m_i. \tag{3}$$

We are interested in the smallest value M such that all values $m \geq M$ can be obtained as (2) when (3) holds and m_0 is given.

For a given value of s (the number of terms), we first show that all values in the interval $m_0 + (m_0+1) + \cdots + (m_0+s-1) \leq m \leq m_0 + 2m_0 + \cdots + 2^{s-1}m_0$ can be obtained as a required sum (2) (clearly, no other values are obtainable). We can obviously get the sum giving the lower bound. Now, for any given feasible sum, we find the (unique, if it exists) position a in the interval $1 \leq a \leq s-1$, such that $m_i = 2m_{i-1}$ for $a \leq i \leq s-1$, and $m_a < 2\mathrm{m}_{a-1}$. If no such position exists, then $m_i = 2m_{i-1}$ for all $1 \leq i \leq s-1$, and we have reached the upper bound. However, if such a value of a exists, then we can increase m_a by one to increase the sum (2) by one. All values in the given interval are thus obtainable.

For the following values of s, at least the given values of m can be obtained as (2) when (3) holds and $m_0 \geq 1$:

$$\begin{aligned}
s = 1{:}&\quad m_0\\
s = 2{:}&\quad 2m_0 + 1, 2m_0 + 2, \ldots, 3m_0\\
s = 3{:}&\quad 3m_0 + 3, 3m_0 + 4, \ldots, 7m_0\\
&\vdots\\
s = t{:}&\quad tm_0 + t(t-1)/2, m_0 + t(t-1)/2 + 1, \ldots, (2^t - 1)m_0.
\end{aligned}$$

When $t \geq 4$ and $m_0 \geq 2$, we have that $tm_0 + t(t-1)/2 \leq (2^{t-1} - 1)m_0$. Hence all values $m \geq 3m_0 + 3$ occur in these sets and are obtainable. To sum up, by repeating the construction, we can get codes with length $n = n_0 q^m$ and codimension $n_0 - (r_0 + 2m)$ for $m \geq 3m_0 + 3$. Finally, if $m_0 = 1$, we get $m \geq 10$. ■

Theorem 4 is a general result. The lower bounds of m can in some cases be slightly improved as we shall see in Section 5. In particular, this holds if we in the first step can also use other values than m_0.

4. LENGTHENING BCH CODES

In the recent papers [3, 9], methods are discussed for lengthening BCH codes with q odd to get new codes with covering radius 2. The codes obtained in [3, 9] are very good for $q = 3$ and the results for $q = 5$ also give improvements on earlier results.

We shall discuss here a construction similar to those in [3, 9]. Here q is even and the parity check matrix is of the form (cf. [9, Eq. (3)])

$$\left[\begin{array}{ccccc|c} 1 & 1 & 1 & \cdots & 1 & \mathbf{0} \\ 0 & \alpha^0 & \alpha^1 & \cdots & \alpha^{q^m-2} & \mathbf{0} \\ 0 & \alpha^0 & \alpha^3 & \cdots & \alpha^{q^m-4} & \mathbf{H}'_m \end{array}\right]. \tag{4}$$

THEOREM 5. *Let $i \geq 2$, $m \geq 1$, and let im be even. Then*

$$l(2m+1, 2; q = 2^i) \leq q^m + \frac{q^m - 1}{q - 1}.$$

Proof. We prove that the matrix (4) has covering radius 2. This is done in a case-by-case proof by showing that each nonzero vector $(a, b, c) \in F_q F_{q^m} F_{q^m}$ can be expressed as a linear combination of at most two columns of that matrix.

For $a = 1$, we take $(a, b, b^3) + u(0, 0, w)$, where $u = (c + b^3)/w$ and w is a column of $\mathbf{H}'_m$, and for $a \neq 0, 1$ we use this result together with $(a, b, c) = a(1, b/a, c/a)$. If $a = 0$ and $b = 0$, we simply take $c/w(0, 0, w)$ where w is a column of $\mathbf{H}'_m$. We are now left with the case $a = 0, b \neq 0$, which is trickier.

If $a = 0$ and $b \neq 0$, we show that the equation $s(1, x_1, x_1^3) + t(1, x_2, x_2^3) = (0, b, c)$, where $s, t \in F_q^*$ and $x_1, x_2 \in F_{q^m}$, has a solution. Clearly $s = t$, so we want to solve the system

$$tx_1 + tx_2 = b$$

$$tx_1^3 + tx_2^3 = c.$$

From these two equations we get (the first equation directly and the second by some manipulation) that

$$x_1 + x_2 = \frac{b}{t}$$

$$x_1 x_2 = \frac{b^3 + ct^2}{t^2 b},$$

so x_1 and x_2 are the solutions of

$$x^2 + \frac{b}{t} x + \frac{b^3 + ct^2}{t^2 b} = 0.$$

By substitution of $x = zb/t$ this equation becomes

$$z^2 + z + \left(1 + \frac{ct^2}{b^3}\right) = 0.$$

This equation has a solution exactly when $\mathrm{Tr}(1 + ct^2/b^3) = 0$. Since $q^m = 2^{im}$ and im is even, $\mathrm{Tr}(1) = 0$. Then $\mathrm{Tr}(1 + ct^2/b^3) = \mathrm{Tr}(ct^2/b^3)$. If $c = 0$, the trace is 0 and we are done. We shall show that for any $\beta = c/b^3$ in $F_{q^m}^*$, we can find a $t \in F_q^*$ such that the required trace is zero. As $q \geq 4$, we can take three distinct elements in F_q^* that sum to zero: $t_1 + t_2 + t_3 = 0$. Then $t_1^2 + t_2^2 + t_3^2 = 0$, from which $\beta t_1^2 + \beta t_2^2 + \beta t_3^2 = 0$ and so

$$\mathrm{Tr}(\beta t_1^2) + \mathrm{Tr}(\beta t_2^2) + \mathrm{Tr}(\beta t_3^2) = 0.$$

Hence for at least one value of $i \in \{1, 2, 3\}$, $\mathrm{Tr}(\beta t_i^2) = 0$. This completes the proof. ■

5. A NEW TABLE

Upper bounds on $l(r, 2; 4)$, $r \leq 25$, are given in Table I. As can be seen, the results in this paper lead to improvements for practically all odd codimensions. The column P in Table I refers to the number of subsets in a $(2, 2)$-partition obtained by the given construction, and the column r_0 to the codimension of the code from which it was constructed.

The code in Theorem 1 has a $(2, 2)$-partition into 16 subsets and length 19. Since $16 \leq 4^2 + 1 \leq 19$, we can apply Theorem 3 with $m = 2$. Theorem 4 says that $l(5 + 2m, 2; 4) \leq 19 \cdot 4^m$ holds for $m \geq 9$, but as can be seen from Table 1, this bound actually holds for $m \geq 8$. To evaluate the quality of these codes, we use the concept of *density*. The density is the average number of codewords that are at distance less than or equal to R (the covering radius) from any word in the space. For the new code family, the density tends to

$$\frac{9 \cdot 19^2}{2^{11}} \approx 1.587$$

TABLE I
Upper Bounds on $l(r, 2; 4)$ for $r \leq 25$

r	$l(r,2;4)$	Reference	P	r_0	r	$l(r,2;4)$	Reference	P	r_0
2	2	Trivial			14	9552	[8]		
3	5	[6]			15	19456	Theorem 3	64	9
4	9	[2]			16	37888	[8]		
5	19	Theorem 1	16		17	77824	Theorem 3	64	9
6	37	[8]			18	151552	[8]		
7	85	Theorem 5			19	316672	Theorem 3	34	11
8	154	[5]			20	611328	[8]		
9	304	Theorem 3	32	5	21	1245184	Theorem 3	128	15
10	592	[8]			22	2424832	[8]		
11	1237	Theorem 2	17	5	23	4980736	Theorem 3	128	15
12	2389	[8]			24	9699328	[8]		
13	4948	Theorem 2	17	5	25	19922944	Theorem 3	128	15

as r tends to infinity. This is slightly worse than the density for the best known quaternary code family with covering radius 2 and even codimensions [8], which is approximately 1.504. However, it is a remarkable improvement on the previous record for the same parameters and odd codimensions [8], which was approximately 1.938.

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